SECOND HARMONIC GENERATION BY AN ELECTROMAGNETIC WAVE INCIDENT ON INHOMOGENEOUS PLASMA

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Waves with nonlinear dispersion are shown to be capable of second harmonic generation in inhomogeneous plasma. The second harmonic power emitted by the region of plasma and electromagnetic wave resonance is determined and it is shown that the coefficient of energy transformation into second harmonic can be of the order of unity for an arbitrarily small incident wave amplitude.

1. Second harmonic generation in nonlinear homogeneous media is possible only if we satisfy the matching conditions[1)][1]

$$\omega_2 = 2\omega_1, \quad \mathbf{k}_2 = 2\mathbf{k}_1.$$

In the case of homogeneous plasma with no magnetic field the electromagnetic wave dispersion precludes the satisfaction of the matching conditions. Nevertheless experiment demonstrates an observable second harmonic generation in the ionospheric reflection of electromagnetic waves as well as a considerable admixture of the second harmonic in the radio emission spectrum of the solar corona[2]. Here the second harmonic band of the solar radio emission roughly repeats the features of the fundamental radio emission band. There were attempts to attribute this effect by Raleigh scattering of the second harmonic of the plasma wave into electromagnetic radiation or by Raman scattering of plasma waves by thermal fluctuations (see also [2,3]).

The present paper suggests another (and it seems to us, a more direct) mechanism of second harmonic generation based on the inhomogeneity of plasma; this mechanism is retained also in cold plasma. The law of conservation of the "quasi-momentum" is no longer exact in inhomogeneous plasma so that second harmonic generation is possible, generally speaking, with any dispersion law. We consider here the quasi-classical case of a medium with slowly varying density (in comparison to the wavelength). In such a case the departure from the exact conservation of the quasi-momentum and the attendant second harmonic generation become exponentially small effects. Thus the neighborhood of points that violate the quasi-classical approximation should provide the main contribution to the second harmonic generation. There is a point in inhomogeneous plasma in which (for an electromagnetic wave whose electric vector lies in the plane of incidence) the quasi-classical property is violated in the greatest degree; at this point the dielectric permittivity turns to zero,[2)] and the electric field turns to infinity in the absence of dissipation.

[1)]These conditions can be interpreted as energy and "quasi-momentum" conservation laws.

[2)]R. Z. Sagdeev told us about the possibility of an anomalous emission of the second harmonic in the region with $\epsilon = 0$.

Second harmonic generation in the region $\epsilon = 0$ was previously discussed by Försterling and Wüster[4]. They considered that the amplitude of the first harmonic is limited by the nonlinear terms (nonlinear contribution of spatial dispersion). However dissipation is the controlling mechanism that limits the first harmonic when the field amplitudes are small enough. Therefore the independence of the second harmonic amplitude from the field as reported by the authors of[4] is possible only in the region of sufficiently high fields where dissipation can be neglected. On the other hand (see below) these conditions give rise to a multi-velocity motion, so that the analysis in[4] is incorrect.

In the present work we compute the second harmonic generation due to nonlinear effects in the region where the quasi-classical approximation is disturbed. Throughout this work the ions are considered fixed, thus assuming that the durations of all processes are sufficiently short.

2. Let us briefly discuss the results of the linear theory (see[3,5]). Let the plasma be inhomogeneous along the z axis, the electric field have the components E_y and E_z, and the magnetic field the component H_x. All quantities are proportional to $\exp(i\omega t - ik_y y)$. The magnetic field obeys the equation

$$\frac{d^2H}{dz^2} - \frac{1}{\varepsilon_1}\frac{d\varepsilon_1}{dz}\frac{dH_x}{dz} + \frac{\omega^2}{c^2}(\varepsilon_1 - \alpha_0^2)H_x = 0. \tag{2.1}$$

Here $\alpha_0 = ck_y/\omega$ is the angle of incidence of the wave, considered small from now on, ϵ_1 is the dielectric permittivity of plasma at the fundamental frequency:

$$\varepsilon_1 = 1 - \frac{4\pi e^2}{m\omega^2}N(z)\left[1 + i\frac{\nu_{\text{eff}}}{\omega}\right],$$

and ν_{eff} is the effective collision frequency.

The solution of (2.1) has the form of a standing wave that attenuates beyond the turning point $z_0(\epsilon_1 z_0 = \alpha_0^2)$ towards the plasma interior. The electric field components

$$E_y = -\frac{ic}{\omega\varepsilon_1}\frac{\partial H_x}{\partial z}, \quad E_z = \frac{ic}{\omega\varepsilon_1}\frac{\partial H_x}{\partial y}$$

at $\nu_{\text{eff}} = 0$ have a singular point at $\operatorname{Re}\epsilon_1 = 0$. The origin of this singularity is due to the fact that the frequency of the wave penetrating the plasma equals the frequency of the longitudinal plasma oscillations at this point. A resonance occurs leading to the oscillation of intense longitudinal fields in the region of the singular point:

$$\operatorname{div} \mathbf{E}_1 = 4\pi e n_1 = -\frac{E_z}{\varepsilon_1}\frac{d\varepsilon_1}{dz}.$$

We translate the origin of the coordinates to the point where $\operatorname{Re}\epsilon_1 = 0$ and assume a linear variation of density

$$N(z) = N_0(1 + z/L); \quad \varepsilon_1 = -(z/L + i\nu_{\text{eff}}/\omega).$$

We then introduce the quasi-classical property parameter $\rho = \omega L/c \gg 1$ and from now on assume that the distance from the turning point to the point where $\operatorname{Re}\epsilon_1 = 0$ is large, $\rho\alpha_0^3 \gg 1$, while the resonance region is narrow $\nu_{\text{eff}} \ll \omega\alpha_0^2$. Under these assumptions the solution of (2.1) near the point $z = 0$ has the form

$$H_x^{(1)} = -\frac{H_0}{\alpha_0}\sqrt{\frac{2}{\pi\rho}}\, e^{i\pi/4 - S_0}$$

$$\times\left[1 + \frac{\rho^2\alpha_0^2}{2}\left(\frac{z}{L} + i\frac{\nu_{\text{eff}}}{\omega}\right)\ln\rho\alpha_0\left(\frac{z}{L} + i\frac{\nu_{\text{eff}}}{\omega}\right)\right] \quad (2.2)$$

Here H_0 is the magnetic field amplitude at the plasma boundary and $S_0 = {}^2/_3\rho\alpha_0^3$. The electric fields have the form

$$E_y^{(1)} = H(0)\rho\alpha_0^2\ln k_y(z + i\lambda), \quad E_z^{(1)} = -\frac{\alpha_0 c\rho H(0)}{\omega(z + i\lambda)}, \quad (2.3)$$

where

$$H(0) = H_x^{(1)}(0), \quad \lambda = L\nu_{\text{eff}}/\omega.$$

It also follows from the theoretical results that the energy flow in the reflected wave does not equal the energy flow in the incident wave; a portion of the energy is absorbed in the region $\epsilon_1 \approx 0$. In the limit of $\nu_{\text{eff}} \to 0$ this energy can be computed from the formula

$$\int mN(z)\,\nu_{\text{eff}}\,|v_z|^2 dz = W_\nu = \frac{c}{2\pi}H_0^2 e^{-2S_0}. \quad (2.4)$$

Thus a finite portion of energy is "captured" in the neighborhood of the singular point.

3. We now consider the nonlinear effects assumed to be small. The smallness criteria are given below. Representing the magnetic field of the second harmonic in the form $H_2 = H_2(z)e^{2i(\omega t - k_y y)}$ we obtain for $H_2(z)$

$$\frac{d^2H_2}{dz^2} - \frac{1}{\varepsilon_2}\frac{d\varepsilon_2}{dz}\frac{dH_2}{dz} + \frac{4\omega^2}{c^2}(\varepsilon_2 - \alpha_0^2)H_2 = F(z), \quad (3.1)$$

where $\epsilon_2 = 1 - \omega_0^2(z)/4\omega^2$ is the dielectric permittivity for the second harmonic and $F(z)$ are nonlinear sources of the second harmonic. The expression for $F(z)$ has the form

$$F(z) = -\frac{4\pi e^3\varepsilon_2}{m^2\omega^2 c}\left[\frac{2ik_y E_z^2}{\varepsilon_1\varepsilon_2}\frac{dN}{dz} + \frac{d}{dz}\left(\frac{E_y E_z}{\varepsilon_1\varepsilon_2}\frac{dN}{dz}\right)\right.$$
$$\left. + \frac{ik_y}{2}(E_y^2 + E_z^2)\frac{d}{dz}\frac{N}{\varepsilon_2}\right]. \quad (3.2)$$

We note that $F(z)$ is proportional to the density gradient and vanishes in a homogeneous medium. Insofar as the width of the resonance region $\lambda = L\nu_{\text{eff}}/\omega \ll L$, we neglect the derivatives of ϵ_2 and separate out the main term in $F(z)$. Finally we obtain

$$\frac{d^2H_2}{dz^2} + k_2^2 H_2 = -\frac{e\alpha_0^3 H^2(0)\rho^2}{m\omega^2}\frac{d}{dz}\frac{\ln k_y(z + i\lambda)}{(z + i\lambda)^2}, \quad (3.3)$$

where $k_2^2 = 4\omega^2(\epsilon_2 - \alpha_0^2)/c^2$.

We set the boundary conditions of radiation for (3.3):

$$H_2 \to C_1 e^{-ik_2 z} \text{ for } z \to \infty \text{ and } H_2 \to C_2 e^{ik_2 z}$$

for $z \to -\infty$. Then C_1 and C_2 are determined by

$$C_1 = -\frac{1}{2ik_2}\int_{-\infty}^{\infty} F(z)e^{ik_2 z}dz, \quad C_2 = \frac{1}{2ik_2}\int_{-\infty}^{\infty} F(z)e^{-ik_2 z}dz. \quad (3.4)$$

We note the fact that the singular point $F(z)$ lies in the lower half-plane. Since the function $e^{ik_2 z}$ decreases in the upper half-plane, $C_1 = 0$. The zero value of C_1 means that the second harmonic is emitted only "backwards" and is present only in the reflected signal. Computation of C_2 yields

$$C_2 = \frac{\pi e\rho^2\alpha_0^3 H^2(0)}{mc^2 k_2}\left[C - 1 + \ln\alpha_0 + i\frac{\pi}{2}\right]e^{-\rho\nu_{\text{eff}}/\omega} \quad (3.5)$$

Here C is the Euler constant.

We now compare the energy flows of the first and second harmonics. The flow ratio

$$\frac{s_2}{s_1} = \left(\frac{eH_0}{mc\omega}\right)^2\frac{\rho^2\alpha_0^2}{\varepsilon_2(0)}\left[\frac{\pi^2}{4} + (1 + \ln\alpha_0 - C)^2\right]\exp\left\{-4S_0 - 2\rho\frac{\nu_{\text{eff}}}{\omega}\right\}. \quad (3.6)$$

According to (3.6) the emission strongly depends on the angle of incidence and has a maximum at the angles of incidence $\alpha_0 \sim \rho^{-1/3}$. We note that the second harmonic emission effect is strongest when $\nu_{\text{eff}}/\omega \lesssim 1/\rho$, i.e., when the width of the resonance region is less than or of the order of the wavelength. The criterion of applicability of the above approximation resides in the smallness of the nonlinear corrections to all the physical quantities. According to computations the most significant nonlinear correction is $\delta E_z^{(1)}$ for the first harmonic of the electrical field E_1. We compute it from the formula

$$\delta\mathbf{E}_1 = -\frac{4\pi}{i\omega\varepsilon_1}\delta\mathbf{j}_1,$$

where

$$\delta\mathbf{j}_1 = e\left[n_0\mathbf{v}_1 + n_1\mathbf{v}_0 + n_2\mathbf{v}_1^* + n_1^*\mathbf{v}_2 - \frac{N}{i\omega}\nabla(\mathbf{v}_1^*\mathbf{v}_2 + \mathbf{v}_0\mathbf{v}_1)\right].$$

In the region $\epsilon_1 \approx 0$ where

$$\frac{|E_2|}{|E_1|} \sim \beta = \frac{e|H_{(0)}|}{mc\omega}\frac{\alpha_0\omega^2}{\rho\nu_{\text{eff}}^2},$$

we obtain

$$\left|\frac{\delta E_z^{(1)}}{E_z^{(1)}}\right| \sim \frac{\beta^2\omega}{\nu_{\text{eff}}}.$$

Hence the criterion of small nonlinearity is:

$$\frac{e|H(0)|}{mc\omega} \ll \frac{\rho}{\alpha_0}\left(\frac{\nu_{\text{eff}}}{\omega}\right)^{5/2}. \quad (3.7)$$

Substituting (3.7) into (3.6) we obtain in our approximation the result for the greatest possible second harmonic generation:

$$\frac{s_2}{s_1} \lessapprox \alpha_0^2\left(\rho\frac{\nu_{\text{eff}}}{\omega}\right)^5\exp\left\{-2\rho\frac{\nu_{\text{eff}}}{\omega} - 2S_0\right\};$$

when $\rho\nu_{\text{eff}}/\omega \sim 1$ we have $s_2/s_1 \lesssim \alpha_0^2 e^{-2S_0}$.

Comparing the last formula with (2.4) we find that in our case only a small portion of the energy dissipated in the region of the singular point converts into the second harmonic. When thermal motion is taken into account, the resulting longitudinal plasma waves carry off the energy from the singular region $\operatorname{Re}\epsilon_1 = 0$ and thus limit the longitudinal field. As noted in[3] we can in this case introduce the effective dissipation

$$\nu_{\text{eff}}^{(n)} = \omega\left(\frac{v_T}{\omega L}\right)^{2/3} = \omega\left(\frac{r_D}{L}\right)^{2/3}.$$

We evaluate the effect of the pressure gradient, using equations of motion:*

$$\frac{\partial \mathbf{v}}{\partial t} - \nu_{\text{eff}}\,\mathbf{v} = -\frac{\nabla p}{mN} + \frac{e}{m}\left(\mathbf{E} + \frac{1}{c}[\mathbf{vH}]\right).$$

Hence

$$\frac{|\nabla p|}{|eN\mathbf{E}|} \lesssim \left(\frac{r^D}{L}\right)^{2/3} \ll 1. \tag{3.8}$$

According to (3.8), when $r_D \ll L$ the role of pressure is reflected in all the above formulas merely by the addition of the term $\nu_{\text{eff}}^{(n)}$ to ν_{eff}.

In our case there is another point where the quasi-classical approximation is disrupted: this is the "turning point." We should therefore expect an increase of the second harmonic emission also in the neighborhood of this point. The contribution to the emission from the "turning point" region is determined from the magnitude of the matrix element

$$V \sim \int_{-\infty}^{\infty} \Phi^2 e^{-ik_2 z} dz,$$

where $\Phi(z)$ is an Airy function characterizing the behavior of the first harmonic in the neighborhood of the "turning point," while the behavior of the second harmonic is described for the sake of simplicity by an exponential factor (we can readily see that V determines the "power" of the nonlinear source of the second harmonic in the neighborhood of the "turning point"). It is well known that matrix elements of this type are exponentially small and the exponent is proportional to L_1 where

$$L_1 \sim |\Phi/\Phi'|_{z=-\alpha_0^2 L}, \quad \Phi' = d\Phi/dz$$

(see for example[6]). Making use of these relations and evaluating $d\Phi/dz$ we find that if $\rho\alpha_0^3 \gg 1$, then $L_1 \approx (c/\omega)\rho^{2/3}$ and for $\alpha_0 < \rho^{-1/9}$ the emission of the second harmonic from the resonance region exceeds that from the neighborhood of the "turning point".

4. It is now convenient to examine some features of the phenomenon in cold plasma in Lagrangian coordinates. We first note the following consideration. According to the preceding discussion the special feature of wave behavior is associated with the longitudinal portion of the field into which the energy is pumped. It is therefore clear that we can directly separate the transverse portion of the field, to be henceforth considered as given by the linear approximation formulas: $\mathbf{E} = -\nabla\varphi + \mathbf{F}$, where $\mathbf{F}$ is determined from the equations

$$\operatorname{rot}\mathbf{F} = -\frac{1}{c}\frac{\partial \mathbf{H}}{\partial t}, \quad \operatorname{div}\mathbf{F} = 0,$$
$$\mathbf{F} = (0, F_y, F_z), \quad \mathbf{H} = (H_x, 0, 0), \tag{4.1}$$

H_x is a known function from (2.1).

According to the linear theory the motion can be considered quasi-onedimensional (see (2.4) and (2.5)). In Lagrangian variables the system of equations required for the perturbation of electron density n and the electron coordinates z and φ then assumes the following form

$$\ddot{z} = -\frac{e}{m}\frac{\partial\varphi}{\partial z} - \nu\dot{z} + \frac{e}{m}F_z,$$
$$n\frac{\partial z}{\partial a} = n_0(a),$$
$$\frac{\partial^2\varphi}{\partial z\,\partial a} = -4\pi e(n - N(z))\frac{\partial z}{\partial a}. \tag{4.2}$$

Here $z = z(a, t)$ and $a = z(a, -\infty)$.

For a linear density behavior we obtain from the system (4.2):

$$\ddot{z} + \omega^2 z\left(1 + \frac{z}{2L}\right) = \omega^2 a\left(1 + \frac{a}{2L}\right) - \nu\dot{z} + \frac{e}{m}F_z. \tag{4.3}$$

The nonlinear terms in (4.3) due to the inhomogeneity of the initial density can in principle lead to a multi-velocity flow. In our case however due to the weak inhomogeneity the multi-velocity flow occurs earlier as the unambiguous relationship between z and a is disturbed. Here $n(z, y, t)$ turns to infinity at a certain time instant. The condition for the absence of multi-velocity flow, i.e., the convergence of the Fourier density series, is based on

$$\frac{eH(0)}{mc\omega} \ll \frac{\rho}{\alpha_0}\left(\frac{\nu_{\text{eff}}}{\omega}\right)^2. \tag{4.4}$$

5. The above discussion concerned the second harmonic emission under conditions when the field in the resonance region is determined either by collision absorption or by energy dissipation by plasma waves. A self-consistent analysis for the case when nonlinear effects determine the field characteristics in the resonance region is difficult and therefore we are limited to some qualitative estimates. We first note the fact that the mathematical problem is reduced to a system of two second-order equations with nonlinear coupling that takes the interaction of the first two harmonics only into account. Nevertheless the analysis of both mechanisms, the energy dissipation by plasma waves and the second harmonic emission, can be limited by the following considerations. The conversion from electromagnetic to plasma waves is in the final analysis due to the $T\nabla n$ term in the electron motion equation (as is noted in[5], a straightforward reasoning comparing this term with the friction force yields the magnitude of the field in the resonance region in the linear case). Similarly the nonlinear terms of the type $(\mathbf{v}_2\nabla)\mathbf{v}_1^*$ determine the energy transfer into the second harmonic.

Under conditions such that $|mN(\mathbf{v}_2\nabla)\mathbf{v}_1^*| > |T\nabla n_1|$ the second harmonic emission is the controlling factor. Substituting the expressions for $v_Z^{(1)}$ and $v_Z^{(2)}$ found above, we obtain the following criterion

$$\frac{e|H(0)|}{mc\omega} > \frac{\nu_{\text{eff}}}{\omega}\frac{1}{\alpha_0}\frac{\omega r_D}{c}. \tag{5.1}$$

Furthermore, according to (2.4), the energy absorbed in the region $\epsilon_1 \approx 0$ does not depend on ν_{eff} in the case of weak dissipation. This quantity is maximum at $S_0 \sim 1$. Under the conditions of solar corona: $L \sim 10^{10}$ cm, $N \sim 10^8$ cm^{-3}, $\nu \sim 10$ sec^{-1}, and $r_D \sim 1$ cm for wavelengths of ~100 cm the energy absorbed in the singular region is of the order of the incident wave energy for angles of $\alpha_0 \sim 10^{-2}$. It is clear that the second harmonic energy flow cannot exceed the total energy dissipated in the region $\epsilon_1 \approx 0$. We can expect that the

*$[\mathbf{vH}] \equiv \mathbf{v} \times \mathbf{H}$.

coefficient of energy transformation into the second harmonic is close to unity when (5.1) holds, $S_0 \sim 1$, and $|(v_2\nabla)v_1^*| > \nu_{eff}^{(n)}|v_1|$. We note that when $S_0 \sim 1$. (5.1) holds and collision dissipation is low, one process competing with the second harmonic emission still remains; in a sense it resembles collision dissipation and consists of the formation of a multi-velocity flow. However according to (3.7) and (4.4) there is a regime where the formula for s_2/s_1 no longer holds and the multi-velocity motion has not yet occurred. Whether under these conditions the emission increasing with the field can reach its maximum value equal to the energy absorbed in the singular region is an open question.

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