NONADIABATIC RESTRAINED GAS EXPANSION WITH ARBITRARY ENERGY LOSS OR RELEASE

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A general solution is presented of the problem of gas expansion restrained by counterpressure, at a specified change of pressure p(t) in the presence of arbitrary energy loss or release. The obtained solution is detailed for possible cases of energy loss and release (bremsstrahlung and recombination radiation, thermal conductivity, Joule heat release) and for different cases of pressure variation p(t). Particular attention is paid to the case of expansion of a plasma fireball inside a powerful shock wave (e.g., a light spark or high-temperature energy release).

INTRODUCTION

In this article we present a general solution of the problem of restrained expansion of gas in the presence of arbitrary loss or release of energy. By restrained expansion we mean expansion with counterpressure, in which the rate of expansion is small compared with the local speed of sound in the gas ($t \gg a/c_s(T)$, where t is the expansion time, a the dimension of the gas volume, c_s the speed of sound in the volume; this condition is equivalent to neglect of hydrodynamics and ensures equalization of pressure in the gas). This condition is usually satisfied in a wide range of velocities, if the gas is heated to high temperatures, and it is precisely in this case when the energy loss can be appreciable. A factor restraining the expansion may be the external pressure of the surrounding gas, the pressure of an external magnetic field, the inertia of a heavy piston, etc. This problem is of interest in various phenomena in the physics of hot plasma, geophysics, thermodynamics, a number of applied-physics problems, etc. In particular, its solution can be used to describe the expansion of a plasma fireball at the center of a shock wave due to high-temperature energy release[1], for example in a light spark[2-6], the expansion of the flare following the action of a laser beam on a target in the atmosphere, to describe the behavior of atmospheric and astrophysical plasma formations.

In many problems the energy-release act (the heating of the gas) is usually separated in time from the more prolonged process of energy loss; in this case we take into account only the energy loss and its influence on the expansion of the gas, specifying the results of the heat release as initial conditions; it is possible also to describe another expansion process in the presence of a continuous energy release. In all these cases we assume for simplicity that either absorption processes or heat-release processes predominate in both expansion and compression of the gas.

1. GENERAL SOLUTION FOR NONADIABATIC EXPANSION OF GAS AT A SPECIFIED CHANGE OF THE EXTERNAL PRESSURE

We assume that the gas expands as a result of a decrease in the external pressure p(t), and that the state of the gas is specified by its volume V and mean temperature T, while its properties are characterized by the effective adiabatic exponent γ. In introducing the mean temperature, we assume that either the temperature of the gas inside the body has time to become equalized (the expansion time is $t \gg a^2/\kappa_T$, as is frequently the case when the temperature conductivity κ_T of a strongly heated or ionized gas is high, or else that it is possible to introduce an effective temperature averaged over the volume, which determines the pressure of the gas and the loss power. The gas pressure is assumed close to the external pressure, assuming the pressure drop to be small as a result of the smallness of inertial effects in the gas itself (restrained expansion, see, e.g.,[1,5,6]). We assume that the energy loss is specified in a general power-law form $dQ/dt \approx -AV^{\alpha}T^{\beta}$, where α and β are coefficients that vary weakly in the range of volume and temperature changes under consideration. Then the equation for the energy balance is

$$CdT = -pdV - AV^{\alpha}T^{\beta}dt,$$

where C is the specific heat of the gas ($C = Nk/(\gamma - 1)$) and $pV = NkT$ for a nearly-ideal gas, N is the total number of gas particles, which we assume to change little. If N(t) is specified, then it must not be taken outside the integral sign. We divide both sides of the equation by CT and integrate

$$\int \frac{dT}{T} + (\gamma - 1)\int \frac{dV}{V} = -\frac{(\gamma-1)A}{Nk}\int V^{\alpha}T^{\beta-1}\,dt,$$

or

$$TV^{\gamma-1} = T_0V_0^{\gamma-1}\exp\left\{-A_1\int_{t_0}^{t} V^{\alpha}T^{\beta-1}\,dt\right\},$$

where

$$A_1 = A/C = (\gamma - 1)A/kN.$$

We can indicate an effective method of solving this equation by using the specified function $p(t) = NkT/V$. This method consists in the following: we multiply and divide the integrand by p^l, first substituting p as a function of t, and then as the ratio NkT/V, and choose l such that T and V enter in the combination $(TV^{\gamma-1})^m$. Equating the powers of T and V, we can determine l and m. Indeed, since

$$V^{(\alpha+\beta)}T^{\beta-1-l} = (TV^{\gamma-1})^m$$

when $m = \beta - 1 - l$ and $\alpha + l = m(\gamma - 1)$, it follows that $m\gamma = \alpha + \beta - 1$ and $l\gamma = (\beta - 1)(\gamma - 1) - \alpha$. This substitution yields an equation for $y = TV^{\gamma-1}$:

$$y = y_0 \exp\left\{-\frac{A_1}{(Nk)^l}\int_{t_0}^{t} p^l y^m \, dt\right\},$$

whence, differentiating, we get

$$\frac{dy}{y^{m+1}} = -\frac{A_1}{(Nk)^l} p^l(t)\,dt,$$

or

$$\frac{1}{m}\left(\frac{1}{y^m} - \frac{1}{y_0{}^m}\right) = \frac{A_1}{(Nk)^l}\int_{t_0}^{t} p^l(t)\,dt,$$

if $m \neq 0$, and

$$y = y_0 \exp\left\{-\frac{A_1}{(Nk)^l}\int_{t_0}^{t} p^l(t)\,dt\right\},$$

if $m = 0$ (this particular case will not be considered here).

Thus, by specifying $p(t)$ and y_0, we obtain $y(t) = TV^{\gamma-1}$, but $y = pV^{\gamma}/kN$; or else $y = (kN)^{\gamma-1}$; therefore

$$V(t) = \left\{\frac{kNy}{p}\right\}^{1/\gamma} \text{ and } T(t) = \{p^{\gamma-1}y/(kN)^{\gamma-1}\}^{1/\gamma},$$

where

$$y = y_0\left\{1 + \frac{A_1 m y_0{}^m}{(kN)^l}\int_{t_0}^{t} p^l(t)\,dt\right\}^{-1/m}$$

and

$$m = \frac{1}{\gamma}(\alpha + \beta - 1) \text{ and } l = \frac{1}{\gamma}\{(\beta - 1)(\gamma - 1) - \alpha\}.$$

It is easy to see that these solutions describe both the case of energy loss and the case of energy release (in the latter case the sign of A_1 should be reversed). The gas can either expand or contract, depending on the type of change of $p(t)$ and change of the energy input.

2. PARTICULAR CASES OF PRESSURE CHANGE

a(Constant external pressure $p(t) = p_0$ and

$$y = y_0\left\{1 + \frac{mA_1y_0{}^m}{(kN)^l}(t - t_0)p_0^l\right\}^{-1/m}$$

$$= y_0\{1 + A_1 m V_0{}^{\alpha} T_0^{\beta-1}(t - t_0)\}^{-1/m}$$

$$= y_0\left\{1 + \frac{(\gamma - 1)}{Nk}\frac{m}{T_0}\dot{Q}(t_0)(t - t_0)\right\}^{-1/m}$$

where $\dot{Q}(t_0)$ is the rate of power loss or release at the instant t_0. This case can occur when laser energy is released on a target in the atmosphere, during the final stages of expansion of a plasma cloud in high-temperature energy release, etc.

b(Change of pressure in accordance with a power law $p(t) \approx B/t^s$. In this case we have

$$\int_{t_0}^{t} p^l\,dt = \frac{B^l}{(sl - 1)}\left\{\frac{1}{t_0^{sl-1}} - \frac{1}{t^{sl-1}}\right\},$$

if $sl \neq 1$, and

$$\int_{t_0}^{t} p^l\,dt = B^l \ln\left(\frac{t}{t}\right) \quad \text{if} \quad sl = 1.$$

For example, in the case of expansion of the plasma cloud (the so-called fireball) inside a shock wave of a high-temperature explosion, the pressure varies like[1]

$$p(t) \approx \frac{\mathscr{E}_0}{V} \approx \mathscr{E}_0{}^{2/5}\rho_0{}^{3/5}/t^{6/5},$$

where $\mathscr{E}_0$ is the energy release and ρ_0 is the initial density of the medium. In this case $s = 6/5$. We shall discuss this case in greater detail later (see also[5,6] for the case of a light spark).

3. PARTICULAR CASES OF ENERGY LOSS AND RELEASE

a) Case of constant energy loss or release $\alpha = \beta = 0$, which yields $m = -1/\gamma$ and $l = -(\gamma - 1)/\gamma$, and consequently

$$y = y_0\left\{1 - \frac{A_1}{\gamma y_0^{1/\gamma}}(kN)^{(\gamma-1)/\gamma}\int_{t_0}^{t}[p(t)]^{-(\gamma-1)/\gamma}\,dt\right\}^{\gamma}.$$

b) In the case of energy loss to bremsstrahlung (case of optically non-dense medium, when the range of the quanta exceeds the dimensions of the volume)

$$\frac{dQ}{dt} \approx -\frac{N^2 r_0 \varepsilon^2 \nu_s(\varepsilon)}{\hbar V} \approx -AT^{1/2}/V,$$

i.e., $\alpha = -1$ and $\beta = 1/2$, yielding $m = -3/2\gamma$ and $l = (3 - \gamma)/2\gamma$.

c) In the case of energy loss to radiation from a spherical black body (optically dense medium)

$$\frac{dQ}{dt} \approx -S\sigma T^4 \approx -AV^{2/3}T^4,$$

i.e., $\alpha = 2/3$ and $\beta = 4$, yielding $m = 11/3\gamma$ and $l = (9\gamma - 11)/3\gamma$.

d) In the case of energy loss to recombination radiation emerging from the plasma under the condition that temperature ionization is subsequently produced again ($N_e \sim$ const at sufficiently high temperatures and low diffusion from the outside) we obtain: in the case of double recombination $dQ/dt \approx A/VT^{1/2}$, i.e., $\alpha = -1$, $\beta = -1/2$ or $m = -5/2\gamma$ and $l = (5 - 3\gamma)/2\gamma$; in triple recombination $dQ/dt \approx -A/T^{9/2}V^2$, i.e., $\alpha = -2$, $\beta = -9/2$, or $m = -15/2\gamma$ and $l = (15 - 11\gamma)/2\gamma$; here $A \approx N_e^3 Z^2 \times 10^{-23}$ erg-deg$^{9/2}$/cm^6 (see, e.g. [1], p. 346, and [6]).

e) In the case of energy loss through heat conduction, provided the temperature-drop zone is commensurate with the dimension of the gas region, we have

$$\frac{dQ}{dt} \approx \chi S\frac{dT}{dx} \sim V^{1/3}T\chi(T).$$

If the heat conduction is electronic, then $\chi \sim T^{15/2}$, yielding $\alpha \approx 1/3$, $\beta = 7/2$ or $m = 17/6\gamma$, $l = (15\gamma - 17)/6\gamma$.

In some cases interest attaches to the process of gas expansion in the presence of energy release processes (e.g., ohmic heating by current, microwave heating, recombination heat release, etc.). To describe this process it is sufficient to reverse the sign of the term describing the energy loss.

f) In the case of current heating of a cylindrical plasma column we have

$$dQ / dt \sim V\sigma E_0^2 \sim VT^{3/2}E_0^2,$$

i.e., $\alpha = 1$, $\beta = 3.2$, yielding $m = 3/2\gamma$ and $l = (\gamma - 3)/2\gamma$. In this case the counterpressure p(t) can be ensured by the pressure of an external gas, an external magnetic field, the magnetic field of the current ($p_H(t) \sim H^2/8\pi \sim I^2(t)/a^2$). In the case of plasma heating by a penetrating microwave we have

$$dQ / dt \sim Vn_e\nu E_0^2 / (\omega^2 + \nu^2) \sim A_3 n_e E_0^2 \nu / \omega^2,$$

if the wave frequency ω exceeds the collision frequency $\nu = n_e \sigma_S v_e$. For Coulomb collisions ($\sigma_S \sim 1/T^2$) we obtain $dQ/dt \sim V^{-1}T^{-3/2}$, i.e., $\alpha = 1$, $\beta = -3/2$, yielding $m = -7/2\gamma$ and $l = (3 - \gamma)/2\gamma$.

It is possible to take into account the time variation of the field intensity or of other parameters, by introducing A(t) under the integral sign.

The results can be used not only to study the change of the volume and of the temperature of a heated volume of gas, but also to determine diamagnetic perturbations resulting from the change, the lifetime of plasma trails, etc. Thus, for example, the diamagnetic perturbation due to expansion of a volume of a conducting gas (in the case of penetration of a magnetic field at $t > 4\pi\sigma a^2/c^2$) produces a diamagnetic moment[2-5]

$$M(t) \sim \sigma(T) R^4 \dot{R} H_0 \approx \sigma(T) \frac{d}{dt} V^{5/3} \approx T^{3/2} V^{2/3} \dot{V},$$

if the plasma conductivity is $\sigma(T) \sim T^{3/2}$. Since $T \approx pV/kN$, we have

$$M(t) \sim p^{3/2} \frac{d}{dt} V^{19/6} \approx p^{3/2} \frac{d}{dt} \left(\frac{y}{p}\right)^{19/6\gamma}$$

where

$$= p^{3/2} \left(\frac{y}{p}\right)^{(19-6\gamma)/6\gamma} \left(\frac{\dot{y}}{p} - \frac{\dot{p}y}{p^2}\right) = p^{3/2} \left(\frac{y}{p}\right)^{19/6\gamma} \left(\frac{\dot{y}}{y} - \frac{p'}{p}\right),$$

$$\dot{y} / y = -A_1 y^m p^l / (kN)^l.$$

It is seen from this that the presence of energy loss or release can greatly change the form of M(t). For example, whereas in adiabatic expansion ($y = 0$) the zeros of M(t) corresponded to $\dot{p} = 0$, now they correspond to the condition

$$\dot{p} = -A_1 y^m p^{l+1} / (kN)^l.$$

The obtained solution can be used also for the case when the temperature is not uniform over the volume, but the loss depends only on the local temperature (e.g., any radiation loss in the case of an optically thin volume). In this case all the expressions pertain to a small volume of the medium.

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269