A DIAGRAM TECHNIQUE NEAR THE CURIE POINT AND THE SECOND ORDER PHASE TRANSITION IN A BOSE LIQUID

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The behavior of a Bose liquid as the phase-transition curve is approached from the side of the normal phase is considered by means of finite-temperature Green's functions. A renormalization method for diagrams is proposed allowing to separate in all quantities their dependence on the distance ξ from the phase-transition curve. It is proved that the zero-frequency Green's functions which determine the static properties of the system and the singularities of the thermodynamic quantities are universal homogeneous functions of the momenta and of ξ. Thus, the momentum distribution of the particles (the one-particle Green's function) has the form $n(p) = \xi^{-\alpha}g(p/\xi^{\beta})$. The exponents of ξ in the thermodynamic correlation functions (the critical indices) can be expressed in terms of α and β. The function $g(x)$ and the numbers α and β are simply related to the renormalized phonon emission vertex, for which the equation does not contain any free parameters and can, in principle, be solved numerically. Various asymptotic behaviors of the Green's function are investigated. It is shown how these results can be generalized to the case of an arbitrary system near its phase-transition curve, using the unitarity and analyticity properties in exactly the same manner as for relativistic particles of mass $M = \xi^{\beta}$.

INTRODUCTION

In a Bose liquid, above its lambda-point, as the temperature T approaches the critical value $T_c(\mu)$ (where μ is the chemical potential) the number of particles with small momenta increases in such a manner as to form the Bose condensate when the temperature is further decreased. These slow particles interact strongly, so that the problem does not contain any small parameters, except $(T - T_c)/T_c$.

The fact that the occupation numbers $n(p) = a_p^+ a_p$ are large in the region of momenta in which we are interested suggests that as $T \to T_c$ the system becomes classical (i.e., the commutator $[a^+, a]$ becomes small compared to a^+ and a). However, since the system has an infinity of degrees of freedom (i.e., is a field), the quasiclassical approximation to this problem will be quite unusual.

L. D. Landau (unpublished) was the first to notice that near the Curie point the term with $\omega_n = 0$ plays a leading role in the sums over quantum frequencies $\omega_n = 2\pi nT$ of the finite-temperature diagram technique (e.g. $G(\omega_n = 0, p \to 0) \gg G(\omega_n \neq 0, p \to 0)$, where G is the boson Green's function; cf. Sec. 2 of[1]). This manifests itself through the fact that the parts of the diagrams which have singular for $T \to T_c$ are produced by the contribution of the region $\omega_n \neq 0$, $|p| \ll 1/r_0$, where r_0 is the range of the interparticle force. The contribution from the region $\omega_n \neq 0$, $|p| \sim 1/r_0$ reduces to constants which renormalize the singular parts, similar to what happens in the case of a Fermi fluid or in the renormalization theory of quantum electrodynamics.

The attempt at separating these singularities undertaken by Patashinskiĭ and Pokrovskiĭ[1] was not consistent and led to errors (cf. the discussion in Sec. 4, Art. C of the present paper), therefore we reconsider the problem below, utilizing renormalization theory methods and methods from a previous paper by Gribov and the author[2], where a similar "strong coupling" problem was solved.

The paper is organized as follows. The formulation of the problem and a discussion of the quantities which are essential for the description of a Bose liquid near its Curie point form the content of Sec. 1. There it will be shown how the usual four-boson interaction can be reduced to an effective interaction with "phonons." In the remainder of the paper the "phonon" interaction is used, with the exception of Sec. 4, where a general method for handling arbitrary interactions and phase transitions in any system is proposed. The transition to a phononic interaction considerably simplifies the diagram analysis, and in addition, as shall be seen, the singular part of the heat capacity C_p has a simple expression in terms of the phonon D-function:

$$C_P = \text{const}\cdot D|_{\omega_n=p=0} + \text{const}.$$

At the end of Sec. 1 dimensional estimates will be made in the original equations (5)—(8). These yield the estimate (11) for the heat capacity, which has been derived earlier from different considerations[3-6].

Section 2 is devoted to a separation of the dependence of all the quantities of $(T - T_c)/T_c$. It will be shown that this dependence is given by a power law and that the relation among the powers corresponds to the similitude hypotheses (scaling laws)[3-5], a fact which may serve as a justification of these hypotheses. It will be seen that for the considered interaction the exponents are universal, i.e., do not depend on the magnitude V_0, and the range r_0 of the initial interaction, and are constant along the phase-transition curve $T = T_c(\mu)$ (or $\mu = \mu_c(T)$), where μ is the chemical potential. Equations are proposed which allow, in principle, to determine the exponents of the powers by means of computers, and these exponents will be determined in zeroth approximation. Their order of magnitude is reasonable.

Scaling laws (similitude formulas) will be derived for the Green's functions and vertex parts, e.g., $G(\mathbf{p}, \mu, T) = G_1\xi^{-\alpha}g(p/\kappa\xi^{\beta})$, where $\xi = \mu - \mu_c(T)$ is the distance from the phase-transition curve; $G_1(T)$, $\kappa(T)$ are non-universal (depend on the point T on the phase-transition curve) and α, β, $g(x)$ are universal quantities.

The system of diagram equations for the determination of G and D for $\omega_n = 0$, $|\mathbf{p}| \ll 1/r_0$ exhibits the following unusual invariance: it admits a whole family of solutions which differ by the values of the dimensional constants $G_1(T)$, $\kappa(T)$, which are not determined by these equations, but must be found by joining the solution to perturbation theory far away from the phase transition curve. This property corresponds to the similitude (scaling) hypotheses of refs.[3-5], where it was assumed that the character of large-scale fluctuations ($r \sim r_c(\xi) = 1/\kappa\xi^{\beta}$) is not changed under a change of the scale of lengths, i.e., κ.

It is proved that this system of equations allows one to represent α, β, $g(pr_c(\xi))$, etc. as a sum of diagrams involving the "relativistic" propagators $1/((kr_c)^2 + 1)$, universal constants, and integrations. All diagrams are of the same order of magnitude and $|\mathbf{k}|r_c \sim 1$ are essential in them. The analytic properties of the diagrams are analogous to those of a two-dimensional relativistic field theory, with the role of the energy-momentum vector played by the three-vector $\mathbf{p}$. The quantity $M = \kappa\xi^{\beta} = 1/r_c$ plays the role of the mass of the relativistic particle, and the singularities in the p^2-plane are situated at the points $\mathbf{p}^2 = -n^2M^2$, $n = 1, 2, 3, \ldots$.

Section 3 is devoted to an investigation of the momentum distribution of the particles in the regions $|\mathbf{p}| < \kappa\xi^{\beta}$ and $\kappa\xi^{\beta} \ll |\mathbf{p}| \ll 1/r_0$, and of the singularity of the heat capacity C_p. The theory discussed in Secs. 1–3 literally yields a power-law (possibly with a small power) singularity in the heat capacity $C_p \sim \xi^{3\beta-2}$, so that the experimentally observed[7] logarithmic behavior of C_p in liquid He I remains unexplained. (This would correspond to $\beta = 2/3$, for which so far there are no theoretical foundations.)

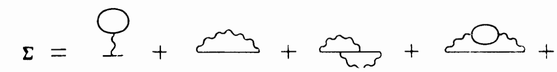
In this connection a more general approach may become necessary, which does not make use of the properties of the initial interaction. Such a semiphenomenological approach is proposed in Sec. 4 and is based on the analogy with the relativistic theory. It is shown that the system of unitarity conditions (45) yields different scaling (similitude) equations, similar to those in[2]. Arguments are advanced in favor of the idea that in the general case too the dimensionless exponents and functions will be universal for a given symmetry of the system, and a method is given for estimating the order of magnitude of the dimensional parameters G_1 and κ.

1. FORMULATION OF THE PROBLEM AND REDUCTION OF THE FOUR-BOSON INTERACTION TO A "PHONONIC" ONE

A. We shall consider the finite-temperature Green's function $G(\omega_n, \mathbf{p}, \mu, T)$ for $T > T_c(\mu)$, which satisfies the Dyson equation[1)]

$$G^{-1} = i\omega_n + \mu - \varepsilon_0(\mathbf{p}) - \Sigma(\omega_n, \mathbf{p}, \mu, T). \tag{1}$$

The self-energy part Σ equals the sum of diagrams:


(1a)

(We assume that the interaction is described by the pair potential V_q represented by the wavy line.)

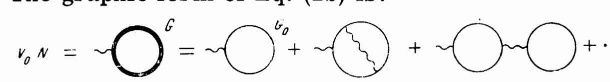
The function $G(\omega_n, \mathbf{p}, \mu, T)$ is directly related to the momentum space particle density $n(\mathbf{p})$:

$$n(\mathbf{p}) = -T\sum_{n=0}^{\infty} G(\omega_n, \mathbf{p}, \mu, T). \tag{2a}$$

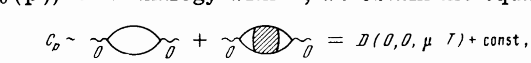
Let us determine the diagrams which are responsible for the singularity of the heat capacity C_p. It is not hard to show (cf., e.g.,[1]) that the singular part of the heat capacity $C_p \propto \partial N/\partial\mu$, where $N(\mu, T)$ is the total particle number related to G via the formula

$$N(\mu, T) = \int n(\mathbf{p})\frac{d^3\mathbf{p}}{(2\pi)^3} = -T\sum_{\omega_n}\int\frac{d^3\mathbf{p}}{(2\pi)^3}G(\omega_n, \mathbf{p}, \mu, T). \tag{2b}$$

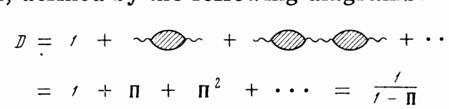
The graphic form of Eq. (2b) is:


(2c)

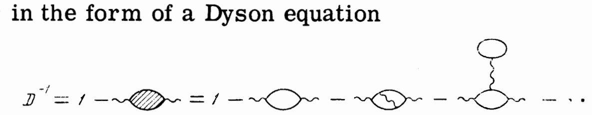
When differentiating these diagrams with respect to μ it is necessary to differentiate only the function G_0 in the internal lines: $\partial G_0/\partial\mu = -G_0^2$, $G_0 = (i\omega_n + \mu - \epsilon_0(\mathbf{p}))^{-1}$. In analogy with[1], we obtain the equation


(3)

where $D(\omega_n, \mathbf{p}, \mu, T)$ is the "phononic" Green's function, defined by the following diagrams:


(3a)

or in the form of a Dyson equation


(4)

Thus the singularities of the thermodynamic quantities are determined by the function D for $\omega_n = \mathbf{p} = 0$. The dependence of D and G on $\mathbf{p}$ for $\omega_n = 0$ yields the character of the momentum distribution of the particles, and the dependence on ω_n determines the relaxation processes, which shall not be considered here.

B. We now investigate which condition determines the phase transition curve $\mu = \mu_c(T)$. As a function of $\mathbf{p}^2$ at $\omega_n = 0$, G has a pole where (1) vanishes. This is not possible for $\mathbf{p}^2 > 0$, since G has the interpretation of a density of particles with momentum $\mathbf{p}$ (cf (cf. (2)), therefore we shall assume that the pole is situated at

$$\mathbf{p}^2 = -M^2(\mu, T) < 0.$$

1)The definition of G and the equations (1), (2) may be found in the book by Abrikosov, Gor'kov, and Dzyaloshinskiĭ [8].

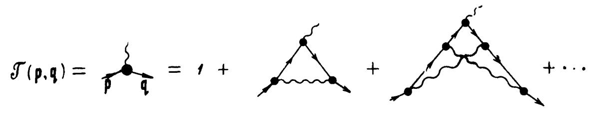
The phase-transition curve $\mu = \mu_c(T)$ (or $T = T_c(\mu)$) will be defined by the condition $M(\mu, T) = 0$, which, as can be seen from (1), is equivalent to the Landau condition: $\mu_c(T) = \Sigma(0, 0, \mu_c, T)$ (cf. Sec. 2 of[1]). It will be seen below that under this condition the thermodynamic quantities, e.g., the heat capacity C_p, have singularities for $\mu \to \mu_c(T)$ ($T \to T_c(\mu)$) and in the momentum distribution of the particles small momenta are essential: $|\mathbf{p}| \sim M \to 0$ (the particles "get ready" to form the Bose-Einstein condensate).

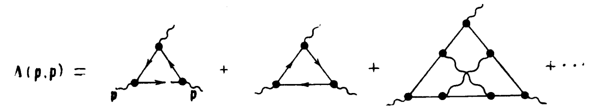
C. We now formulate a system of equations which will determine $G(0, \mathbf{p}, \mu, T) \equiv G(\mathbf{p}, \xi)$ and $D(0, \mathbf{p}, \mu, T) \equiv D(\mathbf{p}, \xi)$ where $\xi = \mu - \mu_c(T)$, $T = \text{const}$. The diagrams (1a) and the equations in (4) are not suitable, since in them the integration regions are $\omega_n \neq 0$, $|\mathbf{p}| \sim 1/r_0$, where the solution is not known. In order to separate from these diagrams the part singular when $\xi = \mu - \mu_c \to 0$. we differentiate G^{-1} and D^{-1} with respect to ξ for $T = \text{const}$. The result of differentiating the diagrams can be written in the form

$$\partial G^{-1} / \partial \xi = \mathcal{T}(\mathbf{p}, \mathbf{p}, \xi) D(0, \xi), \tag{5}$$

$$\partial D^{-1} / \partial \xi = \Lambda(\mathbf{p}, \mathbf{p}, \xi) D(0, \xi), \tag{6}$$

where the vertex functions $\mathcal{T}$ and Λ are defined by the following diagrams


(7)


(8)

Here the straight lines correspond to $G(\omega_n, \mathbf{p})$, the wavy line to $D(\omega_n, \mathbf{k})$ and the vertices correspond to $\mathcal{T}(\omega_n, \omega_n'; \mathbf{p}, \mathbf{p}')$. One sums and integrates as usual over the internal frequencies and momenta:

$$V_0 T \sum_{\omega_n} \int d^3\mathbf{k}/(2\pi)^3.$$

In this form the diagram technique is the same as for interactions with phonons (cf.[8], Eqs. (16), (3)), only in our normalization no additional factors are required in front of the diagrams. We emphasize the fact that in the derivation of the "Ward identities" (5) and (6) we have not neglected the contribution of the regions $\omega_n \neq 0$, $|\mathbf{p}| \sim 1/r_0$ in the diagrams (1) and (4), and such contributions are not yet contained in the diagrams (7) and (8).

C. The equations (5)—(8) for G, D, $\mathcal{T}$ are those initial equations on which our whole subsequent analysis is based. Before starting with this analysis we make several estimates in these equations[2)].

We note first of all that the order of magnitude of the internal momenta in the diagrams (7), (8) is $|\mathbf{k}| \sim M(\xi)$, where $k^2 = -M^2(\xi)$ is the position of the pole of $G(\mathbf{k}, \xi)$. Then the order of magnitude of the diagrams in Eqs. (7), (8) will be

$$\mathcal{T} \sim 1 + \mathcal{T}\lambda + \mathcal{T}\lambda^2 + \ldots, \tag{7a}$$

$$\Lambda \sim \mathcal{T}^3 G^3 M^3 (1 + \lambda^2 + \ldots), \tag{8a}$$

where $\lambda \sim \mathcal{T}^2 D G^2 M^3 V_0 T$ plays the role of an effective coupling constant.

From (5) and (6) follow the estimates

$$G^{-1}\xi^{-1} \sim \mathcal{T} D, \tag{5a}$$

$$D^{-1}\xi^{-1} \sim \Lambda D. \tag{6a}$$

Substituting (8a) into (6a) and utilizing (5q) we see that

$$\xi^{-1} \sim \xi^{-1}\lambda(1 + \lambda^2 + \ldots). \tag{9}$$

Whence

$$\lambda \sim \mathcal{T}^2 D G^2 M^3 V_0 T \sim 1. \tag{10}$$

Substituting here (5a) and remembering that according to (3) the heat capacity $C_p \sim D$ we obtain

$$D \sim C_P \sim \frac{M^3(\xi)}{\xi^2} = \frac{1}{\xi^2 r_c^3(\xi)}. \tag{11}$$

This estimate was first obtained in Refs.[3-5] by means of similitude assumptions, where $r_c(\xi) = 1/M(\xi)$ is the correlation radius.

D. We now notice the following circumstance which will be very essential for us in the sequel. Since $\lambda \sim 1$ it follows from (7a) that $\mathcal{T} \sim 1 + \mathcal{T}$. The case $\mathcal{T} \sim 1$ will be seen to be unrealizable. But this does not imply that necessarily $\mathcal{T} \gg 1$. The reason for this is that owing to the contribution of large momenta $|\mathbf{k}| \sim 1/r_0 \gg M$ and $\omega_n \neq 0$ to Eq. (7) for $\mathcal{T}$ the right-hand side acquires a const ~ 1, which may cancel the original vertex $\mathcal{T}^{(0)} = 1$,[3)] and then the case $\mathcal{T} \ll 1$ becomes possible. In both cases $\mathcal{T} \gg 1$ and $\mathcal{T} \ll 1$ one may omit the original vertex $\mathcal{T}^{(0)} = 1$ as well as the contribution of remote regions $|\mathbf{k}| \sim 1/r_0$, $\omega_n \neq 0$, which shall be done in the sequel.

2. THE RENORMALIZATION OF THE EQUATIONS FOR THE GREEN'S FUNCTIONS AND VERTEX PARTS AND THE "RELATIVISTIC" DIAGRAM TECHNIQUE

We now proceed to a more detailed analysis of Eqs. (5)—(8) for G, D, $\mathcal{T}$. We carry through quantitatively the same line of reasoning as was used in the estimates (5a)—(8a). This will lead to an explicit dependence of all quantities on the distance ξ from the phase transition curve.

A. We first show how to exclude G and D from the equations and how to reduce the problem to the determination of a certain combination of $\mathcal{T}$, G, D, which plays the role of a "renormalized" vertex $\mathcal{T}_c$. The idea of such a renormalization consists in multiplying the diagrams (7) by $(G(p)\, G(q)\, D(p-q))^{1/2}$ and the diagrams (8) by $(D(p) D(p) D(0))^{1/2}$ so that all the

[2)]Such dimensional estimates were made in [2] for an analogous problem and then it was shown in [6] that this method yields the relation (11) for the heat capacity in the Ising model.

[3)]In Sec. 4, Art. B it will be seen that the condition for such a cancellation defines the nonuniversal parameters contained in the solution.

diagrams will contain only the combinations $(G(1)G(2)D(3))^{1/2}\mathcal{T}(1, 2, 3)$.

We equip the renormalized vertex $\mathcal{T}_c$ with some additional factors, the use of which will become clear later, and thus we define $\mathcal{T}_c$ as follows:

$$\mathcal{T}_c(\mathbf{p},\mathbf{q},\xi)=\xi\sqrt{D(0,\xi)D(\mathbf{p}-\mathbf{q},\xi)}$$

$$\times\sqrt{\left(\frac{\mathbf{p}^2}{M^2}+1\right)G(\mathbf{p},\xi)}\sqrt{\left(\frac{\mathbf{q}^2}{M^2}+1\right)G(\mathbf{q},\xi)}\,\mathcal{T}(\mathbf{p},\mathbf{q},\xi). \quad (12)$$

Here the point $p^2 = -M^2(\xi)$ is the position of the pole of $G(\mathbf{p}, \xi)$ which was discussed in Sec. 1, Art. B. For $|\mathbf{p}| \sim |\mathbf{q}| \sim M$, the estimate (5a) yields $\mathcal{T}_c \sim 1$.

We also introduce the "renormalized charge"

$$\lambda=\frac{M^3(\xi)V_0T}{\xi^2D(0,\xi)(2\pi)^3}. \quad (13)$$

According to the estimates (5a)–(11), $\lambda \sim 1$.

We now proceed to the equation for $\mathcal{T}_c$. As was noted at the end of Sec. 1, the term $\mathcal{T}^{(0)} = 1$ may be omitted from Eq. (7) for $\mathcal{T}$, and the contribution of the regions $|\mathbf{k}| \sim 1/r_0$, $\omega_n \neq 0$ may be neglected. Then, going over from $\mathcal{T}$ to $\mathcal{T}_c$ we obtain the following homogeneous equation:

$$\mathcal{T}_c(\mathbf{p},\mathbf{q})=\lambda\int\frac{d^3\mathbf{k}}{M^3}\mathcal{T}_c(\mathbf{p},\mathbf{k})\frac{M^2}{\mathbf{k}^2+M^2}\mathcal{T}_c(\mathbf{k},\mathbf{k}+\mathbf{q}-\mathbf{p})$$

$$\times\frac{M^2}{(\mathbf{k}+\mathbf{q}-\mathbf{p})^2+M^2}\mathcal{T}_c(\mathbf{k}+\mathbf{q}-\mathbf{p},\mathbf{q})+\ldots$$

$$\ldots+\lambda^n\int\left\{\left(\frac{d^3\mathbf{k}}{M^3}\right)^n\mathcal{T}_c^{2n+1}\left(\frac{M^2}{\mathbf{k}^2+M^2}\right)^{2n}\right\}+\ldots \quad (14)$$

In order that the right-hand and left-hand sides of Eq. (14) be of the same order of magnitude for $M(\xi) \to 0$ it is necessary that $\lambda = \text{const} \sim 1$.[4)] Then (14) shows that $\mathcal{T}_c(\mathbf{p}, \mathbf{q}, \xi)$ depends only on the dimensionless momenta $\mathbf{p}/M(\xi)$, $\mathbf{q}/M(\xi)$:

$$\mathcal{T}_c(\mathbf{p},\mathbf{q},\xi)=\mathcal{T}_c\left(\frac{\mathbf{p}}{M(\xi)},\frac{\mathbf{q}}{M(\xi)}\right). \quad (15)$$

B. We now explain how the knowledge of $\mathcal{T}_c$ leads to a determination of G and D, and also how one can find $M(\xi)$ and λ. We start with $G(\mathbf{p}, \xi)$. Setting in (12) $\mathbf{p} = \mathbf{q}$ and substituting $\mathcal{T}(\mathbf{p}, \mathbf{p})$ from the Ward identity (5) we obtain

$$\mathcal{T}_c\left(\frac{\mathbf{p}^2}{M^2}\right)\equiv\mathcal{T}_c\left(\frac{\mathbf{p}}{M},\frac{\mathbf{p}}{M}\right)=\xi\left(\frac{\mathbf{p}^2}{M^2}+1\right)G(\mathbf{p},\xi)\frac{\partial G^{-1}(\mathbf{p},\xi)}{\partial\xi}. \quad (16)$$

We shall consider (16) as an equation for $G(\mathbf{p}, \xi)$. For $\mathbf{p} = 0$ (16) yields

$$G(0,\xi)=G_1\xi^{-\alpha}, \quad (17)$$

where $G_1(T)$ is an arbitrary constant, depending on the point T on the phase transition curve,

$$\alpha=\mathcal{T}_c(0). \quad (18)$$

In order to find a differential equation for $M(\xi)$ we consider (16) for $p^2 \to -M^2(\xi)$, when $G \to Z(p^2)/(p^2 + M^2(\xi))$. Then (16) implies

$$\frac{\partial M^2}{\partial\xi}=\frac{\mathcal{T}_c(-1)}{\xi}M^2(\xi)\quad(T=\text{const}). \quad (19)$$

Hence

$$M(\xi)=\varkappa\xi^\beta, \quad (20)$$

where $\kappa(T)$ is an arbitrary constant, depending on the point T on the phase-transition curve, and

$$\beta={}^1\!/_2\mathcal{T}_c(-1). \quad (21)$$

The general solution of (16) may now be written in the form:

$$G(\mathbf{p},\xi)=G_1\xi^{-\alpha}\exp\left\{\frac{1}{2\beta}\int_0^{\mathbf{p}^2/M^2}\frac{dy}{y}\left[\frac{\mathcal{T}_c(y)}{y+1}-\alpha\right]\right\} \quad (22)$$

We now proceed to $D(\mathbf{k}, \xi)$. Renormalizing according to (12) in Eqs. (6) and (8) for $D(\mathbf{p}, \xi)$, these equations take on the form

$$\frac{\partial D^{-1}(\mathbf{p},\xi)}{\partial\xi}=\frac{D^{-1}(\mathbf{p},\xi)}{\xi}\Lambda_c\left(\frac{\mathbf{p}^2}{M^2}\right), \quad (23)$$

where the vertex Λ_c is defined by the renormalized diagrams

$$\Lambda_c\left(\frac{\mathbf{p}^2}{M^2}\right)=2\lambda\int\frac{d^3\mathbf{k}}{M^3}\frac{\mathcal{T}_c(\mathbf{k}-\mathbf{p},\mathbf{k})\mathcal{T}_c(\mathbf{k},\mathbf{k})\mathcal{T}_c(\mathbf{k},\mathbf{k}-\mathbf{p})}{(\mathbf{k}^2/M^2+1)^2((\mathbf{p}-\mathbf{k})^2/M^2+1)}$$

$$+\ldots+2\lambda^n\int\left\{\left(\frac{d^3\mathbf{k}}{M^3}\right)^n\left(\frac{\mathcal{T}_cM^2}{\mathbf{k}^2+M^2}\right)^{2n+1}\right\}+\ldots \quad (24)$$

The solution of Eq. (23) is

$$D(\mathbf{p},\xi)=D_1\xi^{-\Lambda_c(0)}\exp\left\{\frac{1}{2\beta}\int_0^{\mathbf{p}^2/M^2}\frac{dy}{y}[\Lambda_c(y)-\Lambda_c(0)]\right\}. \quad (25)$$

The condition $\lambda = M^3V_0T/\xi^2D(0, \xi) = \text{const}$ implies a relation for the exponents of the powers in (20) and (25):

$$\Lambda_c(0)=2-3\beta=2-{}^3\!/_2\mathcal{T}_c(-1). \quad (26)$$

This relation determines the parameter λ, which is of the order of unity.

C. We should like to call attention to the rather unusual feature of the strong coupling problem, observed in[2], which has manifested itself here: Equations (14), (16) and (23) for G, D, $\mathcal{T}$ do not determine the scale of the solution, i.e., the dimensional constants κ and G_1 in (20) and (22), but determine only the dimensionless parameters α, β, λ and the dimensionless functions $\xi^\alpha G/G_1 = g(p/\kappa\xi^\beta)$, etc. This scaling property corresponds to the phenomenological hypotheses[3-5] according to which the distribution of large-scale fluctuations, $r_c(\xi) \sim 1/M(\xi)$ does not change under a change of the scale of length, i.e., the correlation functions depend on the variables $r/r_c(\xi)$. The dimensional constants κ and G_1 are not universal (they change along the phase-transition curve $\mu = \mu_C$ $\mu = \mu_c(T)$) and are, in principle, determined from the condition that the solutions join smoothly with the region $\xi \sim 1$ (cf. (49), (50)).

Thus, we have divided our problem into two stages:

a) the determination of the renormalized vertices $\mathcal{T}_c$ and Λ_c and of the renormalized charge λ from the equations (14), (24), (26);

b) the calculation of $G(\mathbf{p}, \xi)$, $D(\mathbf{p}, \xi)$, α, β accord-

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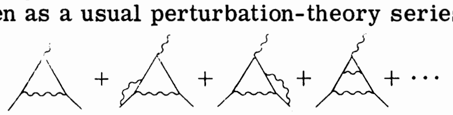
4)More precisely, this condition follows not from (14), but from the condition that all the equations (14), (16), (23), and (24) be self-consistent.

ing to Eqs. (22), (25), (18), (21).

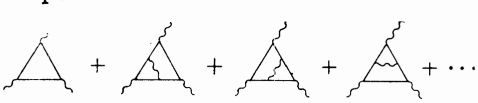
Although it is impossible to obtain a closed form of the solution in this way, one can still clarify a series of general properties, as, for example, the behavior of G, D, $\mathcal{T}$, and of the more complicated vertices for $|\mathbf{p}| \gg \kappa\xi^{\beta}$ and $|\mathbf{p}| \ll \kappa\xi^{\beta}$, as well as the character and the position of the singularities for $p^2 < 0$. These general properties will be analyzed in the next section, and now we develop a diagram technique for the determination of $\mathcal{T}_C$, Λ_C, and of more complicated vertices, a technique which is an exact analog of the relativistic diagram technique for a particle in a two-dimensional space.

D. We show that the renormalized vertices are represented by a sum of diagrams involving the propagators $M^2/(k^2 + M^2)$ and universal numerical factors. For this purpose we shall iterate the equation (14) for $\mathcal{T}_C$ with respect to λ, adopting as the zeroth approximation $\mathcal{T}_C = \mathcal{T}_C(0, 0) = \alpha$. Then the iteration series can be written as a usual perturbation-theory series:

$\frac{1}{\alpha}\mathcal{T}_c(p,q) =$  (27)

The structure of these diagrams coincides with the structure of the original four-boson interaction, but a renormalization is effected in them, namely the original potential $V_0T/(2\pi)^3$ is replaced by the renormalized potential $V_C = \lambda\alpha^2$, and the unrenormalized Green's function $G_0 = (\mu - \epsilon_0(\mathbf{p}))^{-1}$ is replaced by the renormalized one $G_C = M^2/(p^2 + M^2)$, and an integration $\int d^3k/M$ is carried out. There is no need to take into account diagrams which reduce to corrections to internal G- or D-functions (they are already included in the vertex $\mathcal{T}_C$ according to Eq. (12)).

In exactly the same manner the vertex Λ_C in (24) may be represented as a sum of diagrams with the same correspondence rules:

$\frac{1}{2\alpha} =$  (28)

In order to demonstrate the functioning of perturbation theory we consider the first approximation and determine α, β, λ, G and D. In the first approximatino in (27) $\mathcal{T}_C(p, q)$ becomes

$$\frac{1}{\alpha}\mathcal{T}_c^{(1)}(\mathbf{p},\mathbf{q}) = V_c\int\frac{d^3\mathbf{k}}{(\mathbf{k}^2+1)((\mathbf{k}+\mathbf{q}-\mathbf{p})^2+1)} \tag{29}$$

($V_C \equiv \lambda\alpha^2$, $M \equiv 1$) and Λ_C in (28) becomes

$$\frac{1}{\alpha}\Lambda_c^{(1)}(\mathbf{p}^2) = 2V_c\int\frac{d^3\mathbf{k}}{(\mathbf{k}^2+1)^2((\mathbf{p}-\mathbf{k})^2+1)}. \tag{30}$$

From (18), (21), (26) we obtain in this approximation

$$\alpha^{(1)} = 1, \quad \beta^{(1)} = 1/2, \quad \lambda^{(1)} = 1/\pi^2. \tag{31}$$

The functions $G(\mathbf{p}, \xi)$ and $D(\mathbf{p}, \xi)$ are determined from (22) and (25). Since it happens that $T_C^{(1)}(\mathbf{p}, \mathbf{p}) = 1$ in (29), the function $G^{(1)}(\mathbf{p}, \xi)$ turns out to be the same as for an ideal Bose gas

$$G^{(1)}(\mathbf{p},\xi) = G_1\frac{\varkappa^2}{\mathbf{p}^2+\varkappa^2\xi}. \tag{32}$$

$D^{(1)}$ has a more complicated momentum dependence:

$$D^{(1)}(\mathbf{k},\xi) = D_1\xi^{-1/2}d^{(1)}(\mathbf{k}^2/\varkappa^2\xi), \tag{33a}$$

where $d^{(1)}(x)$ can be expressed in terms of $\Lambda_C^{(1)}(x)$ of (30), and according to (25) will have the form

$$d^{(1)}(x) = \exp\left\{\int_0^x\frac{dy}{2y}\left[\frac{2}{\pi\sqrt{y}}\int_0^\infty\frac{dz}{(z+1)^2}\ln\left(\frac{1+(\sqrt{z}+\sqrt{y})^2}{1+(\sqrt{z}-\sqrt{y})^2}\right)-1\right]\right\}. \tag{33b}$$

Such a perturbation theory is of little use, however, for practical calculations, since its expansion parameter $V_C \sim 1$. We have dwelt on it principally to note the analogy with the relativistic theory. This analogy, which shall be useful in the sequel for the analysis of the analyticity properties of G, D, $\mathcal{T}$, etc. consists in the following. The diagrams (27) and (28) represent analytic functions of $\mathbf{p}^2$, $\mathbf{q}^2$, $(\mathbf{p} - \mathbf{q})^2$, having no singularities for $\mathbf{p}^2, \mathbf{q}^2, (\mathbf{p} - \mathbf{q})^2 > 0$, since all denominators are positive there. However for negative $\mathbf{p}^2 = -n^2M^2$ ($n = 2, 3, \ldots$) these diagrams have branch points which are completely analogous to the threshold singularities of a relativistic field theory. (E.g., (29) and (30) have singularities respectively for $(\mathbf{p} - \mathbf{q})^2 = -4M^2$, $\mathbf{p}^2 = -4M^2$.)

E. We summarize the results of this section.

We have shown that the Green's function $G(\mathbf{p}, \xi)$ and $D(\mathbf{p}, \xi)$ which give respectively the momentum-space distribution of the particles $n(\mathbf{p}) \sim G(\mathbf{p}, \xi)$ and the singular part of the heat capacity $C_P \sim D(0, \xi)$ have the following dependence on the distance from the phase transition curve $\xi = \mu - \mu_C(T)$:

$$-T^{-1}n(\mathbf{p}) = G(\mathbf{p},\xi) = G_1\xi^{-\alpha}g(\mathbf{p}/\varkappa\xi^{\beta}), \tag{34}$$

$$D(\mathbf{p},\xi) = D_1\xi^{3\beta-2}d(\mathbf{p}/\varkappa\xi^{\beta}), \tag{35}$$

where $G_1(T)$, $\kappa(T)$ and $D_1(T)$ are nonuniversal constants (they change along the phase-transition curve $\mu_C(T)$) and $\alpha = \mathcal{T}_C(0)$ and $2\beta = \mathcal{T}_C(-1)$ are universal numbers.

The universal functions $g(p/M)$, $d(p/M)$ and $\mathcal{T}_C(p^2/M^2) \equiv \mathcal{T}_C(p/M, p/M)$ are determined from the equations (14), (22), (24)–(26), which are given above. For the determination of these quantities and of the parameters α, β a diagram technique is proposed (cf. (27) and (28)) which coincides with the diagram technique for a two-dimensional relativistic particle.

In the next section we shall consider different asymptotic regions for G, D and $\mathcal{T}$ and their analytic properties, which define the correlations between particles at large distances $rM(\xi) \gg 1$.

3. THE ASYMPTOTIC AND ANALYTIC PROPERTIES OF THE GREEN'S FUNCTIONS AND THE MOMENTUM DISTRIBUTION OF PARTICLES NEAR THE PHASE TRANSITION CURVE

A. It can be seen from the results of the preceding section that after separating in all quantities the dependence on $\xi = \mu - \mu_C(\mathcal{T})$ the problem reduces to the summation of an infinite series of diagrams (27) for the renormalized vertex $\mathcal{T}_C$, which are all of the order of unity for momentum values $|\mathbf{p}| \lesssim \kappa\xi^{\beta}$. Although this problem is simpler than the initial one, and is universal, it can in general form only be solved numerically; therefore we analyze here only different limiting situations.

We start with the simplest limiting case:

a) $|\mathbf{p}| \ll M(\xi) = \kappa\xi^{\beta}$.

Since $G(p^2, \xi)$ is analytic for $p^2 = 0$ the expansion in powers of p^2/M^2 starts with a term of the order p^2/M^2. Then the momentum distribution of the particles for $|p| \ll \varkappa\xi^\beta$ has the form

$$n(\mathbf{p}) = -TG(\mathbf{p}, \xi) + \text{const} \to -TG_1\xi^{-\alpha}\left[1 - a\frac{\mathbf{p}^2}{\varkappa^2\xi^{2\beta}}\right]. \quad (36)$$

With the help of (22) the universal constant a can be expressed in terms of $\mathcal{T}_c(\mathbf{p}^2/M^2)$:

$$a = \frac{\alpha - \mathcal{T}_c'(0)}{2\beta} = \frac{\mathcal{T}_c(0) - \mathcal{T}_c'(0)}{\mathcal{T}_c(-1)}. \quad (37)$$

The function $D(p, \xi)$ has a similar form, as well as the other quantities, for $p^2 \ll \kappa^2\xi^{2\beta}$. In particular, the heat capacity $C_P \sim D(0, \xi)$ has a power-law singularity of the form

$$C_P = \text{const}\cdot D(0, \xi) = \text{const}\cdot \xi^{3\beta-2}. \quad (38)$$

For $\beta \to 2/3$ this singularity goes over into $\ln \xi$, but, unfortunately, we have been unable to prove from our equations that $\beta = 2/3$.

We now consider the inverse limiting case of "large" momenta, or, equivalently, of small ξ:

b) $\kappa\xi^\beta \ll |p| \ll 1/r_0$

For $\xi = 0$ the density $n(p)$ must remain finite (momentum distribution at the Curie point). From the condition that the factors and ξ^α, ξ^β cancel in (34) we obtain

$$G(\mathbf{p}, 0) = \text{const}\cdot(\varkappa/|\mathbf{p}|)^{\alpha/\beta}. \quad (39)$$

In the same manner, for $\xi = 0$ the function $D(p, 0)$ must remain finite at $\xi = 0$, as well as the unrenormalized vertex $\mathcal{T}(p, q, 0)$ and the more complicated unrenormalized vertices.

The following correction to $G(p, \xi)$ is easily derived from the "Ward identity" (5):

$$G^{-1}(\mathbf{p}, \xi) \to \text{const}\cdot\left(\frac{|\mathbf{p}|}{\varkappa}\right)^{\alpha/\beta} + \mathcal{T}(\mathbf{p}, \mathbf{p}, 0)\int_0^\xi D(0, \xi)\,d\xi \quad (40)$$

$$= \text{const}\cdot\left(\frac{|\mathbf{p}|}{\varkappa}\right)^{\alpha/\beta} + \mathcal{T}(\mathbf{p}, \mathbf{p}, 0)\frac{D_1\xi^{3\beta-1}}{3\beta-1}, \quad \xi \to 0.$$

Comparing (40) and (39) we see that

$$\mathcal{T}(\mathbf{p}, \mathbf{p}, 0) = \text{const}\cdot(|\mathbf{p}|/\varkappa)^{(\alpha+1)/\beta-3}. \quad (41)$$

Then (40) and (2a) finally yield the momentum distribution for $1/r_0 \gg |p| \gg \kappa\xi^\beta$:

$$n(\mathbf{p}) = -TG(\mathbf{p}, \xi) + \text{const} \to A + B|\mathbf{p}|^{-\alpha/\beta}\left[1 + b\xi^{3\beta-1}\left(\frac{|\mathbf{p}|}{\varkappa}\right)^{1/\beta-3}\right]. \quad (42)$$

Here $A(T)$ and $B(T)$ are nonuniversal constants; the constant A is added owing to the contribution of the frequencies $\omega_n \neq 0$ to the formula (2a), b is a universal constant and can, in principle be determined from (22).

One can show similarly that always for momenta much larger than the correlation momentum $\kappa\xi^\beta = 1/r_c$ the expansion parameter will be the quantity $(|p|\, r_c)^{1/\beta-3}$. In order to prove this it suffices to note that the differentiation of any many-particle Green's function (correlation function) with respect to ξ is equivalent to adding a phonon line with vanishing momentum, which for $\xi \to 0$ yields the factor $D(0, \xi) \sim \xi^{3\beta-2}$. The other factors have a finite limit for $\xi \to 0$, similar to $\mathcal{T}(p, p, 0)$ in (40) and (41), and integrating over ξ we obtain for the correlation functions an expansion of the type (42) (cf. infra (52)). (Had we differentiated with respect to ξ n times, we would have seen that the coefficient of $(r_c)^{(1/\beta-3)n}$ in this expansion coincides up to a factor with the correlation function having n additional phononic lines with vanishing momenta.)

B. We now consider the analytic properties of G, D, $\mathcal{T}$, and of other vertices.

We have already seen at the end of the preceding section that the renormalized vertex $\mathcal{T}_c(p/M, q/M)$ defined in (12) has a singularity at $(p - q)^2 = -4M^2$. Considering the different diagrams for $\mathcal{T}_c$ it is easy to see that in the "phonon momentum" $p - q \equiv k$ there are branch points at

$$\mathbf{k}^2 = -(2n\varkappa)^2\xi^{2\beta}, \quad n = 1, 2, \ldots \quad (43)$$

The function $\mathcal{T}_c(p, q)$ has branch points in the boson momenta p and q at the points

$$\mathbf{p}^2 = -(2n+1)^2\varkappa^2\xi^{2\beta}, \quad \mathbf{q}^2 = -(2n+1)^2\varkappa^2\xi^{2\beta}. \quad (44)$$

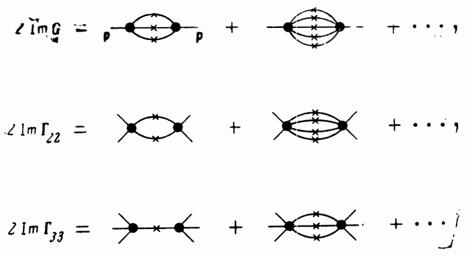
Similarly one can show that an arbitrary vertex has branch points (43) in the phonon variable and (44) in the boson variables, in the same manner as in relativistic field theory. In particular $G(p^2, \xi)$ has branch points (44) in the variable p^2, and $D(k^2, \xi)$ has the branch points (43) in the variable k^2. This can be checked directly from the diagrams (1a) and (3a), since near the singularities of the diagrams the internal momenta $|k_i| \sim M$ and frequencies $\omega_n = 0$ are essential in them. The singularity at $p^2 = -n^2M^2$ is produced by the integration region where the n functions $G(k_i, \xi)$ are situated near the pole $k_i^2 \to -M^2(\xi)$, i.e., one has to rotate the integration contour over k_{iz} to coincide with the imaginary axis (the z axis is along p).

We now wish to prove what was in fact assumed in Sec. 1: the singularity of G at $p^2 = -M^2(\xi)$ is a pole, and not a branch point. This property is analogous to relativistic field theory and is proved invoking the same kind of arguments. Assume the contrary: the singularity at $p^2 = -M^2$, i.e., at the point where $G^{-1}(p, \xi) = 0$ is not a pole but a different kind of singularity. Then it is easy to see that the diagrams for $\Sigma(p, \xi)$ in (1) and (1a) will again have branch points only at the points (44) and for $p^2 = -M^2$ they will be analytic. Therefore the function $G^{-1}(p, \xi)$ will be analytic near the point $p^2 = -M^2$, i.e., the singularity is a simple pole.

4. THE UNITARITY CONDITION AND A MORE GENERAL APPROACH TO THE PHASE TRANSITION PROBLEM

A. The fact that near the singularities (43) and (44) of the diagrams the internal frequencies $\omega_n = 0$ and internal momenta k_i close "mass shell" $k_i^2 = -M^2(\xi)$ are essential in them provides us with the following "semiphenomenological" approach to the phase transition problem, valid for any system near its Curie point.

We shall only assume the fact that a pole of $G(p^2, \xi)$ exists at $p^2 = -M^2(\xi)$, fact which is valid for any system. Then $G(p^2, \xi)$ and the various vertices $\mathcal{T}_{n_1n_2}(p_i, \xi)$ with $n_1 + n_2$ external boson lines will have the branch points (43) and (44), and the discontinuities of these quantities across the cuts for $p_i^2 < 0$ (i.e., the imaginary parts of G and $\mathscr{T}_{n_1n_2}$) will be determined, in analogy to elementary particle theory, by the following "unitarity condition" (cf.[9]):



(45a)

(45b)

(45c)

. . . .

$$2\,\mathrm{Im}\,\Gamma_{n_1n_2} = \sum_{l=1}^{\infty} \int \Gamma_{n_1l}\Gamma_{ln_2} d\tau_l. \tag{45d}$$

Here the lines with crosses correspond to $Z(\xi)2\pi\delta(k^2/M^2(\xi)+1)$ i.e., the contribution of the pole to the imaginary part of the Green's function

$$G(k^2, \xi) \to Z(\xi)\frac{M^2(\xi)}{k^2+M^2(\xi)}, \quad Z = G_1\xi^{-\alpha}, \quad M = \varkappa\xi^\beta.$$

In (45) it is understood that $p^2 < 0$, i.e., $p = (iE, 0, 0)$ and similarly the internal momenta have the form $k_l = (i\epsilon_l, q_l)$. Finally, $d\mathscr{T}_l$ is the phase-space volume, which has the relativistic form

$$d\tau_l = T^{l-1}\delta\Big(E - \sum \varepsilon_i\Big)\,\delta\Big(\sum_i q_i\Big)\prod_{i=1}^{l} d\varepsilon_i d^2q_i (ZM^2)\,\delta(\varepsilon_i^2 - q_i^2 - M^2)$$

$$= (TM^3)^{l-1}Z^lM\delta\Big(E - \sum_i \sqrt{q_i^2+M^2}\Big)M^2\delta\Big(\sum_i q_i \prod_{i=1}^{l}\Big)\frac{d^2q_i}{2M\sqrt{q_i^2+M^2}}$$

(here T is the temperature).

We omit numerical factors from (46), factors which depend on the specific system in which a phase transition occurs. For a Bose liquid these factors can be worked out by means of the equations of the preceding section, and in the general case they follow from the correspondence rules for diagrams. They involve such quantities as factors $2S + 1$ for closed loops with spin S, $1/n!$ for n identical particles, etc.

It follows from (45d) and (46) that the vertices $\mathscr{T}_{n_1n_2}$ with n_1 incoming and n_2 outgoing lines satisfy the scaling law:

$$\Gamma_{n_1n_2} = Z^{-n/2}(TM^3)^{1-n/2}\gamma_{n_1n_2}\Big(\frac{p_i}{M}\Big), \tag{47}$$

where $M = \kappa\xi^\beta$, $Z = G_1\xi^{-\alpha}$, $n = n_1 + n_2$ is the total number of lines. In particular,

$$G^{-1} = \Gamma_{11} = Z^{-1}(\xi)g^{-1}(p^2/M^2(\xi)). \tag{48}$$

These equations could have been obtained, of course, by means of the method of Sec. 2.

The imaginary parts $\mathrm{Im}\,\gamma_{n_1n_2}$ are determined from (45d), and the "energy" conservation law ($E = \Sigma\,(q_i^2 + M^2)^{1/2}$) in (45d) relates with one another the $\gamma_{n_1n_2}$ with small momenta $|q_i| < E = |p|$, i.e. regions of integration in (45) are restricted by the conservation laws. Such a method of separation of "small" momenta $|k_i| \sim M$ by means of unitarity and analyticity conditions is widely used in the theory of strong interactions[9]. For a problem which is analogous to the phase transition problem this method was first used in[2], where "scaling" formulas of the type (47) were derived.

In addition to scaling laws, the proposed method allows one to find the character of the singularities at $(p^2)^{1/2} = inM$, yielding the factor $N(r, \xi)$ in front of the exponential in the exponential decay law for correlation functions, $N(r, \xi)\exp(-rM(\xi))$ (cf.[6]).

The exponents α and β in the formulas $M = \kappa\xi^\beta$ and $Z = G_1\xi^{-\alpha}$ are defined in this method in the following fairly indirect manner. We know that for $\xi = 0$ the functions $G(p, 0)$ and $\Gamma_{n_1n_2}$ in (47), (48) must remain finite. Therefore, for $p^2 \gg M^2$ the function $G(p^2, \xi)$ has the asymptotic behavior $G \to \mathrm{const.}\,|p|^{-\alpha/\beta}$. This shows that $g(x)$ in (48) "knows" about the parameters α and β; similar facts hold for the functions $\gamma_{n_1n_2}(x_i)$ in (47). If the functions $g(x)$ and $\gamma_{n_1n_2}(x_i)$ will in the sequel be <u>universal</u>, then α and β are determined from the solubility condition for the system (45), similar to what was explained in[2], or Sec. 2 of the present paper.

B. We do not have a general proof of universality, but the reasons for expecting universality are the following.

Usually in the unitarity-analyticity method the one universality of a solution is related to the fact that one can always add to G^{-1} and $\Gamma_{n_1n_2}$ so-called subtraction constants, which do not affect the imaginary parts of G^{-1} and $\Gamma_{n_1n_2}$[9]. These constants are determined by unrenormalized quantities and by the contributions of "remote" integration regions—in our case the regions $k_i \sim 1/r_0$ $\omega_n \neq 0$—in the diagrams for G^{-1} or $\Gamma_{n_1n_2}$. Such additions are obviously nonuniversal, but it is important that they do not have singularities for $\xi \to 0$. Therefore, in the case under consideration, when the solution has the form of homogeneous functions (47), (48) where Z and M are singular for $\xi \to 0$, one is not allowed to add such constants without violating the homogeneity condition. For example, for $p = 0$

$$G^{-1}(0, \xi) = \mathrm{const}\cdot\xi^\alpha, \quad \Gamma_{n_1n_2} = \mathrm{const}\cdot\xi^{3\beta+n(\alpha-3\beta)/2} \quad (n = n_1 + n_2),$$

and adding constants to this we would ruin the structure of the solution. The addition of constants to the functions $g^{-1}(x)$ and $\gamma_{n_1n_2}(x_i)$ in (47), (48) would lead to additions to G^{-1} and of the type ξ^α which are singular for $\xi \to 0$, which should not happen.

Reasoning in a less formal manner, the matter consists in the following. Our solution still contains the nonuniversal quantities $\kappa(T)$, $G_1(T)$ and $\mu_c(T)$ in the equations $M = \kappa(T)(\mu - \mu_c(T))^\beta$, $Z = G_1(T)(\mu - \mu_c(T))^{-\alpha}$. These are exactly the nonuniversal quantities which are determined from the requirement that the constant additions to G^{-1} and Γ vanish whenever such constants may distort the solution. For example $\mu_c(T)$ is determined from the requirement that $G^{-1}(0, 0) = 0$, i.e., that the contribution from remote regions to Σ in (1), together with the unrenormalized $G_0^{-1}(0, 0) = \mu_c(T)$ yield zero. For the four-boson vertex Γ_{22}, when $\Gamma_{22} \ll 1$ (i.e., $2\alpha > 3\beta$) the residual (unrenormalized) constant V_0 must vanish together with the contributions from remote regions. This yields yet another condition for nonuniversal constants, etc.

C. We have already mentioned the determination of nonuniversal constans and therefore we shall indicate how one can estimate their order of magnitude. This is done similarly to[2], by means of matching the solutions to perturbation theory.

We denote by $\xi_0(T)$ that distance $\mu - \mu_c(T)$ from the phase transition curve from which onward perturbation theory is already applicable. (For a Bose liquid $\xi_0(T) \sim (V_0T)^2$.) Then for $\xi \sim \xi_0$ and $\mathbf{k} = 0$ we have $G \sim G_0 \sim 1/\xi_0$ and $\Gamma_{22} \sim V_0$. Hence

$$G(0, \xi_0) \sim Z(\xi_0) \sim G_1\xi_0^{-\alpha} \sim \xi_0^{-1},$$
$$\Gamma_{22}(\mathbf{k}_i = 0) \sim \frac{1}{T} Z^{-2}(\xi_0) M^{-3}(\xi_0) \sim V_0.$$

Then

$$G_1 \sim \xi_0^{\alpha-1} \sim (V_0T)^{2(\alpha-1)}, \tag{49}$$

$$\varkappa \sim \xi_0^{1/2-\beta} \sim (V_0T)^{1-2\beta}. \tag{50}$$

D. We finally clarify the question whether the vertices $\Gamma_{n_1n_2}$ can be of the order of unity, i.e., e.g., $2\alpha = 3\beta$, so that $\Gamma_{22} \sim 1$. Such a behavior of Γ_{22} was assumed by Patashinskiĭ and Pokrovskiĭ[1] and led to the following contradictions.

Consider for example $\xi = 0$. Then, if $\Gamma_{22} \sim 1$, i.e., $2\alpha = 3\beta$ then according to (39), $G(p, 0) = \text{const.}\,|\mathbf{p}|^{-3/2}$. Substituting $G \sim |\mathbf{p}|^{-3/2}$ into the diagrams for Γ_{22}, all these diagrams will be logarithmically divergent, and in the next approximation Γ_{22} will be $\Gamma_{22} = V_0(1 + \gamma \ln(1/pr_0))$, where r_0 is the radius of the forces, and p is the largest among the momenta involved in Γ_{22}. It can be checked that γ is a universal number, so that it is impossible to make it vanish by selecting the unrenormalized constants, as was assumed in[1].

Moreover, considering γ small and summing the leading powers of $\gamma \ln(1/pr_0)$ in analogy with what was done in Sec. 3 of[2] 5) we obtain $\Gamma_{22} \sim V_0(pr_0)^{-\gamma}$, $G(p, 0) \sim p^{(\gamma-3)/2}$, in agreement with our Eqs. (47) and (48) with $\gamma = 3 - 2\alpha/\beta$.

Thus we see that if one assumes in the zeroth approximation that $\Gamma_{22} \sim 1$, the successive approximations yield either $\Gamma_{22} \gg 1$ or $\Gamma_{22} \ll 1$, depending on the sign of γ. Here one can also see the universality of α and β which we mentioned before, and the fact that Γ_{22} may be small, since the nonuniversal constants in the solution (in the case under consideration the factor C in the expression $\Gamma_{22} = CV_0(|\mathbf{p}|r_0)^{-\gamma}$) are selected in such a manner that the unrenormalized vertices cancel out together with the contribution of the remote regions $|\mathbf{p}| \sim 1/r_0$.

CONCLUSION

We mention several observable consequences of the proposed theory for a Bose liquid approaching its phase-transition curve from the side of the normal phase.

1) For $|\mathbf{p}| \ll 1/r_0$ (r_0 is the range of the interparticle forces interatomic spacing the momentum space density of particles has the form (34) with the asymptotic behavior (36) and (42).

2) The density correlation function in momentum space

$$D(\mathbf{q}, \xi) \sim \int d^3\mathbf{r} e^{i\mathbf{q}\mathbf{r}} \langle \rho(0)\rho(\mathbf{r})\rangle$$

depends on the distance to the phase-transition curve according to Eq. (35). This function determines the cross section for the scattering of neutrons in He I with momentum transfer $\mathbf{q}$ (cf.[3,8]).

3) The heat capacity C_p has a singularity of the form $\xi^{3\beta-2}$ in agreement with the estimates of[3-5]. If $\beta \to 2/3$ the singularity becomes logarithmic, $\ln\xi$, but it is not clear how one can prove without numerical computations that $\beta = 2/3$.

4) For small momenta the correlation functions can be expanded in powers of $(|\mathbf{p}|r_c)^{2n}$, and for large momenta they can be expanded in powers of $(|\mathbf{p}|r_c)^{-n(3-1/\beta)}$, where $r_c = 1/M = \kappa^{-1}\xi^{-\beta}$ is the correlation radius.

5) The vertex parts Γ_n with zero frequencies, which determine the reaction of the system to external static actions, are subject to the scaling (similitude) relations (47), where the functions $_n\gamma_{n_1n_2}$ are constant along the phase-transition curve. For small momenta $|\mathbf{p}_i| \ll \kappa\xi^\beta$ the behavior of Γ_n is:

$$\Gamma_n(\mathbf{p}_i, \xi) \to G_1^{-n/2}(T\varkappa^3)^{1-n/2}\xi^{3\beta+n(\alpha-3\beta)/2}\left[\gamma_n(0) + \frac{\Sigma(\mathbf{p}_i\mathbf{p}_j)a_{ij}^{(n)}}{\varkappa^2\xi^{2\beta}}\right], \tag{51}$$

where $\gamma_n^{(0)}$ and $a_{ij}^{(n)}$ are universal numbers, G_1 and γ are non-universal constants, which have the order of magnitude (49) and (50).

Near the phase-transition curve, when $\kappa\xi^\beta \ll |\mathbf{p}_i| \ll 1/r_0$, the asymptotic form of Γ_n is:

$$\Gamma_n(\mathbf{p}_i, \xi) \to G_1^{-n/2}(T\varkappa^3)^{1-n/2}\left(\frac{|\mathbf{p}_1|}{\varkappa}\right)^{3+n(\alpha-3\beta)/2\beta} \times\left[\Phi_n\left(\frac{\mathbf{p}_i\mathbf{p}_j}{\mathbf{p}_1^2}\right) + \left(\frac{\varkappa\xi^\beta}{|\mathbf{p}_1|}\right)^{3-1/\beta}\Psi_n\left(\frac{\mathbf{p}_i\mathbf{p}_j}{\mathbf{p}_1^2}\right)\right], \tag{52}$$

where Φ_n and Ψ_n are universal functions of their arguments.

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5)The author has learned from A. A. Abrikosov after this work was finished that such a summation for the example of "parquet" diagrams for Γ_{22} has been effected earlier by L. D. Landau, A. A. Abrikosov and L. P. Gor'kov (unpublished).

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