QUASICLASSICAL THEORY OF BREMSSTRAHLUNG BY RELATIVISTIC PARTICLES

V. N. BAĬER and V. M. KATKOV

Novosibirsk State University

Submitted May 28, 1968

Zh. Eksp. Teor. Fiz. 55, 1542–1554 (October, 1968)

Photon emission by a relativistic particle in an external field is considered. The analysis is performed for fields in which the contribution to the radiation in a given direction is made by the whole particle trajectory. A generalization of the operator quasiclassical method previously proposed by the authors is employed. Emission by particles with spin 0 or $\frac{1}{2}$ in a Coulomb field is investigated in detail. Closed expressions are obtained which are valid for arbitrary values of $Z\alpha$. General formulas are also derived for the photon emission cross sections in the cases when a) the fields decrease more rapidly than a Coulomb field and b) only a small part of the trajectory contributes to the emission in a given direction.

1. IT is well known that relativistic particles radiate into an angle $\sim 1/\gamma$ ($\gamma = E/m$). For this reason in the problem of radiation by a relativistic particle in an external field the relation between the total deflection angle of the particle in the external field and the angle $1/\gamma$ turns out to be important. There are two characteristic cases.

I. The total deflection angle is large in comparison with $1/\gamma$. Then in a given direction the particle radiates from the small portion of its trajectory in which the direction of the particle velocity changes by an angle $\sim 1/\gamma$. This situation occurs in magnetic bremsstrahlung (synchrotron radiation).

II. The total deflection angle of the particle in the field is $\lesssim 1/\gamma$. Then all the radiation of the particle occurs into a narrow cone with angle $\sim 1/\gamma$ and is determined by the entire trajectory of the particle. This situation exists in the case of bremsstrahlung in a Coulomb field.

This situation is preserved also in quantum electrodynamics. This is due to the fact that in the radiation of relativistic particles in external fields the main contribution is from states with large orbital angular momenta, for which the motion has a quasiclassical nature and is quite adequately described in terms of a trajectory.

In our previous papers[1, 2] we have suggested an operator quasiclassical method of considering the radiation process in an external field. The method is based on the fact that there are two types of quantum effects in the motion of a relativistic particle in an external field. The first of these is associated with the quantum nature of the particle's motion itself. The associated noncommutativity of the dynamic variables is of order $1/l$ (where $\hbar l$ is the orbital angular momentum) and decreases with increasing particle energy. The second type of quantum effect is associated with the particle's recoil during radiation and is of order $h\omega/E$, where ω is the frequency of the radiated photon. Since at high energies the effects of the first type are extremely small in comparison with effects of the second type, we can neglect the noncommutativity of the dynamic variables of the particle and consequently can discuss the motion of the particle along a trajectory. On this basis we take into account in the method only the commutators of the dynamic variables of the particle with the field of the radiated photon ($\sim\hbar\omega/E$), which corresponds to considering the recoil during radiation.

Our earlier papers[1, 2] discussed the quantum phenomena in motion of charged particles in a magnetic field (synchrotron radiation, pair production by a photon in a magnetic field, and so forth), i.e., we discussed case I.

In the present paper the proposed method is extended to the case of bremsstrahlung in a Coulomb field (case II). It turns out that in its essential part the method can be used directly in this case also. Here, however, there are extremely important specific properties, namely that in calculation of the matrix element for minimum momentum transfer $q \sim q_{min}$, where the phenomenon of quantum-mechanical diffraction is important, it is necessary to perform a summation over the classical trajectories.

Since large orbital angular momenta are important, we can neglect the contributions of terms of the spin-field interaction type (with an accuracy to terms of order $1/l$). For this reason the discussion is carried on in the same way for particles with any spin, and the calculation of all spin effects, which are determined by the structure of the current, is identical in form with the calculation for free particles.

2. The matrix element for radiation of a photon by a charged particle in an external field has the standard form

$$U_{fi} = \frac{e}{(2\pi)^{3/2}\sqrt{2\omega}} \int dt \int d^3r \times F_{fs'}^{+}(\mathbf{r}) e^{iE_f t} (\varepsilon j) e^{i\omega t - i\mathbf{k}\mathbf{r}} e^{-iE_i t} F_{is}(\mathbf{r}), \quad (1)$$

where $F_{is}(\mathbf{r})$ is the solution of the wave equation in the given field with energy E_i and a spin state s, ε_μ is the photon polarization vector, j_μ is the current operator. We will use the metric $(ab) = a_0b_0 - \mathbf{a}\cdot\mathbf{b}$, and the system of units $\hbar = c = 1$.

For the states with large orbital angular momentum l of interest to us we can use the approximate representation

$$e^{iE_i t} F_{is}(\mathbf{r}) = \Psi_s(\mathbf{P}) e^{i\mathcal{H}t} | i \rangle, \tag{2}$$

where $\Psi_S(\mathbf{P})$ is the operator form of the wave function of the particle in a spin state s in the given field. This form was obtained from the free wave functions by replacement of the variables by operators, $\mathbf{p} \to \mathbf{P}$, $E \to \mathcal{H} = \sqrt{\mathbf{P}^2 + m^2}$. In Eq. (2) we have neglected spin-field interaction terms (for example, for particles with spin $1/2$ we have discarded terms of the type $\boldsymbol{\sigma} \cdot \mathbf{H}$ and $\boldsymbol{\alpha} \cdot \mathbf{E}$).

Converting to Heisenberg operators, we write the matrix element U_{fi} in the form[1]

$$U_{fi} = \frac{e}{(2\pi)^{3/2}\sqrt{2\omega}} \left\langle f \left| \int dt e^{i\omega t} M(t) \right| i \right\rangle \tag{3}$$

where

$$eM = \Psi_{s'}^{+}(\mathbf{P}) \{(\varepsilon j),\ e^{-i\mathbf{k}\mathbf{r}}\} \Psi_s(\mathbf{P}); \tag{4}$$

here $j_\mu(t)$ and $\mathbf{r}(t)$ are respectively the current operators and particle coordinates, and the curly brackets { } indicate the symmetrized product of the operators. The order of writing the operators occurring in $\Psi_S(\mathbf{P})$ is unimportant (with an accuracy to terms of order $1/l$). For example, for a particle with spin zero

$$M_S = \frac{1}{\sqrt{\mathcal{H}}} (\varepsilon \mathbf{P}) e^{-i\mathbf{k}\mathbf{r}} \frac{1}{\sqrt{\mathcal{H}}}, \tag{5}$$

and for a particle with spin $1/2$

$$M_e = \frac{1}{\sqrt{\mathcal{H}}} u_{s'}^{+}(\mathbf{P}) (\alpha\varepsilon) e^{-i\mathbf{k}\mathbf{r}} u_s(\mathbf{P}) \frac{1}{\sqrt{\mathcal{H}}}, \tag{6}$$

where

$$u = \sqrt{\frac{\mathcal{H}+m}{2}} \begin{pmatrix} \varphi(\zeta(t)) \\ \dfrac{\boldsymbol{\sigma}\mathbf{P}}{\mathcal{H}+m} \varphi(\zeta(t)) \end{pmatrix}; \tag{7}$$

here $\varphi(\zeta(t))$ is a two-component spinor describing the spin state of the electron at time t.

The probability of the radiation process, summed over all final states of the particle, has the form[1]

$$dW = \frac{\alpha}{(2\pi)^2} \left\langle i \left| \int dt_1 \int dt_2 e^{i\omega(t_1 - t_2)} M^*(t_2) M(t_1) \right| i \right\rangle \frac{d^3k}{\omega}, \tag{8}$$

where $e^2/4\pi = \alpha = 1/137$. This expression is very convenient for discussion of the radiation of a relativistic particle in any external field.

The further calculation is begun by performing a number of operations with the operators occurring in Eq. (8). We will take out the operator $\exp(-i\mathbf{k}\cdot\mathbf{r}(t_1))$ in $M(t_1)$ to the left, and the operator $\exp(i\mathbf{k}\cdot\mathbf{r}(t_2))$ in $M^*(t_2)$ to the right, for which we will use the relation

$$f(\mathbf{P}) e^{-i\mathbf{k}\mathbf{r}} = e^{-i\mathbf{k}\mathbf{r}} f(\mathbf{P} - \mathbf{k}), \tag{9}$$

which is a consequence of the fact that $\exp(-i\mathbf{k}\cdot\mathbf{r})$ is a displacement operator in momentum space. The variation of the function $f(\mathbf{P})$ in (9) on commutation with $\exp(-i\mathbf{k}\cdot\mathbf{r})$ corresponds to inclusion of the recoil during radiation. After extraction of the operator $\exp(-i\mathbf{k}\cdot\mathbf{r})$ from M only commutating operators (with an accuracy to terms of order $1/l$) remain in the matrix element and the subsequent problem reduces to discussion of the combination $\exp(i\mathbf{k}\cdot\mathbf{r}(t_2)) \times \exp(-i\mathbf{k}\cdot\mathbf{r}(t_1))$ which appears. The noncommutativity of the operators entering into this combination is essential, so that, generally speaking, we cannot limit ourselves to expansion of this combination in the lowest commutators. One of the central points of the present approach is the unfolding of this combination.

The unfolding of the operator product $\exp(i\mathbf{k}\cdot\mathbf{r}(t_2)) \times \exp(i\mathbf{k}\cdot\mathbf{r}(t_1))$ for case I (synchrotron radiation) was carried out in our previous papers.[1,2] It was based on the fact that the radiation in a given direction receives contributions from only a small portion of the trajectory, so that $|\dot{\mathbf{v}}|(t_2 - t_1) \sim 1/\gamma$; taking this into account, we performed an expansion in powers of $|\dot{\mathbf{v}}|(t_2 - t_1)$. In case II the radiation receives contributions from the entire trajectory, and this approach is inapplicable. However, we can make use of the fact that far from the source the field of a particle trajectory is close to rectilinear and the dynamic variables of the particle almost do not change (see Appendix A). We note also that, as in our previous work,[1] the discussion is carried out for fields for which the inhomogeneity index

$$n = |\partial \ln V / \partial \ln r| \tag{10}$$

satisfies the inequality

$$n / \gamma \ll 1 \tag{11}$$

(for a Coulomb field n = 1).

As a result of the unfolding (see Appendix B) we obtain

$$\exp\{-ikx(t_2)\} \exp\{ikx(t_1)\} = \exp\left\{-i\frac{\mathcal{H}}{\mathcal{H}-\omega}[kx(t_2) - kx(t_1)]\right\}. \tag{12}$$

This expression coincides exactly with the result of the unfolding for the synchrotron-radiation case[1,2] and is universal for the problem of radiation in any external field, since it in essence takes into account only the recoil during radiation, and the recoil, naturally, does not depend on the field.

The combination $\exp(i\mathbf{k}\cdot\mathbf{r}(t_2)) \exp(-i\mathbf{k}\cdot\mathbf{r}(t_1))$ commutes with $\mathcal{H}$ and $\mathbf{P}$ with an accuracy to terms of order $1/\gamma$. This follows directly from the representation of $kx(t)$ in the form (A.7) and from the fact that in this representation only kx_0 does not commute with $\mathcal{H}$ and $\mathbf{P}$.

Thus, all operators in expression (8) within our accuracy turn out to be commutative and therefore, after carrying out the unfolding operation, all of them which are in the brackets of the initial state can be replaced by their classical values (c-numbers). Now we can write the square of the matrix element in the form[1]

$$\langle i | M^*(t_2) M(t_1) | i \rangle = \exp\left\{ i\omega(t_2 - t_1) + \frac{E}{E-\omega} i [kx(t_1) - kx(t_2)] \right\} R^*(t_2) R(t_1), \tag{13}$$

where

$$eR(t) = \Psi_{s'}^{+}(\mathbf{p}') {}^1\!/_2 (\varepsilon[j(\mathbf{p}) + j(\mathbf{p}')]) \Psi_s(\mathbf{p}); \tag{14}$$

here $\mathbf{p}' = \mathbf{p} - \mathbf{k}$ and $E' = E(\mathbf{p}') = \sqrt{(\mathbf{p} - \mathbf{k})^2 + m^2}$ already are not operators but c-numbers.

In the unfolding operation the spin characteristics of the particle contained in the function R(t) are completely uninvolved, which is due to the fact that in our approximation the interaction of the spin with the external field is neglected (terms of order $1/l$). The function R(t) which describes them has the form of a transition matrix element for free particles with inclusion of the conservation laws. This permits discussion of the problem in a single way for any spin.

3. For definiteness we will carry out the further calculation for the case of motion of a particle with spin zero in a Coulomb field, then discuss an electron in a Coulomb field, and present a number of formulas for radiation in an arbitrary external field.

After carrying out the unfolding we must perform the integration over time in formula (8), substituting in it expression (13). In case I it was convenient to integrate the square of the matrix element over time, converting to relative time $t_2 - t_1$.[1] In case II it turns out to be convenient to integrate the matrix element directly over time, since the whole trajectory contributes to the radiation. The latter is discussed with the necessary accuracy in Appendix A.

The matrix element for a scalar particle (Eqs. (5), (13), and (14)) is

$$\mathscr{M}_S = \frac{e}{(2\pi)^{3/2}\sqrt{2\omega}} \int \frac{1}{\sqrt{EE'}} \exp\left\{ i \frac{E}{E-\omega} kx(t) \right\} (\boldsymbol{\varepsilon}\mathbf{p})\, dt. \tag{15}$$

In a Coulomb field the function $\mathbf{p}(t)$ (A.1) with inclusion of conditions at $t \to \pm\infty$ has the form

$$\mathbf{p}(t) = \mathbf{p}_1 - \frac{\mathbf{q}}{2}\left(1 + \frac{vt}{\sqrt{\rho^2 + v^2t^2}}\right). \tag{16}$$

Taking into account recoil during radiation (14), we have

$$\mathbf{p}'(+\infty) = \mathbf{p}_f' = \mathbf{p}_f - \mathbf{k} \equiv \mathbf{p}_2; \tag{17}$$

on the other hand, on the basis of (16), (A.6)

$$\mathbf{p}(+\infty) = \mathbf{p}_f = \mathbf{p}_1 - \mathbf{q}. \tag{18}$$

Combining these results we have

$$E_2 = E_1 - \omega, \quad \mathbf{p}_2 = \mathbf{p}_1 - \mathbf{k} - \mathbf{q}. \tag{19}$$

Substituting (16) and $kx(t)$ in the form of (A.7) and (A.10) into formula (15), we obtain

$$\mathscr{M}_S(\rho) = \frac{ie}{(2\pi)^{3/2}E_1}\sqrt{\frac{E_2}{2E_1\omega}}\left[\frac{\boldsymbol{\varepsilon}\mathbf{p}_1}{kv_1} - \frac{\boldsymbol{\varepsilon}\mathbf{p}_2}{kv_f}\right]\eta K_1(\eta), \tag{20}$$

where $v = 1$,

$$\eta = \rho\frac{E_1}{E_2}\sqrt{(kv_1)(kv_f)} > 0, \tag{21}$$

$K_1(\eta)$ is the Macdonald function. Using (17) we can compute (kv_f) with an accuracy to terms of order $1/\gamma^2$:

$$(kv_f) = \frac{E_2}{E_1^2}(kp_2). \tag{22}$$

Further discussion depends substantially on the value of the momentum transfer q. We can distinguish two characteristic regions of q values, depending on the ratio of the quantities q and q_{min}, where

$$q_{min} = |\mathbf{p}_1| - |\mathbf{p}_2| - |\mathbf{k}| \approx \omega m^2 / 2E_1E_2; \tag{23}$$

in the first region $q \gg q_{min}$, and in the second $q \sim q_{min}$. This division is associated with the phenomenon of quantum-mechanical diffraction in the radiation process. The diffraction angle in this case is determined by the uncertainty of the momentum in the portion of the trajectory which provides the main contribution to radiation of the photon. Since the electron velocity $v \sim 1$, the length of this portion is $vT \sim T$, where T is the time interval which is important in the integral (15), the main contribution to which comes from the region $(E_1/E_2)kx(T) \sim 1$. Hence we obtain

$$\Delta p \approx T^{-1} \approx \omega m^2 / 2E_1E_2 = q_{min}. \tag{24}$$

Thus, the phenomenon of quantum-mechanical diffraction becomes important for $q \sim q_{min}$. We note that by definition the vector q_{min} is longitudinal and always $|q_\parallel| \sim q_{min}$, so that for $\mathbf{q}^2 = \mathbf{q}_\parallel^2 + \mathbf{q}_\perp^2 \gg q_{min}^2$ the transfer is mainly transverse.

It turns out to be convenient to use the variable

$$y = \frac{(kp_1)(kp_2)}{E_1E_2q^2} \tag{25}$$

the quantity η ((21) and (22)) being expressed in terms of y in the following way:

$$\eta = \rho q\sqrt{y}. \tag{26}$$

In the region $q \gg q_{min}$, $y \ll 1$ and then $\eta \ll 1$. In this case $\eta K_1(\eta) = 1 + O(\eta)$ and with this accuracy

$$\mathscr{M}_S \to \mathscr{M}_S^0 = \frac{ie}{(2\pi)^{3/2}}\sqrt{\frac{E_2}{2E_1\omega}}\left(\frac{\boldsymbol{\varepsilon}\mathbf{p}_1}{kp_1} - \frac{E_1}{E_2}\frac{\boldsymbol{\varepsilon}\mathbf{p}_2}{kp_2}\right) \tag{27}$$

Taking into account that for $q \gg q_{min}$ the momentum transfer is determined by the angle of scattering in the external field, we can write the cross section for radiation of a photon in the form

$$d\sigma_\gamma = d\sigma(\mathbf{q}_\perp)\, dW_\gamma(\mathbf{q}_\perp, \mathbf{k}), \tag{28}$$

where

$$dW_\gamma = |\mathscr{M}_S^0|^2 d^3k. \tag{29}$$

Actually, when $q \sim q_\perp$, the value of $q_\perp$ is determined only by the action of the external field (A.1) and is not associated with the radiation process. For this reason the scattering process does not depend on the radiation process. We note that Eq. (27) is obtained by integration of (15) with respect to time, if the trajectory is represented in the form of an "angle" (momentum p_1 in the interval $-\infty < t < 0$ and momentum p_f in the interval $0 < t < \infty$). Using the explicit form of $d\sigma(\mathbf{q}_\perp)$ in a Coulomb field,

$$d\sigma(\mathbf{q}_\perp) = \frac{4Z^2\alpha^2}{q_\perp^4} d^2q_\perp, \tag{30}$$

we find that in the region of "classical" momentum transfers $q \gg q_{min}$ the cross section for radiation of a photon (28) is identical to the cross section calculated in the Born approximation.[3]

Let us turn to the region $q \sim q_{min}$. Because of the diffraction phenomenon, in this case the value of momentum transfer already is not determined by the scattering angle of the particle. Therefore, to obtain the matrix element for radiation for a given momentum transfer it is necessary to perform a summation over the trajectories (over the partial amplitudes $\mathscr{M}(\boldsymbol{\rho})$). For this purpose we will use the impact-parameter method (see for example Glauber[4]). Then the expression for the radiation cross section in the case of a Coulomb potential can be written in the form

$$d\sigma_\gamma = |\mathscr{M}(\mathbf{q}_\perp)|^2 d^2q_\perp d^3k, \tag{31}$$

where

$$\mathscr{M}(\mathbf{q}_\perp) = \frac{1}{2\pi i}\int d^2\rho e^{i\mathbf{q}_\perp\rho} e^{i\chi(\rho)}\mathscr{M}(\rho); \quad (32)$$

here $\chi = 2\xi \ln(\rho/a)$, $\xi = Z\alpha$, a is a convergence parameter in calculation of the scattering phase in a Coulomb potential; the cross section (31) does not depend on the quantity a, which we will assume is much larger than all of the characteristic dimensions; for scalar particles $\mathscr{M}_S(\rho)$ is given by Eq. (20).

In calculation of the integral (32) it is necessary to take into account that for $q \sim q_{min}$ all particles are traveling almost along a single straight line, so that $kv_1 \approx kv_f$. Then it is evident that the matrix element $_S(\rho)$ (20) is proportional to $\boldsymbol{\varepsilon}(\mathbf{p}_1 - \mathbf{p}_2) = \boldsymbol{\varepsilon}\mathbf{q}_\perp$ and, using the relation between $\mathbf{q}_\perp$ and ρ (A.1) in a Coulomb field, we obtain

$$\mathscr{M}_S(\rho) = \mathbf{D}_S\frac{\boldsymbol{\rho}}{\rho^2}\eta K_1(\eta), \quad (33)$$

where

$$\mathbf{D}_S = \frac{2ie\xi}{(2\pi)^{3/2}}\frac{1}{m^2\omega}\sqrt{\frac{2E_1E_2}{\omega}}\boldsymbol{\varepsilon}. \quad (34)$$

Substituting expression (33) into (32) and integrating over angle, we obtain

$$\mathscr{M}_S(\mathbf{q}_\perp) = (\mathbf{D}_S\mathbf{q}_\perp)\frac{q\sqrt{y}}{q_\perp}\int\rho(\rho/a)^{2i\xi}K_1(q\rho\sqrt{y})J_1(q_\perp\rho)d\rho. \quad (35)$$

We will take into account that in the region $q \sim q_{min}$ we have (cf. (25)) $y = q^2_{min}/q^2$, $q_{min} = q_\parallel$, so that

$$q_\perp = q\sqrt{1-y}. \quad (36)$$

Calculating this integral (see ref. 5), we obtain

$$\mathscr{M}_S(\mathbf{q}_\perp) = (\mathbf{D}_S\mathbf{q}_\perp)\frac{1}{q^2}\left(\frac{2}{aq}\right)^{2i\xi} \times (1+i\xi)\Gamma^2(1+i\xi)F(-i\xi, 1+i\xi; 2, 1-y), \quad (37)$$

where F is the hypergeometric function. In derivation of (37) we used the standard transformation of the hypergeometric function from an argument $1 - 1/y$ to an argument $1 - y$. Substituting this result into the expression for the cross section (31), we obtain

$$d\sigma_{S\gamma} = d\sigma_{S\gamma}{}^B\Phi(y), \quad \Phi(y) = \frac{V^2(y) + \xi^2y^2W^2(y)}{V^2(0)} \quad (38)$$

where $d\sigma^B_{S\gamma}$ is the radiation cross section in the Born approximation,

$$V(y) = F(i\xi, -i\xi; 1, 1-y), \quad W(y) = \frac{1}{\xi^2}\frac{dV(y)}{dy}. \quad (39)$$

Here we used the relation

$$(1+i\xi)F(-i\xi, 1+i\xi; 2, 1-y) = V(y) + i\xi yW(y). \quad (40)$$

Formula (38) is valid in the region of momentum transfers $q_{min} \le q \ll m$. When $y \ll 1$, the correction factor to the Born cross section $\Phi(y) \to 1$. Taking into account that formula (28) for $q \gg q_{min}$ is the radiation cross section in the Born approximation, we find that formula (38) is applicable for all momentum transfers of interest to us:

$$d\sigma_{S\gamma} = \frac{Z^2\alpha^3}{\pi^2}\frac{1}{q^4}\frac{E_2}{E_1}\left|E_2\frac{\boldsymbol{\varepsilon}\mathbf{p}_1}{kp_1} - E_1\frac{\boldsymbol{\varepsilon}\mathbf{p}_2}{kp_2}\right|^2\Phi(y)\frac{d^3k}{\omega}d\Omega_2. \quad (41)$$

To calculate the spectrum of radiated photons it is convenient to use the variables

$$\zeta = \frac{2E_1}{\omega}\frac{(kp_1)}{m^2}, \quad \eta = \frac{2E_2}{\omega}\frac{(kp_2)}{m^2}, \quad y = \frac{(kp_1)(kp_2)}{q^2E_1E_2}. \quad (42)$$

Summing over photon polarization and replacing the variables in Eq. (41), we obtain

$$d\sigma_{S\gamma} = \frac{4Z^2\alpha^3}{\pi m^2}\frac{E_2}{E_1}\frac{d\omega}{\omega}\int\frac{dy\,d\zeta\,d\eta}{\zeta\eta s\sin\varphi}\left[\frac{1}{y} - 1 - \frac{(\zeta-\eta)^2}{\zeta^2\eta^2\delta^2}\right]\Phi(y), \quad (43)$$

where

$$s\sin\varphi = [2\lambda\zeta\eta(\zeta+\eta-2) - (\lambda\zeta\eta)^2 - (\zeta-\eta)^2]^{1/2},$$
$$\lambda = \delta^2\left(\frac{1}{y} - \frac{1}{\zeta\eta}\right), \quad \delta = \frac{q_{min}}{m} = \frac{\omega m}{2E_1E_2}. \quad (44)$$

With the accepted degree of accuracy we can use the approximate expression

$$\lambda = \delta^2(1-y)/y. \quad (45)$$

The integral in (43) over one of the variables (ζ, η) is taken between zeroes of the function $s\sin\varphi$, and the second integral is computed by elementary means. Performing the integration, we obtain

$$d\sigma_{S\gamma} = \frac{8Z^2\alpha^3}{3m^2}\frac{E_2}{E_1}\frac{d\omega}{\omega}\int_{\delta^2}^{1}\frac{1-y}{y}\Phi(y)dy. \quad (46)$$

Using the hypergeometric equation, we can easily show that

$$\int_{\delta^2}^{1}\frac{1-y}{y}\Phi(y)\,dy = \frac{1-y}{V^2(0)}[VW + V^2 - y(1-y)\xi^2W^2]\Big|_{\delta^2}^{1}. \quad (47)$$

Taking the well known asymptotic forms of the hypergeometric functions and retaining the principal terms of an expansion in δ^2, we obtain

$$d\sigma_{S\gamma} = \frac{16Z^2\alpha^3}{3m^2}\frac{E_2}{E_1}\frac{d\omega}{\omega}\left[\ln\frac{1}{\delta} - \frac{1}{2} - f(\xi)\right], \quad (48)$$

where

$$f(\xi) = \sum_{n=1}^{\infty}(-1)^{n+1}\zeta(2n+1)\xi^{2n} = \xi^2\sum_{n=1}^{\infty}\frac{1}{n(n^2+\xi^2)}. \quad (49)$$

For $\xi \to 0$, $f(\xi) \to 0$ and we obtain the spectrum of radiated photons in the Born approximation. For $\xi \gg 1$, $f(\xi) \to \ln\xi + C$ (C is the Euler constant). In this case

$$d\sigma_{S\gamma} = \frac{16Z^2\alpha^3}{3m^2}\frac{E_2}{E_1}\frac{d\omega}{\omega}\left[\ln\frac{1}{\delta\xi} - \frac{1}{2} - C\right]. \quad (50)$$

This result can be obtained on the basis of purely classical concepts of the collision process. Actually, in the classical theory there is a rigorous connection between the impact parameter ρ and the momentum transfer, $q_\perp = 2\xi/\rho$ (this is evident, for example, from expression (32), where for $\xi \gg 1$ the integrand contains a rapidly oscillating factor). Then $\Phi(y) \to 4\xi^2yK_1^2(2\xi\sqrt{y})$ and the integral in expression (46) receives its main contribution from the region $y \sim 1/4\xi^2 \ll 1$, so that an exponential cutoff occurs of the region with minimum momentum transfer where the quantum effects are important. Substituting this expression for $\Phi(y)$ into (46), we obtain Eq. (50).

Note that for $\xi \gg 1$, formula (28) has the "classical" form:

$$d\sigma_\gamma = \rho\, d\rho\, d\varphi\, dW_\gamma(\rho, k). \tag{51}$$

4. We will now consider the radiation problem for an electron in a Coulomb field. Integration of the matrix element with respect to time is performed in the same way as for scalar particles (20). In summary we obtain

$$\mathcal{M}_e(\rho) = \mathcal{M}_e^0(\rho)\eta K_1(\eta), \tag{52}$$

where $\mathcal{M}_e^0(\rho)$ is obtained by integration over the trajectory in the form of an "angle" (cf. (27)):

$$\mathcal{M}_e^0(\rho) = \frac{ie}{(2\pi)^{3/2} E_1\sqrt{2\omega}}\left(\frac{E_2}{E_1}\right)^{1/2}\left(\frac{R_1}{kv_1} - \frac{R_f}{kv_f}\right). \tag{53}$$

Here $R_{1,f}$ is defined by Eq. (14):

$$R_{1,f} = u_{s'}^+(\mathbf{p}'_{1,f})(\boldsymbol{\varepsilon\alpha})u_s(\mathbf{p}_{1,f}) = \varphi_{s'}^+[A_{1,f} + i\sigma\mathbf{B}_{1,f}]\varphi_s, \tag{54}$$

where

$$A = (\boldsymbol{\varepsilon}\mathbf{p})\left(\sqrt{\frac{E'+m}{E+m}} + \sqrt{\frac{E+m}{E'+m}}\right),$$

$$\mathbf{B} = \sqrt{\frac{E'+m}{E+m}}[\boldsymbol{\varepsilon}\mathbf{p}] + \sqrt{\frac{E+m}{E'+m}}[\boldsymbol{\varepsilon}\mathbf{p}']. \tag{55}$$

In formula (54) it is taken into account that with our accuracy ($\sim 1/\gamma$) for small angles of deflection in the external field we can neglect the time dependence of the spin states $\varphi(\xi(t))$ (7).

As in the discussion of the radiation of a scalar particle it is necessary to discuss separately the regions $q \gg q_{min}$ and $q \sim q_{min}$. In general, all of the discussions relating to the division into regions and the behavior of the cross sections in them do not depend on the spin of the particle.

In the region $q \gg q_{min}$ formula (28) is valid and, as in the case of a scalar particle, it gives a radiation cross section identical with that calculated in the Born approximation. Using formulas (28) and (52)–(54), we can easily write a general expression for the cross section for radiation by polarized particles:

$$d\sigma_\gamma = d\sigma(\mathbf{q}_\perp)\frac{e^2}{(2\pi)^3}\frac{E_2}{2\omega E_1^3}\cdot\frac{1}{4}\mathrm{Sp}[(1+\zeta_i\sigma)O^+(1+\zeta_f\sigma)O]\,d^3k, \tag{56}$$

where

$$O = \frac{A_1}{kv_1} - \frac{A_f}{kv_f} + i\sigma\left(\frac{\mathbf{B}_1}{kv_1} - \frac{\mathbf{B}_f}{kv_f}\right). \tag{57}$$

This cross section, after summing over the spin of the final particle and averaging over the spin of the initial particle, with our accuracy can be written in the form

$$d\sigma_{e\gamma}^B = \frac{Z^2\alpha^3}{\pi^2}\frac{1}{q^4}\frac{E_2}{E_1}$$
$$\times\left\{\left|E_2\frac{\boldsymbol{\varepsilon}\mathbf{p}_1}{kp_1} - E_1\frac{\boldsymbol{\varepsilon}\mathbf{p}_2}{kp_2}\right|^2 + \frac{\boldsymbol{\varepsilon}^*\boldsymbol{\varepsilon}}{4(kp_1)(kp_2)}[\mathbf{k}(\mathbf{p}_1-\mathbf{p}_2)]^2\right\}\frac{d^3k}{\omega}d\Omega_2. \tag{58}$$

In this formula the first term in the curly brackets is identical to the corresponding term in the cross section for radiation by a scalar particle, and the second term is the addition due to spin.

In the case $q \sim q_{min}$ it is necessary to use formulas (31) and (32), in which $\mathcal{M}(\rho)$ is now substituted in the form of Eq. (52). Here the statement that in this region the matrix element is proportional to $\mathbf{q}_\perp$ is still true (this can be seen specifically from formulas (54) and (55)), so that

$$\mathcal{M}_e(\rho) = \mathbf{D}_e\frac{\boldsymbol{\rho}}{\rho^2}\eta K_1(\eta). \tag{59}$$

The further calculations coincide with those performed in the case of a scalar particle, (35)–(38). We finally obtain

$$d\sigma_{e\gamma} = d\sigma_{e\gamma}^B\Phi(y). \tag{60}$$

Thus, only the Born-approximation cross section, which enters into the cross-section formulas (38) and (60), depends on spin, including all spin and polarization effects, and the inclusion of higher orders of interaction with the external field (the function $\Phi(y)$) in general does not depend on spin.

Calculation of the spectrum of radiated photons is carried out in the same way as for a scalar particle (Eqs. (42)–(46)). The final result is

$$d\sigma_{e\gamma} = \frac{4Z^2\alpha^3}{m^2}\frac{E_2}{E_1}\left(\frac{E_1}{E_2} + \frac{E_2}{E_1} - \frac{2}{3}\right)\frac{d\omega}{\omega}\left[\ln\frac{1}{\delta} - \frac{1}{2} - f(\xi)\right]. \tag{61}$$

Equation (60) for unpolarized electrons was obtained for the first time in the well known work of Bethe and Maximon[6] in which the calculations were made with approximate ($l \gg 1$) wave functions for the electron in a Coulomb field. Formula (61) was obtained subsequently by Olsen.[7] Formulas (60) and (61) for electrons were also obtained by Olsen and Maximon[8, 9] with use of quasiclassical wave functions. The summation over impact parameters which occurs in such an approach is performed by a means similar to that used above. All of the results for scalar particles are obtained here for the first time.

5. The approach which we have suggested can be applied to discussion of radiation processes in any external field. We will discuss below fields falling off with distance no more slowly than a Coulomb field (case II).

For $q \gg q_{min}$, formula (28) remains in force; here $dW_\gamma(\mathbf{k})$ is obtained by integration over the trajectory in the form of an "angle" and does not depend on the field shape (the cross section $d\sigma(\mathbf{q}_\perp)$ in (28) depends on field). Therefore we can use for $dW_\gamma(\mathbf{k})$ the expressions obtained in a Coulomb field.

In the region $q \sim q_{min}$ formulas (31) and (32) remain valid, and we must substitute in them the scattering phase for the assumed field:

$$\chi = -\int_{-\infty}^{\infty} V(\rho, z)\,dz. \tag{62}$$

We will now calculate the matrix element $\mathcal{M}(\rho)$ in a centrally symmetric field $V(r)$, taking into account that in this region $R(t)$ can be represented in the form (cf. (33) and (59)):

$$R(t) = \mathbf{D}\mathbf{p}(t), \tag{63}$$

where $\mathbf{D}$ does not depend on time. Substituting into the expression for the matrix element

$$\mathcal{M} = \int_{-\infty}^{\infty} \exp\left\{i\frac{E_1}{E_2}kx(t)\right\}R(t)\,dt \qquad (64)$$

and using the explicit forms of p(t) (A.1) and kx(t) (A.7), we obtain with our accuracy

$$\mathcal{M}(\rho) = \frac{i}{q_{min}}\frac{\mathbf{D}\rho}{\rho}\frac{d}{d\rho}\int_{-\infty}^{\infty} dt\, e^{iq_{min}t}\, V(\sqrt{\rho^2+t^2}). \qquad (65)$$

Substituting this expression and the phase (62) into formula (32), we obtain

$$\mathcal{M}(\mathbf{q}_\perp) = \mathcal{M}_B A(q, y), \qquad (66)$$

where

$$\mathcal{M}_B = \frac{i}{q_{min}}\mathbf{D}\mathbf{q}_\perp,$$

$$A(q, y) = \frac{1}{q_\perp}\int \rho\, d\rho\, J_1(q\rho\sqrt{1-y})\, e^{i\chi(\rho)}\frac{d}{d\rho}\int_{-\infty} e^{iq_{min}t}\, V(\sqrt{\rho^2+t^2})\,dt, \qquad (67)$$

so that the radiation cross section is

$$d\sigma_\gamma = |\mathcal{M}_B|^2 d^3k\, |A(q, y)|^2 d^2q. \qquad (68)$$

In the region $y \ll 1$,

$$|A(q, y)|^2 d^2q \to d\sigma(q^2), \qquad (69)$$

which follows directly from the representation according to Eq. (67).

Since in the region $q^2 \gg q_{min}^2$ the formula

$$d\sigma_\gamma = dW_\gamma d\sigma(q^2) \qquad (70)$$

is valid, and for $q \sim q_{min}$ we have

$$dW_\gamma \to |\mathcal{M}_B|^2 d^3k, \qquad (71)$$

the general expression for the cross section for radiation of a photon by a relativistic particle in an external field V(r) in case II can be represented in the form

$$d\sigma_\gamma = dW_\gamma(\mathbf{k})\, |A(q, y)|^2 d^2q, \qquad (72)$$

where dW_γ is obtained by integration over the trajectory in the form of an "angle" and is identical with the probability for radiation with a given momentum transfer, calculated in the Born approximation. In a Coulomb field, all of the results obtained above follow from this formula.

We can also write out general formulas of the type of (72) for case I. Since in this case only a small portion of the trajectory contributes, the calculation can be carried out in general form as far as the explicit expression for the spectrum of radiated photons (see ref. 1):

$$\frac{d\mathcal{E}_\omega}{d\alpha} = -\frac{e^2m^2}{4\sqrt{3}\pi^2}\int_{-\infty}^{\infty}\frac{\alpha}{(1+\alpha)^3}\left[\int_{2\alpha/3\chi}^{\infty} K_{5/3}(x)\,dx + \delta_{e,s}\frac{\alpha^2}{1+\alpha}K_{2/3}\left(\frac{2\alpha}{3\chi}\right)\right]dt, \qquad (73)$$

where $\alpha = \omega/(E-\omega)$, $\chi = |\mathbf{v}|\gamma^2/m$, $\delta_S = 0$ for particles with spin zero and $\delta_e = 1$ for particles with spin ½. In the classical limit this formula goes over to the well known expression given, for example, by Landau and Lifshitz.[10]

APPENDIX A

THE TRAJECTORY OF A PARTICLE IN AN EXTERNAL FIELD IN THE APPROXIMATION OF SMALL ANGLES

We will consider the trajectory of the particle in the approximation of small angles, which is important for the present work. In the case of a classical trajectory the dependence of momentum on time in an external field V in the first approximation for momentum transfer is determined by the expression

$$\mathbf{p}(t) = \mathbf{p}_1 - \mathbf{q}(t) = \mathbf{p}_1 + \int_{-\infty}^{t} e\frac{\partial V}{\partial\rho^2}2\boldsymbol{\rho}\,dt, \qquad (A.1)$$

where ρ is the impact parameter and $\mathbf{p}_1 = \mathbf{p}(-\infty)$.

With the same accuracy the deflection of the particle trajectory from a straight line in a field V(r) is

$$\mathbf{x}_q(t) = -\int_0^t \frac{1}{E}\mathbf{q}\,dt = \int_0^t \frac{1}{E}dt'\int_0^{t'} e\frac{\partial V}{\partial\rho^2}2\boldsymbol{\rho}\,dt' + \text{const}\cdot t. \qquad (A.2)$$

By interchanging the order of integration this expression can be reduced to the form

$$\mathbf{x}_q(t) = \int_0^t dy\,(t-y)\,e\frac{\partial V(\sqrt{\rho^2+v^2y^2})}{\partial\rho^2}2\boldsymbol{\rho} \qquad (A.3)$$

$$= \frac{2e\boldsymbol{\rho}}{E}\left[t\int_0^t\frac{\partial V}{\partial\rho^2}dy - \frac{V(r)}{2}\right] + \text{const}\cdot t.$$

We will subsequently be interested in the quantity kx(t), which, in consideration of what we have said above, can be written in the form

$$kx(t) = c_1t + c_2\frac{E}{2\rho^2}(\boldsymbol{\rho}\mathbf{x}_q(t)) + kx_0, \qquad (A.4)$$

where kx_0 is a constant, and the factor $E/2\rho^2$ is separated for convenience. The coefficients c_1 and c_2 can be determined from the conditions at $t = \pm\infty$. As $t \to -\infty$, $kx \to (kv_1)t$, from which we can write

$$(kv_1) = c_1 - c_2\int_0 e\frac{\partial V}{\partial\rho^2}dy = c_1 - c_2d; \qquad (A.5)$$

as $t \to +\infty$, $kx \to (kv_f)t$ ($v_f = v(t = +\infty)$), and hence

$$(kv_f) = c_1 + c_2d. \qquad (A.6)$$

Finally we obtain

$$kx = {}^1\!/_2(kv_1 + kv_f)t + {}^1\!/_2(kv_f - kv_1)g(t) + kx_0, \qquad (A.7)$$

$$g(t) = \frac{E}{2\rho^2d}(\boldsymbol{\rho}\mathbf{x}_q(t)). \qquad (A.8)$$

It should be noted that the quantity

$$g(t) = \left(\int_0^\infty\frac{\partial V}{\partial\rho^2}dy\right)^{-1}\left[t\int_0^t\frac{\partial V}{\partial\rho^2}dy - \frac{V(r)}{2}\right], \qquad (A.9)$$

if we set $v^2 = 1$ (with an accuracy to terms of order $1/\gamma^2$), does not depend on the dynamic variables of the particle, but only on the form of the field and the impact parameter ρ.

In the case of a Coulomb field we have

$$q(t)=\sqrt{\rho^2+t^2}. \tag{A.10}$$

The last result is obtained, naturally, from the exact expression for the trajectory in a Coulomb field in the case of small deflection angles.

APPENDIX B

UNFOLDING OF THE COMBINATION $\exp(-ikx(t_2))\exp(ikx(t_1))$.

We will represent $\exp(-ikx(t_2))$ in the form

$$e^{-ikx(t_2)}=Le^{-ikx(t_1)}, \tag{B.1}$$

where L is an operator, the problem reducing to determination of this operator. Substituting for brevity

$$a=-ikx(t_1),\ b=i[kx(t_1)-kx(t_2)], \tag{B.2}$$

we have

$$e^{\xi(a+b)}=L(\xi)e^{\xi a}, \tag{B.3}$$

where ξ is a parameter. The operator $L(\xi)$ satisfies the equation

$$\frac{dL}{d\xi}=L(\xi)e^{\xi a}be^{-\xi a}. \tag{B.4}$$

We will now determine

$$e^{\xi a}be^{-\xi a}=\sum_{n=0}^{\infty}\frac{\xi^n}{n!}[a,[a\ldots[a,b]\ldots]]. \tag{B.5}$$

The commutator [a, b] can be calculated if we use the explicit form of a, b (see formula (A.7)). The operators kv_1 and kv_f commute with each other. The function g(t) (as was noted in Appendix A) with an accuracy to terms of order $1/\gamma^2$ is a c-number. For this reason, calculation of the commutator [a, b] reduces to calculation of $[kx_0, kv_1]$, $[kx_0, kv_f]$. Using the relation

$$\mathbf{v}=\frac{\mathbf{p}}{\sqrt{\mathbf{p}^2+m^2}},\quad [x_{0i},p_k]=i\delta_{ik}, \tag{B.6}$$

we obtain

$$[kx_0,kv_1]=\frac{2i\omega}{\mathcal{H}_1}kv_1,\quad [kx_0,kv_f]=\frac{2i\omega}{\mathcal{H}_f}kv_f. \tag{B.7}$$

Hence with an accuracy to terms of order $1/\gamma^2$ we obtain

$$[a,b]=2\omega b/\mathcal{H}. \tag{B.8}$$

Note that the operators $\mathcal{H}$ and b which enter into Eq. (B.8) commute with each other. If we take into account that

$$[a,\mathcal{H}^{-1}]=\omega/\mathcal{H}^2, \tag{B.9}$$

we obtain

$$e^{\xi a}be^{-\xi a}=\sum_{n=0}^{\infty}\frac{\xi^n}{n!}(n+1)!\left(\frac{\omega}{\mathcal{H}}\right)^n b=\frac{b}{(1-\omega\xi/\mathcal{H})^2}. \tag{B.10}$$

The solution of the differential equation (B.4) with the boundary condition L(0) = 1 has the form

$$L(\xi)=\exp\left\{\frac{\xi b\mathcal{H}}{\mathcal{H}-\omega\xi}\right\}, \tag{B.11}$$

Then, considering (B.1) and (B.2), we obtain

$$e^{-ikx(t_2)}e^{ikx(t_1)}=L(1)=\exp\left\{-i\frac{\mathcal{H}}{\mathcal{H}-\omega}[kx(t_2)-kx(t_1)]\right\}. \tag{B.12}$$

[1] V. N. Baĭer and V. M. Katkov, Zh. Eksp. Teor. Fiz. **53**, 1478 (1967) [Sov. Phys.-JETP **26**, 854 (1968)].
[2] V. N. Baĭer and V. M. Katkov, Phys. Letters **25A**, 492 (1967).
[3] A. I. Akhiezer and V. B. Berestetskiĭ, Kvantovaya élektrodinamika (Quantum Electrodynamics), Fizmatgiz, 1959.
[4] R. Glauber, Lectures in Theoretical Physics, Vol. 1, University of Colorado, Boulder; Interscience Publishers, New York, 1959.
[5] I. S. Gradshteĭn and I. M. Ryzhik, Tablitsy integralov, summ, ryadov i proizvedeniĭ (Tables of Integrals, Sums, Series, and Products), Fizmatgiz, 1962.
[6] H. A. Bethe and L. C. Maximon, Phys. Rev. **93**, 768 (1954).
[7] H. Olsen, Phys. Rev. **99**, 1335 (1955).
[8] H. Olsen, L. C. Maximon, and H. Wergeland, Phys. Rev. **106**, 27 (1957).
[9] H. Olsen and L. C. Maximon, Phys. Rev. **114**, 887 (1959).
[10] L. D. Landau and E. M. Lifshitz, Teoriya polya (Theory of Fields), Nauka, 1967.

Translated by C. S. Robinson
171