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STRONG COUPLING IN THE POMERANCHUK POLE PROBLEM

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The possibility of strong coupling is investigated for the Pomeranchuk pole problem, corresponding to substantial modifications of the trajectory and the residue of this pole under the influence of branch points. It is shown that owing to the non-Hermitian character of the interaction of reggeons this possibility is not self-consistent, and apparently weak coupling is realized, i.e., the coupling constants of the reggeons vanish after renormalization (cf. [4]). For a Hermitian interaction we have determined the form of the reggeon Green's functions, as well as their analyticity properties and asymptotic behavior. Hermitian strong-coupling problems are encountered in the physics of condensed states, when the scattering amplitudes (Green's functions) have closely spaced singularities for small momenta, and the expansion parameter of perturbation theory tends to infinity (phase transitions, etc.). Our investigation may turn out to be useful for the consideration of such problems.

1. INTRODUCTION

In a previous paper of one of the authors[1] it has been shown that the problem of determining the diffraction scattering amplitude at high energies and small momentum transfers $\mathbf{k}^2 = -t$, under the assumption that the Pomeranchuk pole exists, reduces to the problem of determination of a Green's function of a nonrelativistic quasiparticle in a two-dimensional space, where the quasiparticle is capable of decaying into an arbitrary number of similar quasiparticles. The threshold singularities produced by these decays correspond to the Mandelstam branch cuts[1-3] generated by the Pomeranchuk pole. The role of the quasiparticle energy is played by the variable spin of the Pomeranchuk pole, minus one: $\varepsilon = 1 - j = -\omega$. The condition that the Pomeranchuk trajectory pass through the angular momentum value $j = 1$ at $\mathbf{k}^2 = 0$ corresponds to a quasiparticle spectrum without energy gap, i.e., $\varepsilon(\mathbf{k}) \to 0$ as $\mathbf{k}^2 \to 0$. Without taking the decays into account we have $\varepsilon_0(\mathbf{k}) = \alpha'\mathbf{k}^2$, and the free Green's function $D_0(\omega, \mathbf{k})$ has the form $D_0(\omega, \mathbf{k}) = (\omega + \alpha'\mathbf{k}^2)^{-1}$. The absence of a gap in the quasiparticle spectrum leads to n-particle decay thresholds $\varepsilon_n = \alpha'\mathbf{k}^2/n$, which for small $\mathbf{k}^2$ are close to the position of the pole corresponding to the quasiparticle. This is equivalent to the fact that the Pomeranchuk pole and the branch points generated by it approach one another as $\mathbf{k}^2 \to 0$.

The presence of threshold singularities near the pole has as a consequence that the self-energy part which determines the exact Green's function of the quasiparticle–reggeon: $D^{-1} = \omega + \alpha'\mathbf{k}^2 - \Sigma(\omega, \mathbf{k})$ computed according to perturbation theory,[1] is large for small ω and $\mathbf{k}^2$, i.e., $\omega + \mathbf{k}^2\alpha'$ as $\omega, \mathbf{k}^2 \to 0$. This means that in order for such a quasiparticle without a gap in the spectrum to exist, it is necessary that the interaction which is responsible for the decays must substantially renormalize either the vertex parts Γ which determine the interaction strength, or the quasiparticle spectrum $\varepsilon(\mathbf{k})$. It is obvious that perturbation theory is not applicable to this problem, even if the constants that determine the decay amplitudes are small.

This problem has many features in common with problems encountered in investigations of many-particle systems near phase transition points. The essential distinction of the Pomeranchuk pole problem from the latter consists in the fact that the effective potential is non-Hermitian in the case in which we are interested. In the preceding paper[4] of the authors it was shown that owing to the non-Hermiticity of the effective Hamiltonian a solution corresponding to weak coupling can be realized, i.e., the quasiparticle spectrum does not undergo renormalization and the vertex parts are small at small ω and $\mathbf{k}^2$.

In the present paper we investigate the structure of the solution corresponding to strong coupling, when both the spectrum and vertex parts undergo renormalization. We show that in the case of a Hermitian Hamiltonian the solution for the Green's function for small ω and $\mathbf{k}^2$ has the form

$$D(\omega, \mathbf{k}) = Z\omega^{-\mu} f(x/x_0), \tag{1}$$

where $x = -\mathbf{k}^2/\omega^{\nu}$, $f(y)$ is a universal function; μ and ν are unknown universal numerical parameters, which do not depend on the coupling constants, and such that $0 < \mu < \nu < 1$. The quantities Z and x_0 are not universal and are determined from the condition of continuous transition to the region of large ω where perturbation theory is valid. We have also investigated the analyticity properties and the asymptotic behavior of the function $f(y)$. The vertex functions are also homogeneous power-functions, analogous to (1) (cf. (26) and (32)).

In the case under consideration, of an effective non-Hermitian Hamiltonian, a solution of the form (1) encounters contradictions, discussed in detail in the last section of the paper (cf. also [4]). These contradictions are however not of such nature as to enable us to exclude categorically the possible existence of solutions corresponding to strong coupling. The only conclusion which can be reached with relative definiteness is the fact that a solution of this kind will correspond to a total interaction cross section, which does not become constant at high energies, but decreases or increases logarithmically with energy.

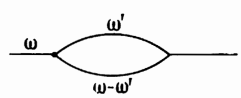
The main body of the paper (Secs. 2–6) is devoted to the investigation of a Hermitian Hamiltonian, since in this case a series of problems can be carried to the end, thus allowing to see what modifications are necessary in the non-Hermitian case. In addition, it seems to us that this investigation is interesting from the theoretical point of view and may be useful in other problems.

2. STRONG COUPLING IN THE PROBLEM WITH A HERMITIAN THREE-REGGEON HAMILTONIAN

The Dyson equation for $D(\omega, \mathbf{k})$ has the form

$$D^{-1}(\omega, \mathbf{k}) = \omega + \alpha' \mathbf{k}^2 + \Sigma(0,0) - \Sigma(\omega, \mathbf{k}), \tag{2}$$

where the self-energy part $\Sigma(\omega, \mathbf{k})$ is defined by the diagram

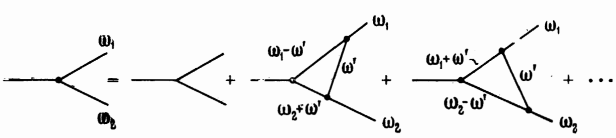


(3)

The analytic expression for this diagram is:

$$\Sigma(\omega, \mathbf{k}) = \frac{\rho^2}{2} \int_{-i\infty}^{+i\infty} \frac{d\omega'}{2\pi i} \frac{d^2 \mathbf{k}'}{(2\pi)^2} D(\omega', \mathbf{k}') D(\omega - \omega', \mathbf{k} - \mathbf{k}') \times \Gamma(\omega', \mathbf{k}'; \omega - \omega', \mathbf{k} - \mathbf{k}'). \tag{3a}$$

Here ρ is a real three-reggeon coupling constant; $\Gamma(\omega_1, \mathbf{k}_1;\ \omega_2, \mathbf{k}_2)$ is the vertex part corresponding to the graphical expression:



(4)

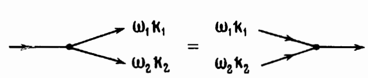
In analytic form this equation has the form:

$$\Gamma(\omega_1, \mathbf{k}_1; \omega_2, \mathbf{k}_2) = 1 + \rho^2 \int_{-i\infty}^{+i\infty} \frac{d\omega'}{2\pi i} \frac{d^2\mathbf{k}'}{(2\pi)^2} \Gamma(\omega', \mathbf{k}'; \omega_2, \mathbf{k}_2) \times D(\omega', \mathbf{k}') \Gamma(\omega', \mathbf{k}'; \omega_1 - \omega', \mathbf{k}_1 - \mathbf{k}') D(\omega_1 - \omega', \mathbf{k}_1 - \mathbf{k}') \times \Gamma(\omega_1 - \omega', \mathbf{k}_1 - \mathbf{k}'; \omega_2 + \omega', \mathbf{k}_2 + \mathbf{k}') D(\omega_2 + \omega', \mathbf{k}_2 + \mathbf{k}') + \ldots + \text{permutation } (1 \leftrightarrow 2). \tag{4a}$$

The vertex part is symmetric in its arguments

$$\Gamma(\omega_1, \mathbf{k}_1; \omega_2, \mathbf{k}_2) = \Gamma(\omega_2, \mathbf{k}_2; \omega_1, \mathbf{k}_1).$$

In addition, the decay vertex equals the "fusion" vertex:



(4b)

The singularities of the quantities Γ and D for $\mathbf{k}_i^2 > 0$ occur at negative frequencies $\omega_i \le 0$, corresponding to the fact that the partial-wave amplitude $f_j(t)$ in the t-channel, related to $D(\omega, \mathbf{k})$ for $t = -\mathbf{k}^2 < 0$, has no singularities to the right of the point $j = 1$ ($\omega = 0$). In the language of the analogy with a nonrelativistic two-dimensional particle, this is equivalent to the fact that in nonrelativistic diagram technique all quantities have singularities only at positive energies $\varepsilon = -\omega > 0$, since there exist no antiparticles.

The integration paths over ω_i^1 in (3) and (4), for $\omega_i > 0$, $\mathbf{k}_i^2 > 0$ are situated along the imaginary axes, so that all singularities of each of the quantities D and Γ are situated to one side of the contour, in such a manner that owing to the law of frequency conservation, the singularities of some of the D and Γ are situated only to the right of the imaginary axis, and the singularities of other are only to the left. Thus the integrals do not vanish. For $\omega_i < 0$ or $\mathbf{k}_i^2 < 0$ the integrals are defined as analytic continuations, i.e., the integration paths are deformed in order not to separate the singularities belonging to the same D or Γ, in the same manner as in the usual diagram technique. The cuts in the ω plane are placed in such a manner that for $\mathbf{k}^2 > 0$, $\omega > 0$ the Green's function $D(\omega, \mathbf{k})$ is real, and similarly for $\Gamma(\omega_1, \mathbf{k}_1;\ \omega_2, \mathbf{k}_2)$.

In the integrals (3) and (4) there are regular parts of the form $A + B\omega + C\mathbf{k}^2 + \ldots$, which are due to the contributions from the region $\omega_i' \sim 1 \sim \alpha' \mathbf{k}_i'^2$ and are not determined from the reggeon diagram technique. In order that $\varepsilon(0) = 0$ it is necessary that $D^{-1}(0, 0) = 0$, therefore we have subtracted from $\Sigma(\omega, \mathbf{k})$ the constant $A = \Sigma(0, 0)$. The terms $B\omega$ and $C\mathbf{k}^2$ represent the renormalization of the unrenormalized Green's function and are therefore inessential. The regular parts in (4) represent the renormalization of the "charge" ρ.

In order to understand the essence of the problem, we first consider the case of small ρ^2/α'. In our case[1] perturbation theory leads to an expansion in powers of $\rho^2/\alpha' \omega$. Therefore for $\rho \ll \alpha'$ there exists a region of frequencies $1 \gg \omega \gg \rho^2/\alpha'$, where perturbation theory is applicable, and where the diagram technique is still meaningful.

As ω decreases the perturbative expansion becomes meaningless. This is connected to the fact that as ω decreases for small $\mathbf{k}^2$ we approach the threshold singularities. The divergence of the perturbation series for small ω means that our selection of the zeroth approximation in the form $D_0 = (\omega + \alpha' \mathbf{k}^2)^{-1}$, $\Gamma_0 = 1$ is incorrect, and that either the trajectory $\varepsilon(\mathbf{k})$ of the pole undergoes an essential modification, or so does the interaction defined by the vertex part Γ. In this case the series expansion in terms of diagrams containing exact vertex parts and Green's functions is meaningful if we assume that there appear no new singularities, except poles and thresholds. The reggeon diagram technique is in fact based on this assumption.

It is not hard to see that the expansion parameter of this new series (in place of $\rho^2/\alpha' \omega$) will be the quantity

$$\eta = \rho^2 \Gamma^2 D^3 \omega \bar{k}^2(\omega),$$

where $\bar{k}^2(\omega)$ is the order of magnitude of the internal momenta in the diagrams, order of magnitude which is determined by the form of the spectrum (pole trajectory) $\omega = -\varepsilon(\bar{k})$. In order that the series be meaningful it is necessary that the parameter η be of the order of magnitude of one, or much smaller than unity, i.e., that at least one of the quantities Γ, D, $\bar{k}^2(\omega)$ should be considerably diminished compared to the original one. As discussed in more detail below, in the Hermitian case under consideration, Γ is defined by a series with positive terms, and cannot become small. As regards D, owing to subtractions in Σ the series for D is not composed only of terms of one sign, and therefore one might expect that $D \ll D_0$, i.e.,

$$\Sigma_c(\omega, \mathbf{k}) \equiv \Sigma(\omega, \mathbf{k}) - \Sigma(0,0) \gg \omega + \alpha' \mathbf{k}^2.$$

After these heuristic considerations we are ready to solve the problem. We shall look for a solution for which $\Sigma_c \gg \omega + \alpha' \mathbf{k}^2$, $D \ll D_0$ (strong coupling). Thus

$$D(\omega, \mathbf{k}) = [\Sigma(0,0) - \Sigma(\omega, \mathbf{k})]^{-1}, \quad \omega, \mathbf{k} \to 0. \tag{5}$$

An unusual peculiarity of Eq. (5) is that it does not in fact contain dimensional constants—the constant α' contained in Eq. (2) has been cancelled in the transition to (5), and the constant ρ which has the dimension of length appears as a common factor in (5), so that $D(\omega, \mathbf{k}) = \rho^{-2/3} D'(\omega, \mathbf{k})$ where the equation for D' does no longer depend on ρ (it is easy to note that in the transition from D to D' the constant ρ disappears also from Eq. (4) for the vertex part). Therefore it is hard to understand at first sight how the equations (3)–(5) can at all have solutions for the Green's function $D(\omega, \mathbf{k})$ that depends on the dimensional constant $\mathbf{k}^2$. A trivial solution on the basis of "dimensionality" of $D(\omega, \mathbf{k}) = (|\mathbf{k}|\rho)^{-2/3} \varphi(\omega)$ does not satisfy the equation, if for no other reason than the presence of a singularity at $\mathbf{k}^2 = 0$, and the equations are constructed in such a manner that there is no such singularity.

We shall show below how a solution of the kind (1) with singularities in $\mathbf{k}^2$ situated only at finite values $\mathbf{k}^2 = -n^{2-\nu} \omega^{\nu} x_0$, with $n = 1, 2, 3, \ldots$, x_0 is an unknown constant of dimension $[\mathbf{k}^2]$, to be determined from the smooth junction of solutions in the region $\omega, \alpha' \mathbf{k}^2 \sim \rho^2/\alpha'$ with those from perturbation theory.

3. A MODEL EQUATION FOR $\Gamma = 1$

We first consider a simplified equation, without corrections to the vertex part, but involving exact Green's functions. This problem can be solved to the end. We have

$$D^{-1}(\omega, \mathbf{k}) = \Sigma(0,0) - \Sigma(\omega, \mathbf{k}),$$
$$\Sigma(\omega, \mathbf{k}) = \frac{\rho^2}{2} \int \frac{d\omega' \, d^2 k'}{(2\pi)^3 i} D(\omega', \mathbf{k}') D(\omega - \omega', \mathbf{k} - \mathbf{k}'). \tag{6}$$

We shall seek the solution of (6) for $\omega, \mathbf{k} \to 0$ in the form of a homogeneous function of ω and $\mathbf{k}^2$:

$$D(\omega, \mathbf{k}) = \rho^{-2/3} \omega^{-\mu} f(-\mathbf{k}^2 / \omega^{\nu}). \tag{1a}$$

We represent $f(x)$ in (1a) by an integral of the Cauchy type:

$$D(\omega, \mathbf{k}) = \rho^{-2/3} \int_{-i\infty}^{+i\infty} dx_1 \frac{f(x_1) \omega^{\nu - \mu}}{2\pi i (x_1 \omega^{\nu} + \mathbf{k}^2)}$$

and substitute it into (6). Thus we obtain the following expression:

$$\Sigma(0,0) - \Sigma(\omega, \mathbf{k}) = \omega^{1+\nu-2\mu} \rho^{2/3} \iint_{-i\infty}^{+i\infty} \Phi(x; x_1, x_2) f(x_1) f(x_2) \frac{dx_1 \, dx_2}{(2\pi i)^2}, \tag{7}$$

where the kernel $\Phi(x; x_1, x_2)$ is a homogeneous function of x, x_1, x_2 of the form

$$\Phi(x; x_1, x_2) = \int_{-i\infty}^{+i\infty} \frac{d\xi}{8\pi^2 i} \xi^{\nu-\mu} \left\{ \frac{J(-x; x_1 \xi^{\nu}, x_2 (1-\xi)^{\nu})}{(1-\xi)^{\mu-\nu}} - \frac{\ln [x_1 \xi^{\nu} / x_2 (-\xi)^{\nu}]}{(-\xi)^{\mu-\nu} [\xi^{\nu} x_1 - (-\xi)^{\nu} x_2]} \right\} \tag{8}$$

$$J(z; z_1, z_2) = \frac{1}{\sqrt{L}} \ln \frac{z + z_1 + z_2 + \sqrt{L}}{z + z_1 + z_2 - \sqrt{L}} \tag{9}$$

$$L = (z_1 + z_2 + z)^2 - 4 z_1 z_2. \tag{10}$$

Substituting (7) into the Dyson equation (6), we obtain a relation between μ and ν:

$$3\mu = 1 + \nu \tag{11}$$

and an equation for $f(x)$:

$$\frac{1}{f(x)} = \iint_{-i\infty}^{+i\infty} \frac{dx_1 dx_2}{(2\pi i)^2} \Phi(x; x_1, x_2) f(x_1) f(x_2), \tag{12}$$

where $\Phi(x; x_1, x_2)$ is determined by (8)–(10) and depends on the parameter ν which so far is a free parameter.

Let us now see how one can apply to this equation dimensional considerations. We assume that $f(x)$ has a singularity at the point x_0. Then it can be seen from (12) that $f^{-1}(x)$ has a singularity at the point $x = 2^{2-\nu} x_0$, due to the singularity of Φ at $x = [x_1 \exp(1/(2-\nu)) + x_2 \exp(1/(2-\nu))] \exp(2-\nu)$ (cf. the Appendix). This new singularity of $f(x)$ in its turn generates singularities at the points $3 \exp(2-\nu) x_0$, etc.: $4 \exp(2-\nu) x_0$, ..., $n \exp(2-\nu) x_0$. However, we obtain no singularity in the right-hand side of (12) at the point x_0, for $\nu \neq 2$ (we show below that $\nu < 1$). We do not obtain a contradiction with Eq. (12) only in the case when the right-hand side has a zero at the point $x = x_0$, so that the singularity of $f(x)$ at $x = x_0$ is a pole. Since Eq. (12) contains only one dimensionless parameter ν, the condition that $f(x)$ have a pole at $x = x_0$:

$$\iint_{-i\infty}^{+i\infty} \frac{dx_1 dx_2}{(2\pi i)^2} \Phi(x_0; x_1, x_2) f(x_1) f(x_2) = 0, \tag{13}$$

determines ν.

In another way one may arrive at the equation for ν in the following manner. We consider the equation

$$\frac{1}{f_{\varepsilon}(x)} = \varepsilon^{1/2} + \iint_{-i\infty}^{+i\infty} \Phi(x; x_1, x_2) f_{\varepsilon}(x_1) f_{\varepsilon}(x_2) \frac{dx_1 dx_2}{(2\pi i)^2} \tag{14}$$

where $\varepsilon \to 0$ is a constant of the dimension of x, which imitates the free Green's function $\omega + \alpha' \mathbf{k}^2$. In this equation everything is unique. The function $f_{\varepsilon}(x)$ has a pole at some point x_0, depending on ν and ε,

$$x_0 = \varepsilon / \varphi(\nu).$$

If now we let simultaneously $\varepsilon \to 0$ and $\nu \to \nu_0$, so that $\varphi(\nu_0) = 0$, we obtain as a result of taking the limit the solution under consideration, and the condition of existence of the pole, $\varphi(\nu_0) = 0$, will be equivalent to (13).

One can now reduce (12) to a form analogous to the usual Dyson equation in the variable $y = x/x_0$:

$$x_0^{1/3} f^{-1}(x) = g^{-1}(y) = Z^{-1}(1-y) - \sigma_c(y), \tag{15}$$

$$\sigma_c(y) = -\iint_{-i\infty}^{+i\infty} \Phi_c(y; y_1, y_2) g(y_1) g(y_2) \frac{dy_1 dy_2}{(2\pi i)^2}, \tag{16}$$

$$\Phi_c(y; y_1, y_2) = \Phi(y; y_1, y_2) - \Phi(1; y_1, y_2) - (y-1) \Phi_y'(1; y_1, y_2), \tag{17}$$

$$Z^{-1} = -\int \Phi_y'(1; y_1, y_2) g(y_1) g(y_2) \frac{dy_1 dy_2}{(2\pi i)^2}. \tag{18}$$

These equations can be easily iterated with respect to the "free Green's function" $g_0(y) = Z/(1-y)$. Substituting into (13) and (18), we find ν and Z. We do not dwell on this here; a detailed analysis of (12) is given in the Appendix.

4. PERTURBATION THEORY FOR Γ

Most of the properties enumerated above are retained also in the exact problem, when the vertex part is not replaced by unity. We show below that in the exact problem $D(\omega, \mathbf{k})$ has the form (1), where $3\mu \neq 1 + \nu$ and Γ equals

$$\Gamma(\omega_1, \mathbf{k}_1; \omega_2, \mathbf{k}_2) = \omega^{-\gamma} F(y; y_1, y_2, \omega_1/\omega_2),$$
$$y_i = -\mathbf{k}_i^2/\omega_i^{\nu} x_0, \quad y = -\mathbf{k}^2/\omega^{\nu} x_0,$$
$$2\gamma = 1 + \nu - 3\mu, \quad \omega = \omega_1 + \omega_2, \quad \mathbf{k} = \mathbf{k}_1 + \mathbf{k}_2.$$

In order to understand how such a vertex part arises, we first start from the approximation $\Gamma = 1$ and assume that the corrections to it are small. Then we show how such a solution is confirmed in the general case.

We consider the right-hand side of (4) in the approximation used before:

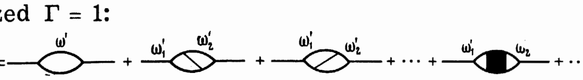
$$\Gamma^{(0)} = 1, \quad D^{(0)} = \omega^{-(\nu+1)/3} \rho^{-2/3} x_0^{-1/3} f(-\mathbf{k}^2/\omega^{\nu} x_0).$$

It is easy to see that all diagrams for Γ will diverge logarithmically, owing to the integration region $\omega \ll \omega' \ll \rho^2/\alpha'$, $\mathbf{k}'^2 \sim \omega'^{\nu} x_0$. As a result, in the next approximation Γ will be equal to

$$\Gamma^{(1)} = 1 + \gamma \ln(\rho^2/\alpha'\omega), \tag{19}$$

where $\Gamma^{(1)}$ is the larger of the variables ω_i and $\mathbf{k}_i^{2/\nu}$, and γ is a universal numerical coefficient, depending on ν but not on ρ and x_0. This approximation would have an exact meaning if $\gamma \ll 1$. In this case our purpose is to find a solution for D and Γ for $\gamma \ln(\rho^2/\alpha'\omega)$ on the order of unity.

We first determine the first correction in $\gamma \ln(\rho^2/\alpha'\omega)$ to D. For this purpose we consider the diagrams for Γ containing exact D, but unrenormalized $\Gamma = 1$:

Σ =  (20)

Let us consider as an example the second diagram. Here there are two regions leading to logarithmic divergence: 1) $\omega_1' \gg \omega_2' \sim \omega$ and 2) $\omega_2' \gg \omega_1' \sim \omega$. The logarithmic integral over each of these yields simply a correction to Γ in the form $\gamma_1 \ln(\rho^2/\alpha'\omega)$, corresponding to the first diagram in (20), since a logarithmic factor can be taken out of the remaining integral, which converges according to a power-law.

A similar reasoning can be carried through for every diagram in Σ. Any diagram can be written as the last diagram in (20). The logarithmic corrections reduce either to the left or the right vertex Γ, and as a result in the first approximation in $\gamma \ln \omega$ the self-energy part Σ becomes

$$\Sigma^{(1)} = \Sigma^{(0)}[1 + 2\gamma \ln(\rho^2/\alpha'\omega)] = \Sigma^{(0)}[\Gamma^{(1)}]^2.$$

This is equivalent to replacing ρ by $\rho\Gamma^{(1)}(\omega)$ in Eq. (6), where $\omega = \max[\omega, \mathbf{k}^{2/\nu}]$. Thus, in first approximation $D(\omega, \mathbf{k})$ equals

$$D^{(1)}(\omega, \mathbf{k}) = \frac{\rho^{-2/3}[\Gamma^{(1)}(\omega)]^{-2/3}}{x_0^{1/3}\, \omega^{(1+\nu)/3}} f(-\mathbf{k}^2/\omega^{\nu} x_0). \tag{21}$$

We now determine the next approximation $\Gamma^{(2)}$. In the equation (4) for Γ the diagram containing $2n + 1$ vertices $\Gamma^{(1)}$ involves $3n$ propagators $D^{(1)}$ and diverges logarithmically during the last integration with respect to ω' $(\omega \ll \omega' \ll \rho^2/\alpha')$. In all the integrations, except the logarithmic integration over ω', the quantities $\Gamma^{(1)}$ in the vertices and propagators do not participate in the integration and are equal to $\Gamma^{(1)}(\omega') = 1 + \gamma \ln(\rho^2/\alpha'\omega')$. As a result one obtains for $\Gamma^{(2)}$ an expression of the form

$$\Gamma^{(2)}(\omega) = 1 + \gamma \int_{\omega}^{\rho^2/\alpha'} \frac{d\omega'}{\omega'} \Gamma^{(1)}(\omega') = 1 + L + \frac{L^2}{2}, \tag{22}$$

where γ is the same universal coefficient as before.

Repeating this iteration process we obtain

$$\Gamma(\omega) = 1 + \gamma \int_{\omega}^{\rho^2/\alpha'} \Gamma(\omega') \frac{d\omega'}{\omega'}, \tag{23a}$$

$$D(\omega, \mathbf{k}) = \omega^{-(1+\nu)/3} (\Gamma^2 \rho^2 x_0)^{-1/3} f(-\mathbf{k}^2/\omega^{\nu} x_0). \tag{23b}$$

Finally,

$$\Gamma(\omega) = \exp[\gamma \ln(\rho^2/\alpha'\omega)] = (\rho^2/\alpha'\omega)^{\gamma}, \tag{24a}$$

$$D(\omega, \mathbf{k}) = \frac{\rho^{-2/3} x_0^{-1/3} (\alpha'/\rho^2)^{2\gamma/3}}{\omega^{(1+\nu-2\gamma)/3}} f(-\mathbf{k}^2/\omega^{\nu} x_0). \tag{24b}$$

5. EXACT EQUATIONS FOR Γ AND D

We can now formulate a recipe for finding solutions in a general form independently of γ. We shall look for both D and Γ in the form of homogeneous functions

$$D(\omega, \mathbf{k}) = (\rho_c^2 x_0)^{-1/3} \omega^{-\mu} f(-\mathbf{k}^2/x_0\omega^{\nu}), \tag{25}$$

$$\rho\Gamma(\omega_1, \mathbf{k}_1; \omega_2, \mathbf{k}_2) = \frac{\rho_c}{\omega^{\gamma}} F\left(\frac{-\mathbf{k}^2}{x_0\omega^{\nu}}, \frac{-\mathbf{k}_1^2}{x_0\omega_1^{\nu}}, \frac{-\mathbf{k}_2^2}{x_0\omega_2^{\nu}}, \frac{\omega_1}{\omega_2}\right), \tag{26}$$

where

$$2\gamma = 1 + \nu - 3\mu, \quad \omega = \omega_1 + \omega_2, \quad \mathbf{k} = \mathbf{k}_1 + \mathbf{k}_2,$$

ρ_c denotes the renormalized charge.

Substituting into any diagram for Σ or Γ the expression of D and Γ in the form of the homogeneous functions (25) and (26) it follows that the singular part of the diagram will again be a homogeneous function of a certain degree, such that, owing to the relation $3\mu = 1 + \nu - 2\gamma$ all diagrams for Γ and Σ_c will have the correct order $\Gamma \sim \omega^{-\gamma}$, $\Sigma_c \sim \omega^{\mu}$.

Let us prove this first for Γ. We consider the diagram $\Gamma_{(2n+1)}$ for Γ, containing $2n + 1$ vertices, i.e., proportional to ρ^{2n}. It is easy to see that this diagram contains $3n$ Green's functions and involves n integrations over $d\omega_i d^2\mathbf{k}_i'$. Therefore the contribution to $\Gamma_{(2n+1)}$ from the integration region $\omega_i' \sim \omega$, $\mathbf{k}_i'^2 \sim \omega^{\nu} x_0$ will be a homogeneous function of the form

$$\Gamma_{(2n+1),\,c} = \frac{\rho_c}{\rho} \omega^{-(2n+1)\gamma - 3n\mu + n\nu + n} F_{2n+1}\left(y, y_1, y_2, \frac{\omega_1}{\omega_2}\right),$$

where F_{2n+1} is independent of ρ_c or x_0. Since $2\gamma = 1 + \nu - 3\mu$,

$$\Gamma_{(2n+1),\,c} = \frac{\rho_c}{\rho} \omega^{-\gamma} F_{2n+1}\left(y, y_1, y_2, \frac{\omega_1}{\omega_2}\right), \tag{27}$$

i.e., the contribution from the region of small internal frequencies and momenta ω_i', $\mathbf{k}_i' \sim \omega$, $\mathbf{k}$ has the required structure (26).

We now note that $\gamma > 0$, i.e., $\Gamma \gg 1$. This is related to the abovementioned positiveness of the terms of the perturbative expansion of Γ for frequencies ω_1 and ω_2 situated to the right of the singularities (e.g., $\omega_1, \omega_2 > 0$, $\mathbf{k}_1 = \mathbf{k}_2 = 0$). The reason for this is that for ω_i situated to the right of the singularities the perturbation theory series converges and therefore Γ is larger than the first term, i.e., unity. We stress the fact that when we talk about positiveness of the perturbation theory series, we have in mind the unrenormalized expansion, since subtractions in Σ violate the positiveness of the terms of the series. Therefore, in order to verify the positiveness of the series one must substitute the original (unrenormalized) Green's function without subtractions into the pole trajectory, i.e., $D_0 = (\omega + \alpha' \mathbf{k}^2 + \varepsilon_0)^{-1}$. Here ε_0 is the position of the pole (unrenormalized), related to $\Sigma(0, 0)$ due to the fact that we consider $\varepsilon(0) = 0$, i.e., $D^{-1}(0, 0) = 0$, hence

$$\varepsilon_0 = \Sigma(0,0) = \int\limits_{-\infty}^{0} \operatorname{Im}\Sigma(\omega', 0)\frac{d\omega'}{\pi\omega'}.$$

Since $\operatorname{Im}\Sigma > 0$ (Hermitian Hamiltonian), we get $\omega_0 > 0$. In any diagram of perturbation theory for Γ, diagram which contains the propagators D_0, one can integrate over all frequencies, after which the diagram will contain only nonrelativistic denominators of the form

$$\omega_{1,2} + n\varepsilon_0 + \mathbf{k}_1'^2 + \mathbf{k}_2'^2 + \ldots + \mathbf{k}_n'^2,$$

corresponding to different intermediate states with n reggeons. Since $\varepsilon_0 > 0$, $\mathbf{k}_i'^2 > 0$ each diagram is positive, hence $\Gamma > 1$.

We finally remark that each diagram of unrenormalized perturbation theory for Γ has the order of magnitude $(\rho^2/\alpha'\varepsilon_0)^n \sim 1$, and not $(\rho^2/\alpha'\omega)^n$, so that no stronger restriction on γ than $\gamma > 0$ can be deduced from here.

Thus $\gamma > 0$. This means that the term equal to one in (4) can be neglected, so that we obtain for Γ a homogeneous equation of the form

$$\rho\Gamma(\omega_1, \mathbf{k}_1; \omega_2, \mathbf{k}_2) = \rho_c \frac{1}{\omega^\gamma}\sum_{n=1}^{\infty} F_{2n+1}\left(y, y_1, y_2, \frac{\omega_1}{\omega_2}\right).$$

For the function F in (26) therefore follows a universal homogeneous equation

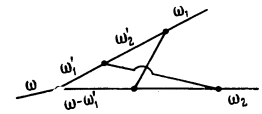
$$F(y, y_1, y_2, z) = \sum_{n=1}^{\infty} F_{2n+1}(y, y_1, y_2, z), \quad z = \omega_1/\omega_2. \tag{28}$$

In order to complete the proof we must verify that all integrals in the diagrams (4) for Γ converge both in frequencies and momenta $(\omega_i', \mathbf{k}_i') \sim (\omega, \mathbf{k})$. We first consider the first diagram in (4). Since it contains only one integration and has the order $\omega^{-\gamma} \gg 1$, it is easy to see that it converges for $\omega' \gg \omega$, like $\int d\omega^{-1}/(\omega)^{1+\gamma}$. This refers to $\omega \ll \omega' \ll \rho^2/\alpha'$ where Eqs. (25) and (26) are still applicable. For larger $\omega' \gtrsim \rho^2/\alpha'$ we have the order of magnitude

$$D \sim (\omega' + \alpha'\mathbf{k}'^2)^{-1}, \quad \Gamma \sim 1.$$

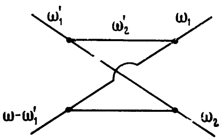
Thus the integral behaves like $\int d\omega'/(\omega')^{1+\gamma}$ for $\omega \ll \omega' \ll \rho^2/\alpha'$ and like $\int d\omega'/(\omega')^2$ for $\omega' \geq \rho^2/\alpha'$. The contribution of the transition region $\omega' \sim \rho^2/\alpha'$ is a constant, since the external ω_1 and ω_2 do not occur in the integral. This constant can be neglected compared to $\Gamma \sim \omega^{-\gamma} \gg 1$.

Similar phenomena will occur in more complicated diagrams in (4). Consider for example the diagram of the form



(29)

The contribution of the regions $\omega_1' \sim \omega_2' \sim \omega$ has, as we know, the order of magnitude $\omega^{-\gamma}$, the contribution of the transition region $\omega_1' \sim \omega_2' \sim \rho^2/\alpha'$ is an inessential constant. Of great interest is the contribution of the region $\omega_1' \sim \rho^2/\alpha'$, $\omega_2' \sim \omega$ or $\omega_1' \sim \omega$, $\omega_2' \sim \rho^2/\alpha'$. The contribution from the region $\omega_2' \sim \rho^2/\alpha'$, $\omega_1' \sim \omega$ is equivalent to the fact that the four-point diagram of the form



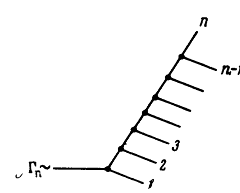
(30)

which occurs in (29) is replaced by some constant λ. Since the contribution of the region $\omega_2' \sim \omega$ to the diagram (30) has the order of magnitude

$$\omega^{-4\gamma-4\mu+1+\nu} \sim \omega^{-2\gamma-\mu} \gg 1,$$

this constant λ may be neglected in comparison with the contribution of the region $\omega_2' \approx \omega$. Similarly one may discard the contribution to the diagram (29) of the region $\omega_1' \sim \rho^2/\alpha'$, $\omega_2 \sim \omega$. Now it is easy to see that in more complicated diagrams the contribution from the regions where part of the frequencies are $\omega_i' \sim \omega$ and the others are $\omega_i'' \sim \rho^2/\alpha'$, is equivalent to replacing some n-point diagram Γ_n inside the diagram by a constant having the meaning of an unrenormalized coupling constant.

It is however easy to see that any n-point diagram is large: $\Gamma \gg 1$. Indeed, since all diagrams for Γ_n are of the same order of magnitude, Γ_n can be estimated from the simplest diagram of the form



(31)

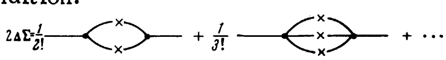
Hence $\Gamma_n \sim \rho\Gamma(\rho\Gamma D)^{n-2} \gg 1$. A more exact expression for Γ_n has the form

$$\Gamma_n = \frac{\rho_c^{(n+1)/3}}{x_0^{(n-2)/3}}\,\omega^{\mu-(n-1)(1+\nu-\mu)/2}\,\Phi_n\left(-\frac{\mathbf{k}_i\mathbf{k}_j}{x_0\omega_j^{\nu}}\right) \tag{32}$$

(here use has been made of the relation $2\gamma = 1 + \nu - 3\mu$).

Thus we have shown that all integrals in the diagrams for Γ_n converge for $\omega_i' \sim \omega$. At the same time it was clear that the same is true for an arbitrary n-point diagram with n + 1 external lines, and that small corrections from the regions $\omega_i \sim \rho^2/\alpha'$ are equivalent to including nonrenormalized vertices Γ_n.

We now consider the equation for D. We must find $\Sigma_c(\omega, \mathbf{k}) = \Sigma(\omega, \mathbf{k}) - \Sigma(0, 0)$. For this it does not suffice to substitute into (3) D and Γ from (25) and (26), since in the integrand the region $\omega' \sim \rho^2/\alpha'$ are essential, where (25) and (26) are invalid. (We have seen on the example of the logarithmic approximation, that, for instance, the contribution to Σ of the second diagram in (20) is incorrectly estimated by replacing one vertex by its asymptotic value $\gamma_1 \ln(\rho^2/\alpha'\omega)$.) In order to avoid this difficulty we find the discontinuity $\Delta\Sigma$ in the variable ω, which discontinuity satisfies the reggeon unitarity condition:[1-3]



(33)

Here a line with a cross denotes the pole part of the D-function

$$\Delta D^{(pol)}(\omega, \mathbf{k}) = 2\pi\rho^{-2/3} x_0^{-1/3} \omega^{-\mu} \delta(\mathbf{k}^2/x_0\omega^\nu + 1).$$

An integration with respect to internal frequencies and momenta is understood, taking into account the conservation laws $\omega = \Sigma\omega_i$, $\mathbf{k} = \Sigma\mathbf{k}_i$.

The vertices describing transitions of one reggeon into n reggeons Γ_n which enter into (33) as well as $\Gamma_2 = \rho\Gamma$ are homogeneous functions of their arguments and have the form (32). The order of magnitude of the n-th term in (33) for $\mathbf{k}^2 \sim \omega^\nu x_0$ will be

$$(\Delta\Sigma)_n \sim \Gamma_n^2 \omega^{n-1} \omega^{(n-1)\nu} \omega^{-\mu n} \rho_c^{-2n/3} x_0^{-n/3} \sim \omega^\mu \rho_c^{2/3} x_0^{1/3}.$$

Consequently the right-hand side of (33) can be written as

$$\Delta\Sigma(\omega, \mathbf{k}) = \sum_{n=2}^{\infty} (x_0\rho_c^2)^{1/3} \Delta[\omega^\mu \sigma_n(-\mathbf{k}^2/\omega^\nu x_0)].$$

The Dyson equation (5) now reduces to an equation for f(y) in (25):

$$f^{-1}(y) = -\sum_{n=2}^{\infty} \sigma_n(y). \tag{34}$$

In the same manner as in (6), in Eq. (34) the absence of a dimensional parameter implies that the condition for the existence of a pole

$$\sum_{n=2}^{\infty} \sigma_n(1) = 0 \tag{35}$$

determines the parameter ν for given μ. From the condition of solubility of the homogeneous equation (28) one determines the remaining parameter μ, since $F(y_i, z)$ and f(y) depends only on μ and does not involve any other parameters.

Thus the Green's function and vertex part have the forms (25) and (26), with f(y) and $F(y, y_1, y_2, z)$ universal functions, and μ, ν, γ—universal numerical parameters $\rho\Gamma$ and D depend on the dimensional parameters ρ_c and x_0, which are determined by joining the solution with that in the region ω, $\alpha'\mathbf{k}^2 \sim \rho^2/\alpha'$.

We now estimate the order of magnitudes of the parameters ρ_c and x_0 as functions of ρ and α'. Consider $\mathbf{k}^{2/\nu} \ll \omega \sim \rho^2/\alpha'$, i.e., at the boundary of applicability of perturbation theory. In this region we must have

$$D^{-1}(\omega, \mathbf{k}) \sim \omega, \quad \Gamma(\omega_i, \mathbf{k}_i) \sim 1,$$

i.e.,

$$(x_0\rho_c^2)^{1/3}\omega^\mu \sim \omega, \quad \rho_c\omega^{-\gamma} \sim \rho, \quad \omega \sim \rho^2/\alpha';$$

whence

$$x_0\rho_c^2 \sim (\rho^2/\alpha')^{3(1-\mu)}, \quad \rho_c^2 \sim \rho^2(\rho^2/\alpha')^{2\gamma},$$

or

$$x_0 \sim \rho^{-2}(\rho^2/\alpha')^{3-3\mu-2\gamma} = \rho^{-2}(\rho^2/\alpha')^{2-\nu}.$$

Finally

$$x_0 \sim (\rho^2/\alpha')^{1-\nu}/\alpha', \tag{36}$$

$$\rho_c \sim \rho(\rho^2/\alpha')^{\gamma}. \tag{37}$$

Let us now find out what follows from general considerations for the parameters μ, ν, γ and the function f(y). We shall show that

$$0 < \mu < \nu < 1, \quad 2\gamma = \nu + 1 - 3\mu > 0,$$

and that f(y) has the qualitative form represented by the solid line in Fig. 1.

First of all, it is clear that $\nu > 0$, since the trajectory of the pole (the spectrum) $\varepsilon(\mathbf{k}) = -(-\mathbf{k}^2/x_0)^{1/\nu}$ has no "gap" at $\mathbf{k} = 0$: $\varepsilon(0) = 0$. In addition, in the Dyson equation (2) $D^{-1}(0,0) = 0$, hence $\mu > 0$. From the fact that $\Sigma_c \gg \omega + \alpha'\mathbf{k}^2$ follow the inequalities

$$\mu < 1, \quad \mu < \nu,$$

and from the fact that $\Gamma \gg 1$ one can derive

$$2\gamma = 1 + \nu - 3\mu > 0.$$

The function $D(\omega, \mathbf{k})$ corresponds to a Hermitian Hamiltonian, implying the inequality

$$\nu < 1.$$

This is related to the fact that for $\mathbf{k}^2 > 0$ the Green's function of the Hermitian problem should not have complex singularities in the ω-plane. According to (25) and the definition of x_0 the Green's function $D(\omega, \mathbf{k})$ has a pole for

$$\omega = (-\mathbf{k}^2/x_0)^{1/\nu} \equiv -\varepsilon(\mathbf{k});$$

For $\mathbf{k}^2 < 0$ this pole is situated on the physical sheet for $\omega > 0$ (cf. Fig. 2). As the phase of $-\mathbf{k}^2$ varies from 0 to π the trajectory of the pole $\varepsilon(\mathbf{k})$ acquires a phase π/ν. The pole and all the branch points at the points $\omega_n = -n\varepsilon(\mathbf{k}/n)$ are on the unphysical sheet if $\nu < 1$. Then for $\mathbf{k}^2 > 0$ the function $D(\omega, \mathbf{k})$ is analytic in the ω-plane with a cut from $\omega = 0$ to $\omega = -\infty$ (Fig. 3).

The second general assertion following from the

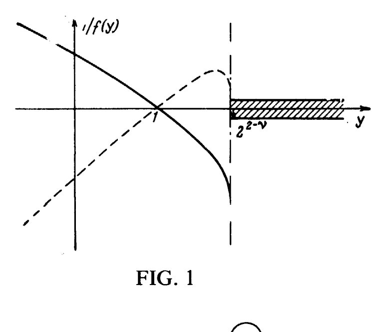


FIG. 1

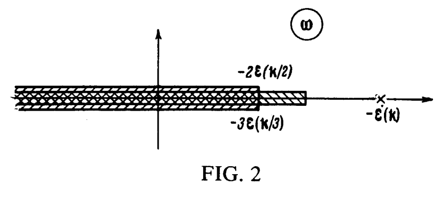


FIG. 2

Hermiticity of the Hamiltonian is the following: the discontinuity

$$\Delta_\omega D(\omega, \mathbf{k}) = [D(\omega - i0, \mathbf{k}) - D(\omega + i0, \mathbf{k})] / 2i$$

is positive for all singularities. For $\mathbf{k}^2 < 0$ this reduces to the positivity of the discontinuity on the cut from $\omega = 0$ to $\omega = -\infty$. For $\mathbf{k}^2 < 0$, i.e., $t = -\mathbf{k}^2 > 0$, the singularities of $D(\omega, \mathbf{k})$ are situated at the points $\omega_n = -n\varepsilon(\mathbf{k}/n)$, or, what amounts to the same,

$$y_n = -\mathbf{k}^2 / x_0\omega_n^\nu = n^{2-\nu}, \quad n = 1, 2, \ldots \tag{38}$$

This is clear from the condition for reggeon unitarity (33).

The condition that the discontinuities ΔD be positive can be formulated in terms of the variable $t = -\mathbf{k}^2$ or $y = -\mathbf{k}^2/x_0\omega^\nu$. In the complex y plane the singularities of $D(\omega, \mathbf{k})$ are situated at the points (38). The point $y = 1$ is a pole and the points $y = y_n = n^{2-\nu}$ are threshold branch points; the cut goes from $y = y_2 - 2^{2-\nu}$ to $y = +\infty$ (Fig. 4). The discontinuity

$$\Delta_{\mathbf{k}} D(\omega, \mathbf{k}) \equiv [D(\omega, \mathbf{k}^2 - i0) - D(\omega, \mathbf{k}^2 + i0)] / 2i = \omega^{-\mu}\Delta f(y)$$

across this cut is positive. This means that $f(y)$ in (1) has a positive jump, i.e., all derivatives $f^{(n)}(y)$ are positive for $y < 1$:

$$f^{(n)}(y) = n! \int_1^\infty \frac{\Delta f(y')\,dy'}{\pi(y' - y)^{n+1}} > 0 \quad \text{for } y < 1. \tag{39}$$

The graph of $f^{-1}(y)$ has the qualitative form represented in Fig. 1 by the solid line.

We now consider the singularity of $D(\omega, \mathbf{k})$ for $\omega = 0$, generated from an accumulation of branch points at the points

$$D^{-1}(\omega, \mathbf{k}) \to C\mathbf{k}^{2\mu/\nu} - \Sigma_1(\omega, \mathbf{k}), \quad \omega \to 0.$$

($\omega_n \to 0$ for $n \to \infty$). When $D(\omega, \mathbf{k})$ is written in the form form (1) it would seem that the singularity at $\omega = 0$ is due to the factors ω^μ and ω^ν and has a power-law character. We shall see however that this is not so: $f(x)$ in (1) behaves for $\omega \to 0$, i.e., $x \to \infty$ like $x^{-\mu/\nu}$, so that the factors ω^μ and ω^ν disappear, and the actual behavior for $\omega \to 0$ is the following:

$$\omega_n = -n\varepsilon(\mathbf{k}/n) = n^{1-2/\nu}(-\mathbf{k}^2/x_0)^{1/\nu} \tag{40}$$

Here $\Sigma_1(\omega, \mathbf{k})$ has an essential singularity for $\omega = 0$ and $\omega \to -0$

$$\Delta_\omega \Sigma_1(\omega, \mathbf{k}) \sim \exp(-\beta\mathbf{k}^{2/\nu}/\omega).$$

The assertion (40), namely that $D^{-1}(\omega, \mathbf{k}) \to C\mathbf{k}^{2\mu/\nu}$ for $\omega \to 0$ follows from the fact that for $\omega \to 0$ each term of the series for Σ and Γ converges to a finite limit, since in all integrals one can neglect ω compared to the internal frequencies $\omega_i^i \sim \mathbf{k}^{2/\nu}$.

The character of the singularity of Σ_1 can be clarified using the example considered above, when $\Gamma = 1$ (Eqs. (6)–(12)). We shall consider that $x_0\omega^\nu \ll \mathbf{k}^2$, and write $D^{-1}(\omega, \mathbf{k})$ in the form

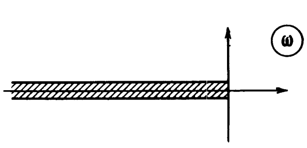


FIG. 3

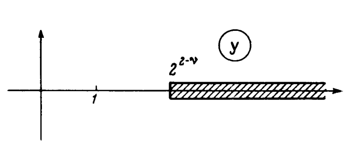


FIG. 4

$$D^{-1}(\omega, \mathbf{k}) = \Sigma(0, 0) - \Sigma(0, \mathbf{k}) - \Sigma_1(\omega, \mathbf{k}),$$

where

$$\Sigma_1(\omega, \mathbf{k}) = \Sigma(\omega, \mathbf{k}) - \Sigma(0, \mathbf{k})$$

or

$$D^{-1}(\omega, \mathbf{k}) = D^{-1}(0, \mathbf{k}) - \Sigma_1(\omega, \mathbf{k}).$$

The existence of the limit

$$D^{-1}(0, \mathbf{k}) = \frac{\rho^2}{2} \int \frac{d\omega'\,d^2\mathbf{k}'}{(2\pi)^3 i} D(\omega', \mathbf{k}')[D(-\omega', -\mathbf{k}') - D(-\omega', \mathbf{k} - \mathbf{k}')] = C(x_0\rho^2)^{1/3}(\mathbf{k}^2/x_0)^{\mu/\nu},$$

where C is a number, can be seen in the present case from the explicit form of the kernel $\Phi(x; x_1, x_2)$ of Eq. (12) for $f(x)$. For $x_0\omega^\nu \ll \mathbf{k}^2$ we have $\Sigma_1(\omega, \mathbf{k}) \ll D^{-1}(0, \mathbf{k})$. Then

$$D(\omega, \mathbf{k}) \to D(0, \mathbf{k}) + D^2(0, \mathbf{k})\Sigma_1(\omega, \mathbf{k}), \quad \omega \to 0 \tag{41}$$

We transform $D(\omega, \mathbf{k})$ to $D(\xi, \mathbf{k})$ by means of a Laplace transformation

$$D(\xi, \mathbf{k}) = \int_{-i\infty}^{+i\infty} \frac{d\omega}{2\pi i} e^{\omega\xi} D(\omega, \mathbf{k}) = \int_{-\infty}^{0} \frac{d\omega}{\pi} e^{\omega\xi} \Delta_\omega D(\omega, \mathbf{k}). \tag{42}$$

According to (41)

$$\Delta_\omega D(\omega, \mathbf{k}) = D^2(0, \mathbf{k})\Delta_\omega\Sigma(\omega, \mathbf{k}). \tag{42a}$$

In our example $\Sigma(\omega, \mathbf{k})$ has the form (6) and the discontinuity of this expression equals

$$\Delta_\omega\Sigma(\omega, \mathbf{k}) = \rho^2 \int_{-\infty}^{0} \frac{d\omega'}{\pi} \int \frac{d^2\mathbf{k}'}{(2\pi)^2} \Delta_\omega D(\omega', \mathbf{k}')\,\Delta_\omega D(\omega - \omega', \mathbf{k} - \mathbf{k}'). \tag{43}$$

Substituting (43) into (42a) and then into (42) we obtain the following equation for $D(\xi, \mathbf{k})$:

$$D(\xi, \mathbf{k}) = D^2(0, \mathbf{k})\Sigma(\xi, \mathbf{k}) = D^2(0, \mathbf{k})\rho^2 \int \frac{d^2\mathbf{k}'}{(2\pi)^2} D(\xi, \mathbf{k}')D(\xi, \mathbf{k} - \mathbf{k}'). \tag{44}$$

To the asymptotic behavior of $D(\omega, \mathbf{k})$ for $x_0\omega^\nu \ll \mathbf{k}^2$ corresponds the asymptotic behavior of $D(\xi, \mathbf{k})$ for $\xi^\nu\mathbf{k}^2 \gg x_0$. The equation (44) is valid for $\xi^\nu\mathbf{k}^2 \gg x_0$, $\xi \gg 1$. We note now that since $D(\omega, \mathbf{k})$ is a homogeneous function of the form (1), it follows that $D(\xi, \mathbf{k})$ is a homogeneous function of the form

$$D(\xi, \mathbf{k}) = \xi^{\mu-1}B(\mathbf{k}^2\xi^\nu)\rho^{-2/3}. \tag{45}$$

It is shown in the Appendix that the asymptotic solution of Eq. (44) for $\mathbf{k}^2\xi^\nu \gg x_0$ is

$$D(\xi, \mathbf{k}) \to \rho^{-2/3}N\exp(-f|\mathbf{k}|\xi^{\nu/2})(\xi^\nu\mathbf{k}^2)^{1/4}\mathbf{k}^{2(1-\mu)/\nu}, \tag{46}$$

where the numbers f and N are related via Eq. (A.11).

In order to find $D(\xi, \mathbf{k})$ in the general case, it is in principle necessary to utilize the unitarity condition (33 for $\Delta\Sigma$. Our approximation (43) is equivalent to retaining in Γ_n in (33) only diagrams of the form

 (47)

where the heavy line with a cross denotes the discontinuity of the exact Green's function $\Delta_\omega D(\omega, \mathbf{k})$. It is to be expected that the asymptotic behavior of type (46) remains valid in the general case, but owing to lack of space we do not discuss this question.

Thus we have analyzed the structure of the Green's functions and vertex parts, corresponding to a Hermitian three-reggeon interaction. The Green's function has the form (25) and the vertex functions have the forms (26), (32), with f, F, and Φ_n universal functions, and μ and ν universal numerical parameters satisfying

$$0 < \mu < \nu < 1, \qquad 2\gamma = 1 + \nu - 3\mu > 0; \qquad (48)$$

f(y) has singularities for $y = n^{2-\nu}$, n = 1, 2, ... The singularity at y = 1 is a simple pole and the remaining singularities are branch points, such that the discontinuity $\Delta f(y)$ across the cut from $y = 2^{2-\nu}$ to $y = +\infty$ is positive. For $y \to -\infty$ $f(y) \to C(-y)^{-\mu/\nu}$.

6. THE UNIQUENESS PROBLEM FOR THE SOLUTION OF THE STRONG COUPLING PROBLEM

Equations (28) and (34), which determine f(y) and $F(y, y_1, y_2, z)$ may, a priori, do this either in a unique manner, or not. The uniqueness problem is interesting several reasons. Let us imagine that the interaction Hamiltonian contains in addition to the $\rho\psi^3$ interaction also a term $\lambda\psi^4$. Formally, from the viewpoint of the considerations given above in Sec. 4, nothing will be changed, since even without the additional interaction there was a contribution from the integration region over internal frequencies ω_i and momenta $\mathbf{k}'_i$ which are much larger than the external ones, which leads to an effective $\lambda\psi^4$ interaction (cf. Sec. 4) without modifying the result.

In reality, only the condition of smooth joining of solutions is modified in a region where the solutions (25) and (26) are not valid. Then, if the solution of the strong coupling problem is unique, the modification of the smooth-joining conditions leads only to a change of the solutions in the transition region. If the solution is not unique, e.g., depending on arbitrary dimensionless parameters, then these conditions will determine the dimensionless parameters, the solution being nonuniversal, and dependent on the coupling constants ρ and λ. The problem of uniqueness of solutions (25) and (26) will also be very important in discussing the possibility of a solution of the form (25) and (26) for a non-Hermitian Hamiltonian. We do not really possess an answer to the uniqueness problem. In this connection we can only make some general considerations.

The simplest reason for nonuniqueness of solutions can, obviously, consist in the fact that the solutions are represented by series, and the series may diverge. But his reason is to a certain degree trivial, since the divergence of the series, signifies the appearance of new singularities, in addition to the singularities at $y_n = n^{2-\nu}$. If such a new singularity is a pole it will in turn generate the corresponding threshold branch cuts, and we arrive at a problem in which there are two vacuum reggeons instead of one. It is obvious that there is no way to exclude such a nonuniqueness. If the singularity produced by the divergence of the series is not a pole, then, as shown above, this singularity does not agree with the equations, and therefore cannot occur. Singularities (analogous to those which occur in the problem of falling into a center) for which the position in the ω plane does not depend on $\mathbf{k}$ have been excluded by us from the beginning, before going over to functions of only $x = -\mathbf{k}^2/\omega^\nu$.

The fact that the number of poles in Eqs. (28) and (34) is indeed not fixed means that these equations have a dispersion character, i.e., that they reduce to the construction of functions which have given singularities and satisfy the unitarity condition involving all channels. Such dispersion equations, as is well known, admit a non-uniquenesses of the CDD type,[5] which reduce to, say, the fact that $D^{-1}(\omega, \mathbf{k})$ can contain an arbitrary number of poles of the type $C/(\omega + \beta\mathbf{k}^2)$. In our case, however, one may not add poles of the type $C/(y_1 - y)$ to the function $f^{-1}(y)$, since this leads to a power singularity in $D(\omega, \mathbf{k})$ at $\omega = 0$, since the asymptotic behavior of f(y) for large y is changed, and the factor ω^μ does not cancel out in (25) as $\omega \to 0$. This singularity is not related to accumulations of branch points, and therefore contradicts the reggeon unitarity condition. Since the singular part of $D^{-1}(\omega, \mathbf{k})$ for $\omega \to 0$ is exponentially small (cf. (40)), any finite number of such poles does not modify the situation.

These considerations speak in favor of the fact that for a given structure of the singularities the equations (28) and (34) may not have ambiguities, i.e., they do not involve any nonuniqueness containing dimensionless constants, to be determined from boundary conditions. In terms of the interaction Hamiltonian the nonuniqueness which is related to the fact that the singularity structure is unknown, may lead to a discontinuous (finite-jump type) dependence of the solution on the coupling constants. For certain values of the coupling constants the solution with one pole will be "joined" with perturbation theory, for others—one with several poles. In the following section we show on the example of the $\lambda\psi^4$ interaction how the continuity and smoothness conditions lead to the necessity of new poles.

7. STRONG COUPLING FOR THE $\lambda\psi^4$ INTERACTION AND THE SMOOTH JOINING OF SOLUTIONS WITH PERTURBATION THEORY

We now investigate how a solution corresponding to strong coupling appears if one considers high frequencies and momenta, where perturbation theory is applicable, and then approaches the region of small frequencies and momenta. We shall consider a Hermitian four-reggeon interaction, assuming that there is no three-reggeon interaction. We denote the unrenormalized vertex describing the transition of one reggeon into three by λ_3, and the unrenormalized vertex for the transition of one reggeon into two reggeons by λ_2. We stress that $\lambda_2 \neq \lambda_3$. Accordingly, we shall denote the exact vertices by λ_3 and λ_2, respectively.

In first approximation of perturbation theory one may retain in the reggeon unitarity condition for $\Delta\Sigma$ only the term involving three reggeons in the intermediate state, and the corresponding contribution to $D^{-1}(\omega, \mathbf{k})$ is

$$D^{-1}(\omega, \mathbf{k}) = \omega + \alpha'\mathbf{k}^2 - \lambda_3^2\left(\omega + \frac{\alpha'\mathbf{k}^2}{3}\right)\ln\left(\omega + \frac{\alpha'\mathbf{k}^2}{3}\right) \qquad (49)$$

(we have lumped with λ_3 inessential numerical factors). The criterion for applicability of perturbation theory for D reduces thus to

$$\lambda_3^2 \ln \omega \ll 1. \qquad (50)$$

Similarly, considering diagrams of the type (53) (cf. infra), one can find the corrections Λ_3 and Λ_2 to the vertices in the form

$$\Lambda_3 \sim \lambda_3 - 3\lambda_3\lambda_2 \ln\omega, \qquad\qquad (51)$$
$$\Lambda_2 \sim \lambda_2 - \lambda_2^2 \ln\omega - 4\lambda_3^2 \ln\omega.$$

These corrections will be small, if

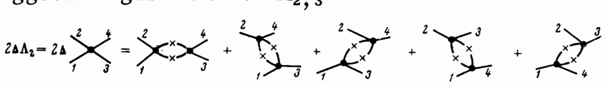
$$\lambda_2 \ln\omega \ll 1, \quad \lambda_3^2 \ln\omega \ll \lambda_2. \qquad (52)$$

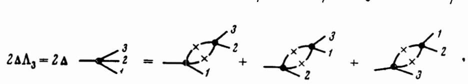
We shall assume that λ_3 and λ_2 have been selected so small that the criteria (50) and (52) are satisfied for $\omega \ll 1$, where the reggeon diagram technique is no longer applicable. Decreasing ω, **k** we enter a region where $\lambda_3^2 \ln\omega \sim \lambda_2$ and $\lambda_2 \ln\omega \sim 1$. In this region the test (50) is still valid and $D = D_0$, but Λ_2, Λ_3 already vary substantially. Here for Λ_3 and Λ_2 one may retain only diagrams with two-reggeon intermediate states, which contain higher powers of $\ln\omega$.

The vertices Λ_2 and Λ_3 depend in this approximation on $l = \ln(1/\omega_{max})$ where ω_{max} is the largest among the variables entering into the vertex. The derivatives $\Lambda_3'(l)$ and $\Lambda_2'(l)$ are related to the discontinuities $\Delta\Lambda_3$ and $\Delta\Lambda_2$:

$$\Delta\Lambda_{2,3} = \Lambda_{2,3}'(l)\Delta \ln\omega^{-1} = \pi\Lambda_{2,3}'(l),$$

where $\Delta\Lambda_{2,3}$ is the sum of discontinuities over all two-reggeon singularities of $\Lambda_{2,3}$:


(53a)


(53b)

The analytic expressions for Eqs. (53) are

$$\Lambda_2'(l) = \pi^{-1}\Delta\Lambda_2 = \Lambda_2^2(l) + 4\Lambda_3^2(l), \qquad (54a)$$
$$\Lambda_3'(l) = \pi^{-1}\Delta\Lambda_3 = 3\Lambda_3(l)\Lambda_2(l). \qquad (54b)$$

The corresponding initial conditions are

$$\Lambda_2(0) = \lambda_2, \quad \Lambda_3(0) = \lambda_3. \qquad (55)$$

If $\lambda_3 \ll \lambda_2$ the solution of these equations is trivial:

$$\Lambda_2(l) = \frac{\lambda_2}{1-\lambda_2 l}, \quad \Lambda_3(l) = \frac{\lambda_3}{(1-\lambda_2 l)^3}. \qquad (56)$$

In the general case the solution has the form

$$\Lambda_2 = \frac{\lambda_2}{|\lambda_2|}(2\Lambda_3^2 + C\Lambda_3^{2/3})^{1/2}, \quad C = \frac{\lambda_2^2 - 2\lambda_3^2}{\lambda_3^{2/3}} \qquad (57a)$$

$$l = \int_{\lambda_3}^{\Lambda_3} \frac{d\Lambda_3}{3\Lambda_2\Lambda_3} = \int_{1/\Lambda_3}^{1/\lambda_3} \frac{dx}{3(2+Cx^{4/3})^{1/2}} \frac{\lambda_2}{|\lambda_2|}. \qquad (57b)$$

(The integral in (57b) reduces to an elliptical integral of the second kind.)

One can find the qualitative dependence of Λ_2 and Λ_3 on l in a convenient manner, utilizing the analogy between (57b) and the a known formula of mechanics, which relates the "coordinate" $x = \Lambda_3^{-1}$ to the "time" $t = -3l\lambda_2/|\lambda_2|$ for the motion of a particle in the one-dimensional potential $U(x) = -Cx^{4/3}$. If $\lambda_2 < -\sqrt{2}|\lambda_3|$ Λ_2 and Λ_3 tend monotonically to zero as l increases, so that with the decrease of ω the interaction weakens and the solution (57) remains valid for all $l = \ln(1/\omega)$. In particular, for $l \to +\infty$ $(\omega \to 0)$

$$\Lambda_2 \to \frac{1}{\ln\omega}, \quad \Lambda_3 \to \frac{\lambda_3}{(\lambda_2^2 - 2\lambda_3^2)^{3/2}} \frac{1}{(\ln\omega)^3}. \qquad (58)$$

Such a solution with $\lambda_2 < -\sqrt{2}|\lambda_3|$ can indeed be realized in nonrelativistic two-dimensional problems and corresponds to particle repulsion. However, as the authors have shown in [4], this solution does not present interest for the Pomeranchuk pole, since (58) contradicts the unitarity condition in the s-channel for the transition amplitude of two particles into two reggeons. Thus, if the three-reggeon interaction is indeed absent, then either the four-reggeon interaction is also absent, i.e., $\lambda_2 = \lambda_3 = 0$ (cf. [4]), or the four-reggeon interaction corresponds to attraction, i.e., $\lambda_2 > -\sqrt{2}|\lambda_3|$.

For the attractive case, the solution for small ω has nothing in common with the repulsive case considered above. Roughly speaking, the reason for the difference consists in the fact that for the two-dimensional system which we consider, an arbitrarily weak attraction leads, as is well known, to the formation of a bound state. In Eq. (57) this manifests itself through the fact that for $\lambda_2 > -\sqrt{2}|\lambda_3|$ as $l = \ln(1/\omega)$ increases, the vertices $\Lambda_{2,3}$ start growing in absolute value, and formally tend to infinity for $l = \ln(1/\omega) = l_0$, where

$$l_0 = \int_{\lambda_3}^{\infty} \frac{d\Lambda_3}{3\Lambda_3\Lambda_2} > 0.$$

The qualitative behavior of $\Lambda_2(\omega)$ is shown by the dotted line in Fig. 5.

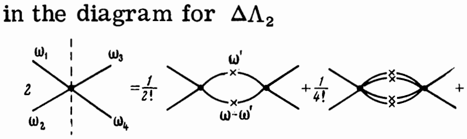
The fact that the position of the pole in this approximation turned out to correspond to a meaningless negative energy $\varepsilon = -\omega = -e^{-l_0} < 0$ means that this solution is incorrect for $\omega \lesssim e^{-l_0}$ and that as ω decreases and Λ_2 and Λ_3 increase all diagrams for Λ_2, Λ_3 and D become essential, so that if ω is decreased further, in particular for $\omega < e^{-l_0}$, strong coupling sets in, as considered in Secs. 1–4. The position of the pole corresponding to the bound state must move in such a way that for $\mathbf{k}^2 > 0$ it is situated in the left ω half-plane $(\varepsilon > 0)$. Here, as above, the solution for D will have the form (1) and Λ_2 and ΛW will have the form (32) with $n = 3$.

We shall not give a detailed proof, which is analogous to that in Sec. 4, but consider only the equation for Λ_2 in order to clarify the bound state problem.

For simplicity we consider the vertex Λ_2 at $\omega_1 = \omega_2 = \omega_3 = \omega_4 \equiv \omega/2$, $\mathbf{k}_1 = \mathbf{k}_2 = \mathbf{k}_3 = \mathbf{k}_4 \equiv \mathbf{k}/2$ and $\omega \to -2\varepsilon(\mathbf{k}/2)$. We consider the discontinuity $\Delta_{\mathbf{k}}\Lambda_2$ in $\mathbf{k}^2$ at the two-reggeon threshold for

$$\omega \to -2\varepsilon(\mathrm{k}/2) = 2^{1-2/\nu}(-\mathrm{k}^2/x_0)^{1/\nu}, \quad \mathrm{k}^2 < 0.$$

Then in the diagram for $\Delta\Lambda_2$


(59)

The values $\omega' \to \omega/2$, $\mathbf{k}' \to \mathbf{k}/2$ are essential, and

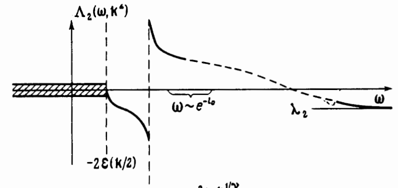


FIG. 5

$\Lambda_2^2(\omega, \mathbf{k})$ may be taken outside the integral. The remaining integral reduces to the function $\Phi(x; x_0, x_0)$ in (7)–(10) with $x = -\mathbf{k}^2/\omega^\nu > 0$, namely

$$\Delta_k \Lambda_2(\omega, \mathbf{k}) = \Lambda_2^2(\omega, \mathbf{k})\, \omega^{1+\nu-2\mu} \Delta_x \Phi(x; x_0, x_0). \qquad (59a)$$

Here $\Delta_x\Phi(x; x_0, x_0)$ is the discontinuity of $\Phi(x; x_0, x_0)$ across the cut from $x = 2^{2-\nu}x_0$ to $x = +\infty$.

It is easy to see that $\Phi(x; x_0, x_0) \to A \ln(2^{2-\nu}x_0 - x)$, where $A > 0$, $x \to 2^{2-\nu}x_0$. Then it follows from (59a) that

$$\Lambda_2(\omega, \mathbf{k}) \to \omega^{2\mu-1-\nu} \frac{B}{\ln(2^{2-\nu} + \mathbf{k}^2/\omega^\nu x_0)} \to -0; \qquad B > 0, \quad \omega^\nu x_0 \to -2^{\nu-2}\mathbf{k}^2 > 0. \qquad (60)$$

We consider $\Lambda_2(\omega, \mathbf{k})$ as a function of ω. In our case, when the incoming and outgoing reggeons have identical frequencies and momenta $(\omega/2, \mathbf{k}/2)$ all terms in the unitarity condition for $\Delta\Lambda_2$ have the same sign, and consequently the derivative $\partial\Lambda_2/\partial\omega$ is negative for $\omega \geq -2\varepsilon(\mathbf{k}/2)$. The vertex $\Lambda_2(\omega, \mathbf{k})$ has the form illustrated in Fig. 5 ($\mathbf{k}^2 < 0$).

Since $\partial\Lambda_2/\partial\omega < 0$ and $\Lambda_2 \to -0$ for $\omega \to -2\varepsilon(\mathbf{k}/2)$, the solution for small ω can be smoothly joined in the region $\omega \sim e^{-l_0}$ with the solution (57) (dotted line) only with the help of a pole of $\Lambda_2(\omega, \mathbf{k})$. Since Λ_2 has the form

$$\Lambda_2(\omega, \mathbf{k}) = \omega^{2\mu-1-\nu}\varphi(-\mathbf{k}^2/\omega^\nu x_0),$$

the position of that pole will be at

$$-\mathbf{k}^2/\omega^\nu = Cx_c \equiv x_0'. \qquad (61)$$

If $x_0' = x_0$, i.e., the bound state is the same reggeon, then the problem is equivalent to the three-reggeon interaction considered in Secs. 1–4. If $x_0' \neq x_0$, this is a different problem, involving two kinds of reggeons, interacting with one another through three-reggeon vertices. Such a problem can be treated along the lines of Secs. 1–4.

8. STRONG COUPLING IN A PROBLEM WITH ANTI-HERMITIAN INTERACTION

We now consider an anti-Hermitian three-reggeon interaction, corresponding to the problem[1] of the Pomeranchuk trajectory, i.e., we shall consider that in Secs. 1–3 the three-reggeon coupling constant is purely imaginary $\rho = ir$. If we formally set $\rho = ir$ in Eqs. (25) and (26) we are led directly to the following contradiction.

Since the dependence of D in (25) on ρ reduces to the factor $\rho^{-2/3}$ the substitution $\rho^2 = -r^2$ reduces, at first sight, to a change of sign of D. If this were indeed so, the discontinuity $\Lambda_\omega D(\omega, \mathbf{k})$, which is positive for a Hermitian Hamiltonian ($\rho^2 > 0$) would become negative in our case. But the total scattering cross section, which is expressed in terms of $\Delta D(\omega, 0)$ by means of a Sommerfeld–Watson integral

$$\sigma_t(s) = \frac{1}{s}\,\mathrm{Im}\, A(s, 0) = g^2 \int_{-i\infty}^{+i\infty} \frac{d\omega}{2\pi i} s^\omega D(\omega, 0) = g^2 \int_{-\infty}^{0} \frac{d\omega}{\pi} s^\omega \Delta_\omega D(\omega, 0) = -\frac{g^2}{r^{2/3}} f(0) \frac{(\ln s)^{\mu-1}}{\Gamma(\mu)} \qquad (62)$$

(g is the coupling constant between the particle and the reggeon) would then be negative, which is absurd. In reality the substitution $\rho = ir$ is not equivalent to a sign change of D, but leads to an essential modification of the solution. This is explained by the modification of the smoothness conditions (for joining solutions to perturbation theory) when going over to an anti-Hermitian Hamiltonian. (In particular, if we formally set $\rho^2 = -r^2$ in Eqs. (36) and (37), we are led to complex values for ρ_c, x_0.

Let us investigate how the junction with perturbation theory occurs. We first consider the case $\mathbf{k} = 0$. For $\omega \gg r^2/\alpha'$ perturbation theory is applicable and $D^{-1}(\omega, 0) = \omega$. For $0 < \omega \ll r^2/\alpha'$ in the Hermitian case $D^{-1}(\omega, 0)$ is positive and has the form given in Fig. 6.

In the anti-Hermitian case, if we simply modified the sign of D, we would obtain $D^{-1}(\omega, 0) < 0$ (dotted curve), i.e., then $D^{-1}(\omega, 0)$ should change sign for $\omega \sim r^2/\alpha'$. But for this either $D^{-1}(\omega, 0)$ or $D(\omega, 0)$ must become infinite, in contradiction with the initial equations, in which neither D, nor $D^{-1} = \omega - \Sigma$ has any singularities for $\mathbf{k} = 0$, $\omega > 0$ ($j > 1$). We therefore see that in order to be able to join the solution with perturbation theory, $D^{-1}(\omega, 0)$ must become positive for $\omega > 0$. Then $f(0) < 0$ in (62) and the contradiction with the negative cross section disappears.

This is only possible if either the solution of Eqs. (28) and (34) is nonunique, in spite of the arguments given in Sec. 5, or γ in (26) can be negative in the anti-Hermitian case, so that these equations are incorrect. In the first case, if the solutions are nonunique, the joining conditions require that there exist a solution $\tilde{f}(y)$ with other values for $\tilde{\mu}$ and $\tilde{\nu}$, for which the function $1/\tilde{f}(y)$ has the form illustrated in Fig. 1 by the dotted line. (The slope of the curve near the point $y = 2^{2-\nu}$ is given by the sign of the discontinuity at the two-reggeon threshold.) We cannot exclude this possibility, however, we note that in the model example considered in Sec. 1, there are no such solutions. It is shown in the Appendix that in this example $f(0) > 0$ for any solution.

If in the anti-Hermitian case ($\rho = ir$) the parameter $2\gamma = 1 + \nu - 3\mu$, which determines the magnitude of the vertex part, is negative, i.e., $\Gamma \ll 1$, this means that the nonsingular parts of the diagrams for Γ have cancelled the unrenormalized (original) vertex part, 1. If then $1 + \nu - 2\mu < 0$ is still positive, the effective four-reggeon interaction which comes from high frequencies in the diagrams (29) and (30) is still small and we are led to the previous equations (28) for Γ.

If $1 + \nu - 2\mu < 0$, everything is determined by an effective Hermitian four-reggeon interaction. As was indicated in the preceding section, depending on the magnitude and signs of λ_2 and λ_3 we obtain either a solution corresponding to weak coupling $D \to D_0$, $\Lambda_{2,3} \to 0$, considered in our preceding paper,[4] or a solution c

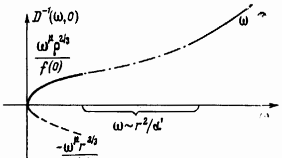


FIG. 6

corresponding to strong coupling, which does not contain additional poles. (The latter assertion about additional poles is, unfortunately, correct only for small λ_2 and λ_3, since if λ_2 and λ_3 are not small, the problem is in its entirety non-Hermitian, and we cannot be confident that the derivative $\partial\Lambda_2/\partial\omega$ which was used in the previous section is positive.)

Thus, the possibility of strong coupling in the non-Hermitian case with one pole runs into difficulties. We have been unable to prove that strong coupling is impossible in the anti-Hermitian case. One can only say that if such a solution does nevertheless exist, it follows from the structure of (1) that the total cross section determined by Eq. (62) does not tend to a constant, but behaves like $\sigma_t \sim (\ln s)^{\mu-1}$.

The case $\mu = 1$, which would correspond to a constant total cross section, does not agree with the equation $D = -1/\Sigma_c$, since for $\mu = 1$ the unitarity condition implies $\mathrm{Im}\,\Sigma \sim \omega$, i.e., $\Sigma_c \sim C\omega \ln \omega + \omega\sigma(\mathbf{k}^2/\omega^\nu)$. The cancellation of logarithms, i.e., the condition $C = 0$ could could only be "accidental," since C is determined by the region of small ω and $\mathbf{k}$, where there are no free parameters.

In conclusion we would like to express our sincere gratitude to K. A. Ter-Martirosyan for numerous useful discussions, and also to I. T. Dyatlov, B. L. Ioffe, E. M. Levin and A. M. Polyakov for a series of useful remarks.

APPENDIX

We consider Eq. (12) in more detail. We express it in terms of the discontinuity $\Delta f(x) = [f(x+i0) - f(x-i0)/2i$:

$$\frac{1}{f(x)} = \int_1^\infty\int_1^\infty \Phi(x; x_1, x_2)\Delta f(x_1)\frac{dx_1}{\pi}\Delta f(x_2)\frac{dx_2}{\pi} \tag{A.1}$$

The kernel $\Phi(x; x_1, x_2)$ defined by (8)–(10) is a homogeneous function of the form $x^{-1}\Phi(x_1/x, x_2/x)$ and is analytic in x in a plane with a cut from $x = (x_1 \times \exp(1/(2-\nu)) + x_2 \exp(1/(2-\nu)) \exp(2-\nu)$ to $x = +\infty$. For $x \to -\infty$

$$\Phi(x; x_1, x_2) \to (-x)^{\mu/\nu}\frac{1}{4\pi}\int_{-i\infty}^{+i\infty}\frac{d\varphi}{2\pi i}\frac{1}{\varphi^{\mu-\nu}(-\varphi)^{\mu-\nu}}\cdot$$
$$\cdot\left\{J(1; x_1\varphi^\nu, x_2(-\varphi)^\nu) - \frac{1}{\varphi^\nu x_1 - (-\varphi)^\nu x_2}\ln\frac{x_1\varphi^\nu}{x_2(-\varphi)^\nu}\right\}. \tag{A.2}$$

We can now investigate the analytic properties of $f(x)$. For $x = 1$ the function $f(x)$ has a pole. The condition for the constants μ and ν arising from this has the form

$$\frac{1}{f(1)} = \int_1^\infty\int_1^\infty \Phi(1, x_1, x_2)\Delta f(x_1)\Delta f(x_2)\frac{dx_1\,dx_2}{\pi^2} = 0. \tag{A.3}$$

For $1 < x < 2^{2-\nu}$ the function $f(x)$ is analytic, since in (A.1) $x_1, x_2 > 1$, and $\Phi(x; x_1, x_2)$ is analytic for $x \le (x^1 \exp(1/(2-\nu)) + x_2 \exp(1/(2-\nu)) \exp(2-\nu)$.

The next singularity of $f(x)$ appears from a coincidence of the singularity for $x_1 = 2\exp(2-\nu)$ in $f(x_1)$ and the pole for $x_2 = 1$ in $f(x_2)$ in Eq. (A.1) (or, conversely, of a pole in $f(x_1)$ and a branch point in $f(x_2)$) with a singularity of $\Phi(x; x_1, x_2)$. This happens for

$$x = (2+1)^{2-\nu} = 3^{2-\nu}$$

etc. Thus, f^{-1} has the branch points

$$x_n = n^{2-\nu}$$

and is analytic in the x plane with a cut from $x = 2\exp(2-\nu)$ to $x = +\infty$. The discontinuity $\Delta f(x)$ is positive at all singularities, including the pole, and the graph of $1/f(x)$ has approximately the form in Fig. 1 (solid line).

If as a zeroth approximation one retains in $f(x)$ only the pole term $f_1 = Z/(1-x)$, then the next approximation yields for $f(x)$ the expression

$$\frac{1}{f_2(x)} = Z^2\int_{-i\infty}^{+i\infty}\frac{d\varphi}{8\pi^2 i}\frac{J(-x; \varphi^\nu, (1-\varphi)^\nu) - J(-1; \varphi^\nu, (1-\varphi)^\nu)}{\varphi^{\mu-\nu}(1-\varphi)^{\mu-\nu}},$$

where J is defined by Eqs. (9) and (10)

$$Z^{-3} = \int_{-i\infty}^{+i\infty}\frac{d\varphi}{8\pi^2 i}\frac{J_x'(-1; \varphi^\nu, (1-\varphi)^\nu)}{\varphi^{\mu-\nu}(1-\varphi)^{\mu-\nu}}. \tag{A.4}$$

In this approximation, for $\nu = 3\mu - 1$ the equation has the form $\Phi(1; 1, 1) = 0$, or

$$\int_{-i\infty}^{+i\infty} d\varphi\left\{\frac{J(-1; \varphi^\nu, (1-\varphi)^\nu)}{(1-\varphi)^{\mu-\nu}\varphi^{\mu-\nu}} - \frac{\ln[\varphi^\nu/(-\varphi)^\nu]}{\varphi^{\mu-\nu}(-\varphi)^{\mu-\nu}[\varphi^\nu - (-\varphi)^\nu]}\right\} = 0. \tag{A.5}$$

In order to clarify the behavior of $D(\xi, \mathbf{k})$ in (42) we must solve (44) with $D(\xi, \mathbf{k})$ of the form (45), for $\xi^\nu\mathbf{k}^2 \to +\infty$. Thus, we must find $B(\mathbf{k}^2\xi^\nu)$ in (45) for $\mathbf{k}^2\xi^\nu \to +\infty$. We look for the solution in the form

$$B(z^2) = B_0(z^2)e^{-fz}, \quad z = |\mathbf{k}|\xi^{\nu/2} \gg 1, \quad f = \mathrm{const},$$

i.e.,

$$D(\xi, \mathbf{k}) = \rho^{-2/3}\xi^{\mu-1}B_0(\mathbf{k}^2\xi^\nu)\exp\{-f\sqrt{\mathbf{k}^2\xi^\nu}\}, \tag{A.6}$$

where $B_0(z^2)$ varies slowly compared with the exponential.

Then, if in the integral (44) one aligns k_x' along $\mathbf{k}$, the integration region $0 < k_x' < |\mathbf{k}|$, $k_y' \ll k_x'$ will be important, since the exponent $\exp\{-f\xi^{\nu/2}(|\mathbf{k}'| + |\mathbf{k} - \mathbf{k}'|)\}$ has a sharp sharp maximum there. The integral over k_y' can be taken by means of the method of steepest descent, which reduces (44) to the form

$$D(\xi, \mathbf{k}) = D^2(0, \mathbf{k})\rho^{2/3}\xi^{2(\mu-1)}\exp(-f\sqrt{\mathbf{k}^2\xi^\nu})\cdot$$
$$\cdot\int_0^{|\mathbf{k}|}\frac{dk_x'}{(2\pi)^{3/2}}B_0(\xi^\nu k_x'^2)B_0(\xi^\nu(|\mathbf{k}| - k_x')^2)\frac{\sqrt{k_x'(|\mathbf{k}| - k_x')}}{\xi^{\nu/4}\sqrt{f|\mathbf{k}|}}. \tag{A.7}$$

If we now remember Eq. (1a) of Sec. 3 and represent $D(0, \mathbf{k})$ in (A.7) in the form

$$D(0, \mathbf{k}) = A k^{-2\mu/\nu}\rho^{-2/3}, \tag{A.8}$$

we obtain for $B_0(z^2)$ in (A.6) the equation

$$B_0(z^2) = \frac{A^2}{\sqrt{f}}z^{-4\mu/\nu}\int_0^z\frac{dy}{(2\pi)^{3/2}}B_0(y^2)B_0((z-y)^2)\frac{\sqrt{y(z-y)}}{\sqrt{z}}. \tag{A.9}$$

Its solution is trivial:

$$B_0(z^2) = Nz^{4\mu/\nu - 3/2} = Nz^{2(1-\mu)/\nu + 1/2}, \tag{A.10}$$

where the number N is determined from the relation

$$\frac{(2\pi)^{3/2}\sqrt{f}}{A^2 N} = \int_0^1 dy\,[y(1-y)]^{2(1-\mu)/\nu+1} = \frac{\Gamma^2(2+2(1-\mu)/\nu)}{\Gamma(4+4(1-\mu)/\nu)}. \tag{A.11}$$

Substituting (A.10) into (A.6) we are led to the formula (46) in the main text.

Finally, we show that even if Eq. (12) has not one, but several solutions, then for any solution $f(0) > 0$. (This assertion is mentioned in Sec. 7.) For this purpose we consider the function $\Sigma(\xi, \mathbf{k})$ introduced in (44). Since

$$\omega^{\mu} f^{-1}(0) \rho^{2/3} = \Sigma(0,0) - \Sigma(\omega, 0),$$

$$\Sigma(\omega, \mathbf{k}) = \int_0^\infty d\xi\, e^{-\omega\xi}\, \Sigma(\xi, \mathbf{k}),$$

we have

$$f^{-1}(0) = \rho^{-2/3} \int_0^\infty d\xi\, \Sigma(\xi, 0)(1 - e^{-\xi}). \tag{A.12}$$

Setting $\mathbf{k} = 0$ in (44) substituting $D(\xi, \mathbf{k}')$ according to (45) into the integral, we obtain

$$\Sigma(\xi, 0) = \rho^{2/3} \xi^{-1-\mu} \int_0^\infty B^2(z^2) \frac{dz^2}{4\pi}. \tag{A.13}$$

Hence

$$f^{-1}(0) = \frac{\Gamma(1-\mu)}{4\pi\mu} \int_0^\infty B^2(z^2)\, dz^2. \tag{A.14}$$

We now note that $\mu < 1$, since otherwise $\nu = 3\mu - 1 > 2$, and then the position of the singularities of $D(\omega, \mathbf{k})$ according to (38) would be at $\mathbf{k}^2 > 0$ in the right ω half-plane, in contradiction with the initial analyticity properties of $D(\omega, \mathbf{k})$. Furthermore $\mu > 0$, since in (6) $D^{-1}(0, 0) = 0$.

Finally, $B(z^2)$ is real for $z^2 > 0$, since $D(\xi, \mathbf{k})$ is real for $\mathbf{k}^2 > 0$, as defined by the integral (42), where owing to the Riemann–Schwarz symmetry principle for $\mathbf{k}^2 > 0$ the values of $D(\omega, \mathbf{k})$ on the positive and negative imaginary semiaxes are mutually complex conjugate. Then it follows from (A.14) that $f(0) > 0$.

[1] V. N. Gribov, Zh. Eksp. Teor. Fiz. **53**, 654 (1967) [Sov. Phys. JETP **26**, 414 (1968)].

[2] S. Mandelstam, Nuovo Cimento **30**, 1113, 1127, 1143 (1963).

[3] V. N. Gribov, I. Ya. Pomeranchuk, and K. A. Ter-Martirosyan, Yad. Fiz. **2**, 361 (1965) [Sov. J. Nucl. Phys. **2**, 258 (1965)].

[4] V. N. Gribov and A. A. Migdal, Yad. Fiz. **8**, 1002 (1968) [Sov. J. Nucl. Phys. **8** (1969)].

[5] L. Castillejo, R. H. Dalitz, and F. J. Dyson, Phys. Rev. **101**, 453 (1956).

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