FURTHER RESTRICTIONS ON THE HIGH-ENERGY BEHAVIOR OF THE SCATTERING AMPLITUDE

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The third theorem of Meĭman and its corollary are presented and used to establish a number of new restrictions on the high-energy behavior of the scattering amplitude f(s) on a certain sequence S of physical points s extending to infinity. Rather weak conditions on possible oscillations of the amplitude are established which ensure that these restrictions are satisfied for all sufficiently large physical s. One of these restrictions is equivalent to the result of Khuri and Kinoshita, but is valid under weaker conditions on the possible oscillations of the amplitude. The results can be extended to arbitrary binary reactions. The advantages of introducing asymptotic amplitudes $f_\infty(s)$ are pointed out.

1. INTRODUCTION

RECENTLY, Martin[1] succeeded in obtaining a rigorous upper limit for the forward scattering amplitude within the framework of axiomatic theory:

$$|f(s)| \leqslant O(s\ln^2 s). \tag{1}$$

On the basis of this result, Khuri and Kinoshita[2] and afterwards, Vernov[3] showed that this limit can be lowered to some extent if one makes certain additional assumptions on the behavior of the function

$$\mathrm{H}(s) = \operatorname{Im} f(s) / \operatorname{Re} f(s) \tag{2}$$

for large physical s.

Below we use the third theorem of Meĭman (the first and second theorems of Meĭman were used in[2]) to establish new restrictions on the high-energy behavior of f(s). One of these restrictions is equivalent to the result of[2] but is valid under much weaker conditions on the possible oscillations of the amplitude. The importance of weakening the assumptions on the possible oscillations of the amplitude follows from the fact that up to now it has not been possible[4,5] to find restrictions on these oscillations starting from the standard postulates of quantum field theory.

2. A NEW RESTRICTION ON THE AMPLITUDE

Let us first formulate the general restrictions on the amplitude f(s) = f(s, t = 0) for the elastic scattering of a truly neutral spinless particle, which we shall need in the following.

1. The amplitude f(s) is holomorphic in the upper s plane (possibly with some finite part cut out) and bounded there by an arbitrary linear exponential.[1)] This condition follows from the locality principle of Meĭman[6] as well as from the principle of microcausality in the formulation of Lomsadze and Krivskiĭ.[7,8]

2. The amplitude f(s) is continuous in the region of holomorphism and on its boundary (i.e., on a sufficiently distant part of the real axis). This condition, together with condition 1, allows one to apply the generalized maximum principle of Phragmén-Lindelöf-Nevanlinna to the auxiliary functions which will be constructed below on the basis of the amplitude f(s). This maximum principle guarantees that the conditions of the third theorem of Meĭman are satisfied by the auxiliary functions (for finer details connected with the applicability of this principle to high-energy physics, see[9]).

3. The amplitude f(s) satisfies crossing symmetry in the form

$$f^*(-s^*) = f(s). \tag{3}$$

In order to obtain a new restriction on the high-energy behavior of f(s), we need the following theorem.

<u>Third Meĭman Theorem.</u>[10] Assume that the function G(s) a) is holomorphic in the upper s plane (possibly with a finite part cut out), continuous in this region and on its boundary (i.e., on a sufficiently distant part of the real axis), bounded in this region by an arbitrary linear exponential (cf. footnote[1)]) and tends to zero for $s \to \pm\infty$ along the real axis; and b) satisfies crossing symmetry in the form

$$G^*(-s^*) = G(s). \tag{4}$$

Then for physical $s \to \infty$

$$b(s)/a < (\pi^3/4)\ln^{-1}(s/s_0), \tag{5}$$

where a = sup | Re G(s) | in the upper half-ring (cf. Fig. 1) bounded by the half-circles with sufficiently large radii s_0 and $s > s_0$, and b(s) = inf | ln G(s) | on the line $[s_0, s]$.

It follows from Meĭman's third theorem that however slowly the function G(s) goes to zero, Im G(s) → 0 more rapidly than $\ln^{-1}s$ on a certain sequence of points. This theorem and its corollary have been used by N. N. Meĭman for a new proof of the Pomeranchuk theorem under the condition that the

[1)]This condition of boundedness by an arbitrary linear exponential can be replaced by the even weaker condition (7.1) of [6].

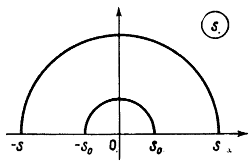

FIG. 1

amplitude f(s) increases more slowly than s ln s (private communication from N. N. Meĭman). We show now that the third Meĭman theorem and its corollary allow one to obtain a new restriction on the high-energy behavior of the scattering amplitude f(s).

We assume that we know (for example, from some general considerations or from experiment) that for large physical s

$$|f(s)| < O(\varphi(s)) \tag{6}$$

and that there exists a function $\Phi^{-1}(s)$ satisfying conditions a) and b) of Meĭman's third theorem so that for large physical s

$$|\Phi(s)| \sim \varphi(s). \tag{7}$$

Assume that the amplitude f(s) possesses the general analytic properties 1 to 3. Then the auxiliary function

$$G(s) = f(s)\Phi^{-1}(s) \tag{8}$$

will satisfy all conditions of the third theorem of Meĭman, and hence the following theorem is true:

Theorem 1. On a certain sequence S of physical points s extending to infinity (9)

$$|\operatorname{Im} f(s) \operatorname{Re} \Phi(s) - \operatorname{Re} f(s) \operatorname{Im} \Phi(s)| < O(|\Phi(s)|^2 \ln^{-1} s), \quad s \in S.$$

This inequality evidently has a real meaning when $\varphi(s)$ approximates $|f(s)|$ for physical $s \to \infty$ up to a factor which increases more slowly than ln s.

3. A FEW CONSEQUENCES

We obtain a number of important consequences from theorem 1 by imposing on the amplitude f(s) certain requirements which can be directly verified by experiment.

If the amplitude f(s) satisfies the general analytic conditions 1 to 3 and if

$$O(\varphi(s) \ln^{-1} s) \leqslant |f(s)| < O(\varphi(s)), \tag{10}$$

where $\varphi(s) = s^a \ln^b s$, then for physical $s \to \infty$

$$\mathrm{H}(s) \to \operatorname{tg}(\pi a/2), \quad s \in S. \tag{11}$$

If the amplitude f(s) satisfies the general analytic requirements 1 to 3 and has the upper limit

$$|f(s)| < O(s^a \ln^b s), \quad a \neq 0, 1 \tag{12}$$

and if $\mathrm{H}(s) \to \tan(\pi a/2)$ but not very rapidly so that

$$|\mathrm{H}(s) - \operatorname{tg}(\pi a/2)| > O(\ln^{-1+\delta} s), \tag{13}$$

where $\delta > 0$ can be chosen arbitrarily small, then

$$|f(s)| < O(s^a \ln^{-M} s), \quad s \in S \tag{14}$$

with arbitrarily large $M > 0$.

Let us prove this assertion. It follows from the inequality (9) and the upper limit (12) that

$$|\operatorname{Re} f(s)|\,|\mathrm{H}(s) - \operatorname{tg}(\pi a/2)| < O(s^a \ln^{b-1+\varepsilon} s), \quad s \in S, \tag{15}$$

and

$$|\operatorname{Im} f(s)|\,|\mathrm{H}^{-1}(s) - \operatorname{ctg}(\pi a/2)| < Q(s^a \ln^{b-1+\varepsilon} s), \quad s \in \mathrm{S}, \tag{16}$$

where we have chosen $\epsilon = \delta/2$ on the right-hand side, With (13) this yields

$$|f(s)| < O(s^a \ln^{b-\delta/2} s), \quad s \in S. \tag{17}$$

Repeating this iteration process an infinite number of times, we obtain the desired result (14).

The other consequences of theorem 1 refer to the cases where a = 1 or a = 0.

Let the amplitude f(s) satisfy the general requirements 1 to 3 and have the upper limit

$$|f(s)| < O(s \ln^b s). \tag{18}$$

Then

$$|\operatorname{Re} f(s)| < O(s \ln^{b-1} s), \quad s \in S. \tag{19}$$

In order to show this we choose for $\varphi(s)$ the following function: $\varphi(s) = s \ln^b s$. Then

$$\Phi(s) = s e^{i\pi/2} (\ln s - i\pi/2)^b. \tag{20}$$

The inequality (9) yields in this case

$$|-\operatorname{Re} f(s) + O(\operatorname{Im} f(s) \ln^{-1} s)| < O(s \ln^{b-1} s), \quad s \in S, \tag{21}$$

from which we obtain at once the required result (19).

If the amplitude f(s) satisfies the general requirements 1 to 3 and has the upper limit

$$|f(s)| < O(\ln^b s). \tag{22}$$

then

$$|\operatorname{Im} f(s)| < O(\ln^{b-1} s), \quad s \in S. \tag{23}$$

This assertion is proved in analogy to the previous case. We note that the inequalities (19) and (22) hold without any assumptions on the behavior of H(s). In particular, it follows from (22), with account of the upper limit of Martin (1), that

$$|\operatorname{Re} f(s)| < O(s \ln s). \tag{24}$$

If the general requirements 1 to 3 and the upper limit (18) are satisfied and if, moreover,

$$|\mathrm{H}(s)| < O(\ln^{1-\delta} s), \tag{25}$$

where $\delta > 0$ is arbitrarily small, then

$$|f(s)| < O(s \ln^{-M} s), \quad s \in S \tag{26}$$

with arbitrarily large $M > 0$.

If the upper limit (22) holds and if

$$|\mathrm{H}(s)| > O(\ln^{-1+\delta} s), \tag{27}$$

where $\delta > 0$ is arbitrarily small, then

$$|f(s)| < O(\ln^{-M} s), \quad s \in S \tag{28}$$

with arbitrarily large $M > 0$.

FIG. 2

4. ACCOUNT OF THE CONDITIONS ON THE OSCILLATIONS OF THE AMPLITUDE

We note that all the results obtained above are true for physical $s \in S$. In order to extend these results to all sufficiently large physical s, one must introduce a restriction on the possible oscillations of f(s). We assume (cf. Fig. 2) that no matter how rapidly the function $|\operatorname{Im} G(s)|$ oscillates, one can find a point $s'(s)$ which goes to infinity no more rapidly than some finite power of s for $s \to \infty$:

$$s'(s) \leqslant O(s^a), \tag{29}$$

such that for all physical $s'' \geq s'$, the value of the function does not exceed b(s):

$$|\operatorname{Im} G(s'')| \leqslant b(s), \quad s'' \geqslant s'. \tag{30}$$

It is clear that our assumption is rather weak. If it is satisfied, it can easily be shown that (5) yields for all sufficiently large physical s

$$|\operatorname{Im} G(s)| < O(\ln^{-1} s). \tag{31}$$

The meaning of this assertion is that however slowly the function G(s) decreases, Im G(s) must fall off more rapidly than $\ln^{-1} s$.

If $|\operatorname{Im} G(s)|$ oscillates so strongly that the inequality (30) can not be satisfied with condition (29), but can be guaranteed by the condition

$$s'(s) \leqslant O(\exp(Bs^a)), \tag{32}$$

then (5) yields the less restrictive inequality

$$|\operatorname{Im} G(s)| < O(\ln^{-1} \ln s). \tag{33}$$

However, we exclude the possibility of such a pathological behavior of the function G(s). Then theorem 1 and all its consequences will hold for all sufficiently large physical s. Combining in this case the inequality (25) with theorem 2 of Khuri and Kinoshita,[2] we conclude that the limit (26) holds for all physical s under very weak conditions on the possible oscillations of the amplitude.

5. CONCLUDING REMARKS

We note that the assumption of the neutrality of one of the scattering particles is inessential, and instead of the crossing symmetry (3) one could use the crossing symmetry in the form (cf., for example,[11])

$$f^{I*}(-s^*) = f^{II}(s), \tag{34}$$

where the indices I and II refer to the reaction and the crossed reaction reaction, respectively. In this case one would have to introduce the symmetric and antisymmetric amplitudes

$$f_+(s) = 2^{-1/2}[f^I(s) + f^{II}(s)], \quad f_-(s) = 2^{-1/2} i [f^I(s) - f^{II}(s)], \tag{35}$$

each of which will also satisfy the general requirements 1 to 3 and, in particular, the crossing symmetry in the form (3).

The extension of these results to the case of scattering of particles with spin presents no difficulties either if one works with invariant amplitudes. Consider, for example, the case of πN scattering where there exist four (with account of the isotopic spin variables) invariant amplitudes $A^{\pm}(s)$ and $B^{\pm}(s)$, which satisfy the crossing relations[12]

$$A^{\pm}(s) = \mp A^{\pm *}(-s^*), \quad B^{\pm}(s) = \pm B^{\pm *}(-s^*). \tag{36}$$

The results obtained above are applicable to each of the four functions $iA^+(s)$, $A^-(s)$, $B^+(s)$, and $iB^-(s)$, which satisfy crossing symmetry in the required form (3).

Finally, we note that in our entire consideration, the exact amplitude f(s) could be replaced by the asymptotic amplitude $f_\infty(s)$ introduced by Meiman.[6] However, in this case one would have to require[9,11] that $\sigma^{\perp} < 0(s)$. Besides the known advantages[6,8] the introduction of asymptotic amplitudes has the important feature[9] that its analyticity does not depend on the neglect of the electromagnetic interaction, which becomes especially important for large energies. Moreover, the introduction of asymptotic amplitudes allows one to extend all our results to the case of arbitrary nonbinary reactions.

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