COHERENT STATES OF A CHARGED PARTICLE IN A MAGNETIC FIELD

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We introduce coherent states of a charged (relativistic or non-relativistic) particle in a uniform magnetic field and also in uniform crossed electric and magnetic fields. We find the wave functions of these states in explicit form. We discuss the physical properties of the coherent states.

INTRODUCTION

In the papers by Schwinger and Glauber[1,2] coherent states of electromagnetic radiation were introduced which were afterwards studied in a number of papers.[3,4] Coherent states in quantum optics are constructed by analogy with the coherent states of a quantum oscillator which were first considered by Schrödinger.[5] We consider the problem of a charge in a magnetic field. This problem was considered in the nonrelativistic case in papers by Kennard,[6] Darwin,[7] and Landau[8] and in the relativistic case in papers by Rabi,[9] and Johnson and Lippmann.[10] Since it is well known[11] that the problem of a charged particle in a uniform magnetic field or in crossed uniform electrical and magnetic fields ($\mathbf{EH} = 0$, $E^2 - H^2 < 0$) reduces to solving an equation for the wave function of the oscillator type (Landau was the first to obtain this result when evaluating the spectrum of the problem under consideration[8]) it is natural, as was noted earlier,[12] to introduce coherent states into this problem and to consider them. The aim of the present paper is a consideration of the properties of coherent states of a charged particle (relativistic or non-relativistic) in a uniform magnetic field. Since in what follows it is necessary to use the problem of a quantum oscillator we briefly remind ourselves of the results referring to the coherent states of an oscillator. A similar derivation and discussion of these results is given in Glauber's book.[13]

The equation for the wave function of a one-dimensional oscillator has the form[11]

$$\omega(a^+a + 1/2)\psi = E\psi \quad (\hbar = 1,\ m = 1), \tag{1}$$

where

$$a = \frac{\omega^{1/2}q - i\omega^{-1/2}p}{\sqrt{2}}, \quad a^+ = \frac{\omega^{1/2}q + i\omega^{-1/2}p}{\sqrt{2}}$$

are the creation and annihilation operators with the commutation relations

$$[a, a^+] = 1. \tag{2}$$

The energy is then given by the equation

$$E_n = \omega(n + 1/2), \tag{3}$$

and the eigenfunction $|n\rangle$ corresponding to this energy has the form

$$|n\rangle = \frac{(a^+)^n}{\sqrt{n!}}|0\rangle, \quad \langle m|n\rangle = \delta_{mn}, \tag{4}$$

where $|0\rangle$ is the vacuum, i.e., the normalized function satisfying the condition

$$a|0\rangle = 0. \tag{5}$$

The operators a and a^+ are not Hermitean, their matrix elements in the complete orthornomal base (4) have the form

$$a|n\rangle = \sqrt{n}|n-1\rangle, \quad a^+|n\rangle = \sqrt{n+1}|n+1\rangle. \tag{6}$$

The coherent states $|\alpha\rangle$ of the oscillator are introduced as the eigenfunctions of the operator a:

$$a|\alpha\rangle = \alpha|\alpha\rangle, \tag{7}$$

where α is a complex number.

One verifies easily that the normalized function $|\alpha\rangle$ has the form

$$|\alpha\rangle = \exp\left(-\frac{|\alpha|^2}{2}\right)\sum_{n=0}^{\infty}\frac{\alpha^n}{\sqrt{n!}}|n\rangle. \tag{8}$$

The scalar product of two coherent states is then equal to

$$\langle\beta|\alpha\rangle = \exp[\beta^*\alpha - 1/2(|\alpha|^2 + |\beta|^2)],$$
$$|\langle\beta|\alpha\rangle|^2 = \exp(-|\alpha-\beta|^2). \tag{9}$$

The coherent states of an oscillator are thus not orthogonal. The coherent states form a complete set since the unit operator is equal to

$$1 = \pi^{-1}\int|\alpha\rangle\langle\alpha|d^2\alpha, \quad d^2\alpha = d\,\mathrm{Re}\,\alpha\, d\,\mathrm{Im}\,\alpha. \tag{10}$$

Moreover, the coherent states form an overcomplete set of functions in the sense that if we have any convergent sequence of complex numbers $\alpha_n \to \alpha_0$, the coherent states $|\alpha_n\rangle$ themselves form a complete set.[14] The expansion of a quadratically integrable function in terms of coherent states is thus generally speaking not unique.

Let $|f\rangle$ be a state such that

$$|f\rangle = \sum_{n=0}^{\infty}c_n|n\rangle, \quad \sum_{n=0}^{\infty}|c_n|^2 = 1. \tag{11}$$

The expansion formulae in terms of coherent states

$$|f\rangle = \pi^{-1}\int|\alpha\rangle f(\alpha^*)\exp\left(-\frac{|\alpha|^2}{2}\right)d^2\alpha, \tag{12}$$

where

$$f(\alpha^*) = \exp\left(\frac{|\alpha|^2}{2}\right)\langle\alpha|f\rangle, \tag{12'}$$

are then valid and these formulae give uniquely an expansion if we require that the coefficients $f(\alpha^*)$ in (12) be analytical functions of the variable α^*.

The coherent states $|\alpha\rangle$ can be obtained by operating on the vacuum $|0\rangle$ with the unitary operator

$$D(\alpha) = \exp(\alpha a^{+} - \alpha^{*} a). \qquad (13)$$

Thus

$$|\alpha\rangle = D(\alpha)|0\rangle = \left(\frac{\omega}{\pi}\right)^{1/4} \exp\left\{-\left(\sqrt{\frac{\omega}{2}}q - \alpha\right)^2 + \frac{\alpha^2 - |\alpha|^2}{2}\right\}. \qquad (13')$$

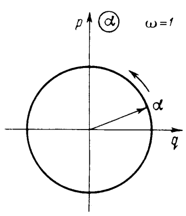
One verifies easily that the operator $D(\alpha)$ satisfies the conditions

$$D(\alpha_1)D(\alpha_2) = D(\alpha_1 + \alpha_2)\exp(i\,\mathrm{Im}\,\alpha_1\alpha_2^{*}),$$
$$D^{-1}(\alpha)aD(\alpha) = a + \alpha. \qquad (14)$$

A coherent state changes in time as follows:

$$|\alpha(t)\rangle = |\alpha_0 e^{-i\omega t}\rangle. \qquad (15)$$

The physical meaning of the coherent states consists in that they are just those states of the given quantum system which are closest to the states considered classically. Indeed, the motion of a classical oscillator which can conveniently be described as the motion in the phase plane (p, q) or in the complex plane of $\alpha = 2^{-1/2}(q + ip)$



is the motion along a circle with radius $\sqrt{2}|\alpha|$ and frequency ω. The complex number $\alpha(t)$ thus completely gives the state of the classical oscillator. It is clear from (13) that the quantum oscillator in the coherent state is completely analogous to a classical one. In a coherent state the indeterminacy of the coordinate and the momentum is as small as possible:

$$(\Delta p)^2 = \langle p^2\rangle - \langle p\rangle^2, \quad (\Delta q)^2 = \langle q^2\rangle - \langle q\rangle^2, \qquad (16)$$
$$(\Delta p)^2(\Delta q)^2 = 1/4.$$

The quantity $|\alpha|$ is the "classical" amplitude of the oscillations of a quantum oscillator and the phase $\varphi(\alpha)$ is the "classical" phase of the oscillations of the same oscillator. However, a quantum oscillator in the state $|\alpha\rangle$ in the language of classical mechanics corresponds to the motion along a circle in the phase plane of a whole set of classical oscillators each of which oscillates with its own amplitude which determines its energy. The energy distribution in the coherent state $|\alpha\rangle$ is a Poisson distribution

$$|\langle n|\alpha\rangle|^2 = \exp(-|\alpha|^2)\frac{|\alpha|^{2n}}{n!}. \qquad (17)$$

1. COHERENT STATES OF A NON-RELATIVISTIC SPIN-½ CHARGED PARTICLE IN A UNIFORM MAGNETIC FIELD

We turn now to a consideration of the behavior of a charged particle in a magnetic field which is determined by the Hamiltonian[11]

$$H = (2m)^{-1}(\mathbf{p} - e\mathbf{A})^2 - \mu\boldsymbol{\sigma}\mathbf{H} \quad (c = \hbar = 1). \qquad (18)$$

We choose a gauge in which $\mathbf{A} = [\mathbf{H}\times\mathbf{r}]/2$. We introduce the variables[15]

$$\xi = \frac{\sqrt{m\omega}}{2}(x + iy), \quad x = \frac{\xi + \bar{\xi}}{\sqrt{m\omega}}, \quad y = i\frac{\bar{\xi} - \xi}{\sqrt{m\omega}}, \qquad (19)$$

where the frequency $\omega = eH/m$. The motion along the direction of the magnetic field $\mathbf{H}$ which we chose along the z-axis is a free one so that the problem considered is essentially a planar one.

The Hamiltonian (18) commutes with the Hermitean operators of the "coordinates of the center of the circle"

$$x_0 = x + (m\omega)^{-1}(p_y - eA_y), \quad y_0 = y - (m\omega)^{-1}(p_x - eA_x), \qquad (19')$$

and also with the z-components of the angular momentum and the spin

$$M_z = [\mathbf{rp}]_z, \quad \sigma_z. \qquad (20)^{*}$$

We consider the following operators

$$b = (x_0 - iy_0)\sqrt{m\omega/2}, \quad b^{+} = (x_0 + iy_0)\sqrt{m\omega/2}, \qquad (21)$$

which in the variables ξ have the form

$$b = \frac{1}{\sqrt{2}}\left(\bar{\xi} + \frac{\partial}{\partial\xi}\right), \quad b^{+} = \frac{1}{\sqrt{2}}\left(\xi - \frac{\partial}{\partial\bar{\xi}}\right). \qquad (22)$$

They satisfy the creation and annihilation operators commutation relations

$$[b, b^{+}] = 1 \qquad (23)$$

and are integrals of motion.

We also introduce other creation and annihilation operators

$$a = \frac{p_x - eA_x - i(p_y - eA_y)}{\sqrt{2m\omega}}, \quad a^{+} = \frac{p_x - eA_x + i(p_y - eA_y)}{\sqrt{2m\omega}}, \qquad (24)$$

which in the variables ξ have the form

$$a = -\frac{i}{\sqrt{2}}\left(\xi + \frac{\partial}{\partial\bar{\xi}}\right), \quad a^{+} = \frac{i}{\sqrt{2}}\left(\bar{\xi} - \frac{\partial}{\partial\xi}\right) \qquad (25)$$

and satisfy the relations

$$[a, a^{+}] = 1, \quad [a, b] = [a, b^{+}] = 0. \qquad (26)$$

Using these operators we can rewrite the Hamiltonian (18) in the usual form:[10]

$$H = \omega(a^{+}a + 1/2) - \mu H\sigma_z + p_z^2/2m. \qquad (27)$$

We now apply the technique described in the Introduction to the problem considered. Since the motion along the field takes place independently the wave function which is an eigenfunction for the Hamiltonian (27) has the form

$$\psi_{Ep_z s_z} = \Phi_{n_1}(\xi, \bar{\xi})\exp(ip_z z)\chi^{1/2}_{s_z}, \qquad (28)$$

where the relations

$$\omega(a^{+}a + 1/2)\Phi_{n_1}(\xi, \bar{\xi}) = \omega(n_1 + 1/2)\Phi_{n_1}(\xi, \bar{\xi}),$$
$$\sigma_z\chi^{1/2}_{s_z} = s_z\chi^{1/2}_{s_z}, \qquad (29)$$

are satisfied while the energy is given by the usual formula:

$$E = \omega(n_1 + 1/2) - \mu H s_z + p_z^2/2m. \qquad (30)$$

*$[\mathbf{rp}]_z \equiv \mathbf{r}\times\mathbf{p}_z$.

One can easily construct a state with a given energy and angular momentum component along the magnetic field. To do this one must construct eigenfunctions of the operators a^+a and b^+b which reduces to constructing the eigenfunctions of a two-dimensional quantum oscillator. Let us consider the "vacuum" state $\Phi_{00}(\xi, \bar{\xi}) = |00\rangle$ such that

$$a|00\rangle = 0, \quad b|00\rangle = 0, \quad \int \Phi_{00}^* \Phi_{00} d\xi\, d\bar{\xi} = 1. \tag{31}$$

Using the form of the operators a and b we easily obtain[10]

$$\Phi_{00} = \sqrt{\frac{m\omega}{2\pi}} \exp(-\xi\bar{\xi}). \tag{32}$$

We determine the normalized states in the usual way:

$$\Phi_{n_1n_2}(\xi, \bar{\xi}) = |n_1n_2\rangle = \frac{(a^+)^{n_1}(b^+)^{n_2}}{\sqrt{n_1!\,n_2!}}|00\rangle; \tag{33}$$

then

$$\int \Phi^*_{n_1n_2}\Phi_{m_1m_2} d\xi\, d\bar{\xi} = \delta_{m_1n_1}\delta_{m_2n_2}. \tag{34}$$

In the variables ξ we have the following explicit form for the functions $\Phi_{n_1n_2}$:[10]

$$\Phi_{n_1n_2} = \sqrt{\frac{m\omega}{2\pi}} \frac{i^{n_1}}{\sqrt{n_1!\,n_2!}} + 2^{(n_2-n_1)/2}\left(\xi - \frac{\partial}{\partial\xi}\right)^{n_1} \xi^{n_2} \exp(-\xi\bar{\xi}). \tag{35}$$

One verifies easily that this state is an eigenstate for the angular momentum z-component operator

$$M_z\Phi_{n_1n_2} = (n_1 - n_2)\Phi_{n_1n_2}. \tag{36}$$

A state with a given energy, momentum, components of the angular momentum $l = n_1 - n_2$ and of the spin s_z along the field direction has thus the form

$$|n_1n_2s_zp_z\rangle = \Phi_{n_1n_2}(\xi, \bar{\xi}) \exp(ip_zz)\chi^{1/2}_{s_z}, \tag{37}$$

where $\Phi_{n_1n_2}(\xi, \bar{\xi})$ is given by Eq. (35). The solution (37) occurs naturally when we consider the problem under discussion in cylindrical coordinates.[11]

We now introduce states which are coherent states of a charged particle in a magnetic field, i.e., such states $|\alpha\beta\rangle$ which satisfy the relations

$$a|\alpha\beta\rangle = \alpha|\alpha\beta\rangle, \quad b|\alpha\beta\rangle = \beta|\alpha\beta\rangle. \tag{38}$$

By a direct check one verifies easily that

$$\Phi_{\alpha\beta} = |\alpha\beta\rangle$$

$$= \sqrt{\frac{m\omega}{2\pi}} \exp\left\{-\xi\bar{\xi} + \sqrt{2}\beta\xi + i\sqrt{2}\alpha\bar{\xi} - i\alpha\beta - \frac{|\alpha|^2}{2} - \frac{|\beta|^2}{2}\right\}, \tag{39}$$

while

$$\int \Phi_{\alpha\beta}^*\Phi_{\alpha\beta} d\xi\, d\bar{\xi} = 1. \tag{40}$$

One can check that $\Phi_{\alpha\beta}$ is a generating function for the functions (35). If we expand the function $\exp\{(|\alpha|^2 + |\beta|^2)/2\}\Phi_{\alpha\beta}$ in a power series in α and β, the expansion coefficients determine the functions (35):

$$\exp\left(\frac{|\alpha|^2 + |\beta|^2}{2}\right)\Phi_{\alpha\beta} = \sum_{n_1n_2=0}^{\infty} \frac{\alpha^{n_1}\beta^{n_2}}{\sqrt{n_1!n_2!}}\Phi_{n_1n_2}. \tag{41}$$

According to the general theory the coherent state $\Phi_{\alpha\beta}$ is obtained from the function (32) by the application of unitary operators obtained from Eq. (13)

$$|\alpha\beta\rangle = D(\alpha)D(\beta)|00\rangle, \tag{42}$$

where

$$D(\alpha) = \exp(\alpha a^+ - \alpha^* a), \quad D(\beta) = \exp(\beta b^+ - \beta^* b),$$

$$[D(\alpha),\ D(\beta)] = 0. \tag{43}$$

The functions of the coherent states (39) satisfy the condition

$$\int \Phi_{\alpha\beta}^*\Phi_{\alpha'\beta'} d\xi\, d\bar{\xi}$$

$$= \exp\{-\tfrac{1}{2}(|\alpha|^2 + |\alpha'|^2 + |\beta|^2 + |\beta'|^2) + \alpha^*\alpha' + \beta^*\beta'\}. \tag{44}$$

The functions $|\alpha\beta\rangle$ introduced here (cf. the Gaussian wave packets studied in[6,7]) correspond to coherent states with respect to the operators a and b. One can consider coherent states with respect to one operator. We can, for instance introduce the function $\Phi_{n_1\beta}(\xi, \bar{\xi})$ corresponding to a well-defined energy and satisfying the conditions

$$a^+a\Phi_{n_1\beta} = n_1\Phi_{n_1\beta}, \quad b\Phi_{n_1\beta} = \beta\Phi_{n_1\beta}. \tag{45}$$

Direct calculation gives

$$\Phi_{n_1\beta} = |n_1\beta\rangle = D(\beta)|n_10\rangle = \sqrt{\frac{m\omega}{2\pi}} \frac{i^{n_1}}{\sqrt{n_1!}} (\sqrt{2}\bar{\xi} - \beta)^{n_1}$$

$$\times \exp\left[-\xi\bar{\xi} + \sqrt{2}\beta\xi - \frac{|\beta|^2}{2}\right]. \tag{46}$$

This function satisfies the condition

$$\int \Phi^*_{n_1\beta}\Phi_{m_1\beta'} d\xi\, d\bar{\xi} = \delta_{n_1m_1} \exp\left\{-\frac{1}{2}(|\beta|^2 + |\beta'|^2) + \beta^*\beta'\right\}. \tag{47}$$

The coherent state $|\alpha n_2\rangle$ is determined by equations similar to Eqs. (45), (46), and (47), and has the form

$$|\alpha n_2\rangle = D(\alpha)|0n_2\rangle,$$

$$|\alpha n_2\rangle = \sqrt{\frac{m\omega}{2\pi}} \frac{2^{n_2/2}}{\sqrt{n_2!}} \left(\xi - i\frac{\alpha}{\sqrt{2}}\right)^{n_2} \exp\left(-\xi\bar{\xi} + i\sqrt{2}\alpha\bar{\xi} - \frac{|\alpha|^2}{2}\right), \tag{48}$$

while

$$\int \Phi^*_{\alpha n_2}\Phi_{\alpha' m_2} d\xi\, d\bar{\xi} = \exp\left[-\frac{1}{2}(|\alpha|^2 + |\alpha'|^2) + \alpha^*\alpha'\right]\delta_{n_2m_2}. \tag{49}$$

Expanding the function $\Phi_{n_1\beta}$ in a power series in the variable β or the function $\Phi_{\alpha n_2}$ in a power series in the variable α we can obtain the states $|n_1n_2\rangle$, i.e., these functions are generating functions to determine the functions (35). Using Eqs. (11), (12), and (12′) one easily obtains an expansion of the functions (35) in terms of coherent states. This expansion is the inverse of the expansion (41):

$$|n_1n_2\rangle = \pi^{-2}\int (\alpha^*)^{n_1}(\beta^*)^{n_2}(n_1!\,n_2!)^{-1/2}$$

$$\times \exp\left[-\frac{|\alpha|^2 + |\beta|^2}{2}\right]|\alpha\beta\rangle\, d^2\alpha\, d^2\beta, \tag{50}$$

or

$$|n_1n_2\rangle = \pi^{-1}\int (\beta^*)^{n_2}(n_2!)^{-1/2} \exp\left\{-\frac{|\beta|^2}{2}\right\}|n_1\beta\rangle\, d^2\beta. \tag{51}$$

We now discuss the physical meaning of the coherent states. As an example of the solutions with a given energy we consider the functions introduced by Landau[8] (see[10]) which in the variables ξ have the form

$$\Phi_{n_1c} = |n_1c\rangle = i^{n_1}2^{-n_1/2}\sqrt{\frac{m\omega}{2\pi}}\,(n_1!)^{-1/2}$$
$$\times\exp\left[-\xi\bar{\xi}-(\xi-c\sqrt{m\omega})^2+\frac{c^2m\omega}{2}\right]H_{n_1}(u),$$
$$u=\xi+\bar{\xi}-c\sqrt{m\omega}, \tag{52}$$

where the quantity c is a real parameter. The function $|n_1c\rangle$ satisfies the conditions

$$a^+a|n_1c\rangle = n_1|n_1c\rangle,$$
$$x_0|n_1c\rangle = c|n_1c\rangle,$$
$$\int \Phi^*_{n_1c}\Phi_{n_2c'}\,d\xi\,d\bar{\xi} = 2\pi\delta_{n_1n_2}\delta((c-c')m\omega). \tag{53}$$

The physical interpretation of the solutions (52) is the following: these solutions correspond to states in which there is an equally probable distribution of orbits with the same energy, the centers of which lie on the straight line $x = c$ ($p_z = 0$). The solutions (35) describe a state in which there is an equally probable distribution of orbits whose centers lie on a circle with radius $R = (m\omega)^{-1/2}[2(n_1 - n_2) + 1]^{1/2}$.[10] It is clear that these solutions do not correspond to the classical picture of the motion. One gets a description which is closest to the classical one, as in the case of the problem of the quantum oscillator, by introducing just the coherent states of a charged particle in a magnetic field. The coherent state (46) is then such a state with a given energy when the indeterminacy in the coordinates of the center of the orbit x_0 and y_0 is the smallest possible, as is clear from Eq. (16) since in the problem of a charged particle in a magnetic field the operators of the coordinates of the center of the circle (19) play the same role as the operators of the momentum and the coordinate in the case of the one-dimensional oscillator. From the fact that the operators b and b^+ commute with the Hamiltonian (18) follows that the parameter β is independent of the time; the parameter β also gives the position of the center of the circle in the x,y-plane. The center of the circle does not move in the classical description of the problem considered. In that sense the coherent state function $|n_1\beta\rangle$ is the best approximation to the classical description of a stationary state with a given energy. However, the function $|n_1\beta\rangle$ is not yet the completest possible approximation to the classical description of the state considered (in which the charged particle rotates along a circle with an exactly fixed center) since although the coordinates of the center of the circle are in this state determined with the smallest possible quantum indeterminacy, the coordinates of the motion around the center do not satisfy this condition.

It is completely clear that this condition will be satisfied if we consider the state (39) which is coherent in both operators a and b. Indeed, this state changes in time in such a way that $\alpha \to \alpha e^{-i\omega t}$ (see (15)). The complex variable α determines in the classical description of the problem discussed the quantities $p_y - eA_y$ and $p_x - eA_x$ and thereby it determines by virtue of the equations

$$x-x_0=-\frac{p_y-eA_y}{m\omega},\quad y-y_0=\frac{p_x-eA_x}{m\omega} \tag{54}$$

(see (19)), which are true also in the classical consideration, the position of the electron on the circle with center in the point x_0y_0. In the coherent state $|\alpha\beta\rangle$ the indeterminacy of the coordinates of the charged particle relative to the center of the circle has by virtue of Eq. (16) its smallest possible value. The quantity β has thus in the coherent state the meaning of the center of the circle while the quantity α has the meaning of the coordinates of the charged particle rotating around this center.

In the coherent state $|n_1\beta\rangle$ the z-component of the angular momentum l satisfies a Poisson distribution

$$|\langle n_1\beta|n_1n_2\rangle|^2 = e^{-|\beta|^2}\frac{|\beta|^{2n_2}}{n_2!},\quad l=n_1-n_2. \tag{55}$$

In the classical coherent state $|\alpha\beta\rangle$ we have the following distribution in energy and angular momentum (see (41)):

$$|\langle\alpha\beta|n_1n_2\rangle|^2 = \exp\{-(|\alpha|^2+|\beta|^2)\}\frac{|\alpha|^{2n_1}|\beta|^{2n_2}}{n_1!\,n_2!}. \tag{56}$$

The action of the operator of rotation of the system of coordinates over an angle φ in the xy-plane on the coherent state leads to the following: $T_\varphi\xi = e^{i\varphi}\xi$,

$$T_\varphi|n_1\beta\rangle = e^{-in_1\varphi}|n_1\beta e^{i\varphi}\rangle \tag{57}$$

and

$$T_\varphi|\alpha\beta\rangle = |\alpha e^{-i\varphi}\beta e^{i\varphi}\rangle. \tag{58}$$

The quantity $|\alpha|$ gives the "classical" radius of the circle along which the charged particle rotates, and the phase φ gives the phase of the rotation. The quantity $|\beta|$ gives the distance of the center of the circle from the coordinate origin and the phase $\varphi(\beta)$ determines the azimuth of the center of the circle.

We consider the operator of the square of the distance of the center of the circle from the coordinate origin

$$r_0^2 = x_0^2 + y_0^2. \tag{59}$$

Then

$$r_0^2|n_1n_2\rangle = \frac{1}{\omega m}(2n_2+1)|n_1n_2\rangle. \tag{60}$$

It is clear that we have for this quantity in the coherent state $|n_1\beta\rangle$ the Poisson distribution (55). The same statement is also valid in the case of the state $|\alpha\beta\rangle$. To emphasize once again that the coherent state $|\alpha\beta\rangle$ is the one closest to the classical one, we note that in the classical problem one can choose as canonical coordinates the quantities $q_1 = x_0$, $q_2 = p_x - eA_x$ and for the momenta conjugate to them, respectively, $p_1 = y_0$ and $p_2 = p_y - eA_y$. The classical equations of motion then take the form

$$\frac{d}{dt}(p_x-eA_x)=\omega(p_y-eA_y),\quad \dot{x}_0=0,$$
$$\frac{d}{dt}(p_y-eA_y)=-\omega(p_x-eA_x),\quad \dot{y}_0=0. \tag{61}$$

One sees thus easily that the motion in the phase planes (q_1p_1) and (q_2p_2) is in the classical picture very simple. The particle is at rest in the q_1p_1-plane and moves along a circle in the q_2p_2-plane which corresponds to the motion of the charged particle along a circle in the real space around a non-moving center. The transition from the classical quantities to operators does not change Eqs. (61). We also note that by virtue of an easily checked formula for the square of the radius of the circle[10]

$$(x-x_0)^2+(y-y_0)^2=2(a^+a+1/2)\,/\,m\omega \tag{62}$$

we have in the coherent state $|\alpha\beta\rangle$ for this quality the Poisson distribution (56) and in the state $|\alpha n_2\rangle$ the distribution

$$|\langle \alpha n_2|n_1 n_2\rangle|^2=\exp(-|\alpha|^2)\,\frac{|\alpha|^{2n_1}}{n_1!}. \tag{63}$$

2. COHERENT STATES OF A SPIN-ZERO RELATIVISTIC CHARGED PARTICLE IN A UNIFORM MAGNETIC FIELD

The results obtained in Sec. 1 for a non-relativistic charged particle can easily be carried over to the case of a spinless relativistic charged particle ($\pi^{\pm}$ mesons). The behavior of such a particle in a magnetic field is described by the equation

$$\left[\frac{\partial^2}{\partial t^2}+m^2+p_z^2+(p_x-eA_x)^2+(p_y-eA_y)^2\right]\psi=0. \tag{64}$$

It is clear that the operators (19), (20), and (21) commute with this equation. Since Eq. (64) differs little from the non-relativistic equation with the Hamiltonian (18) one gets easily its spectrum

$$E_{n_1}=\pm(m^2+p_z^2+\omega m(2n_1+1))^{1/2}. \tag{65}$$

One can completely carry over to the case considered now the discussion of the preceding section; the functions (32), (35), and (46) are solutions of Eq. (64) corresponding to the fixed energy (65). The dependence of the solutions on the time is given by the exponential factor $\exp(\pm iEt)$ which distinguishes between states with positive and negative energies.

We consider the coherent states $|n_1\beta\rangle$ (see (46)). As in the case of the non-relativistic problem these states are the closest to the classical state in the sense that in them the indeterminacy in the coordinates of the center of the circle is smallest. Since the operators b and b^+ commute with the Eq. (64), the quantity β, which gives the position of the center of the circle, does not change in time. Equations (57) and (55) remain valid also in the relativistic case. The complex quantity β as in the non-relativistic case has the meaning of the classical coordinates of the center of the circle. However, if we consider coherent states with respect to both operators a and b (see (39)) in the case of the relativistic Eq. (64) there appear well-defined difficulties. One is dealing here with the fact that such a state as $|\alpha\beta\rangle$ does not change with time following the old law since due to relativistic effects the frequency depends on the energy of the state. The wave packet (39) therefore smears out. If we had initially the state

$$|\alpha_0\beta\rangle=\sum_{n_1=0}^{\infty}\frac{\alpha_0^{n_1}}{\sqrt{n_1!}}\exp\left(-\frac{|\alpha_0|^2}{2}\right)|n_1\beta|\rangle \quad (E_{n_1}>0), \tag{66}$$

at time t we shall have the state

$$\psi(t\beta)=\sum_{n_1=0}^{\infty}\frac{\alpha_0^{n_1}}{\sqrt{n_1!}}\exp\left(-i\omega_{n_1}t-\frac{|\alpha_0|^2}{2}\right)|n_1\beta\rangle, \tag{67}$$

where

$$\omega_{n_1}=[m^2+p_z^2+\omega m(2n_1+1)]^{1/2}, \tag{68}$$

which is not an eigenfunction for the operator a. The physical meaning of the coherent states $|\alpha\beta\rangle$ is thus not clear to us for the case of a relativistic particle. Perhaps we should when elucidating the physical meaning of these states take into account the indeterminacy relation for the energy and time $\Delta t\Delta E \gtrsim h$. All the same, the coherent states $|\alpha\beta\rangle$ can be useful for performing calculations.

3. A RELATIVISTIC SPIN-½ PARTICLE IN A UNIFORM MAGNETIC FIELD

The behavior of a spin-½ charged particle in an electromagnetic field is described by the equation

$$H\psi_E=\{\boldsymbol{\alpha}(\mathbf{p}-e\mathbf{A})+\beta m\}\psi_E \tag{69}$$

($\boldsymbol{\alpha}$ and β are the standard four-by-four Dirac matrices). Following Johnson and Lippmann's paper[10] we use a "squaring" method to solve Eq. (69). We rewrite this equation in the form

$$O_+\psi_E=(H-E)\psi_E=0. \tag{70}$$

We considered the "squared" equation

$$O_-O_+\Phi_E=(H+E)(H-E)\Phi_E=0, \tag{71}$$

which leads to the form

$$\left[\frac{(p_x-eA_x)^2+(p_y-eA_y)^2}{2m}+\frac{p_z^2}{2m}-\frac{\omega\sigma_z}{2}-\frac{E^2-m^2}{2m}\right]\Phi_E=0. \tag{72}$$

The solutions of Eqs. (69) and (72) are then connected as follows:[10]

$$\psi_E=O_-\Phi_E, \tag{73}$$

where we have chosen the four-component function Φ_E in the form

$$\Phi_E=\begin{pmatrix}0\\ \varphi_E\end{pmatrix}, \tag{74}$$

while the function φ_E is a two-component one. We have further

$$\psi_E=\begin{pmatrix}R\varphi_E\\ C\varphi_E\end{pmatrix}, \tag{75}$$

where the operators R and C are given by the formulae

$$R=\boldsymbol{\sigma}(\mathbf{p}-e\mathbf{A}),\quad C=H-m, \tag{76}$$

while

$$R^2=H^2-m^2,\qquad C^2=H^2+m^2-2Hm. \tag{77}$$

Since the "squared" Eq. (72) is similar to Eq. (64) one can easily write down the energy spectrum

$$E_{n_1 s_z}=\pm[\omega m(2n_1+1-s_z)+p_z^2+m^2]^{1/2}. \tag{78}$$

The function φ_E in (74) and (75) can be chosen in the form[1)]

$$\varphi_E^{n_2}=|n_1 n_2\rangle\chi_{s_z}^{1/2}\exp(ip_z z), \tag{79}$$

$$\varphi_E^{\beta}=|n_1\beta\rangle\chi_{s_z}^{1/2}\exp(ip_z z), \tag{80}$$

$$\varphi_E^{c}=|n_1 c\rangle\chi_{s_z}^{1/2}\exp(ip_z z), \tag{81}$$

where the functions $|n_1 n_2\rangle$, $|n_1\beta\rangle$, and $|n_1 c\rangle$ are given by Eqs. (35), (46), and (52). If we use the function (80) to

1) A different choice of function φ_E determine different solutions of the relativistic Eq. (69) (see (75)).

obtain a solution corresponding to a given energy $E_{n_1 s_z}$ this solution will describe a coherent state of a relativistic charged spin-½ particle in a uniform magnetic field. This state is an eigenstate of the annihilation operator b (see (22)) which in the relativistic case is defined as in the non-relativistic case. The physical meaning of the coherent state of a relativistic charged particle in a magnetic field for a given energy E is the same as the physical meaning of a coherent state $|n_1\beta\rangle$ of a non-relativistic particle. In this state both coordinates of the center of the circle of the classical motion are determined with the smallest possible indeterminacy (see (16)).

The operator

$$\sigma_z' = \begin{pmatrix} R\sigma_z R^{-1} & 0 \\ 0 & C\sigma_z C^{-1} \end{pmatrix}. \tag{82}$$

is an integral of motion of Eq. (69) (see[12]). On the set of solutions (75) its eigenvalues are the same as s_z (see (79)–(81)). We write down a coherent relativistic solution with given energy $E_{n_1 s_z p_z}$, spin z-component $s_z = 1$, momentum along the field p_z, and center of the circle β:

$$\psi_{n_1}^{\beta}(xyzt) = \frac{e^{\pm iEt + ip_z z}}{\sqrt{2E(E-m)}} \begin{bmatrix} p_z|n_1\beta\rangle \\ \sqrt{2m\omega n_1}\,|n_1-1,\beta\rangle \\ (E-m)|n_1\beta\rangle \\ 0 \end{bmatrix}. \tag{83}$$

The function $|n_1\beta\rangle$ in (83) is given by Eq. (46). The solution (83) is normalized as follows:

$$\int (\psi_{n_1}{}^{\beta})^+ \psi_{n_2}{}^{\beta}\, d\xi d\bar{\xi} = \delta_{n_1 n_2}, \tag{84}$$

while

$$b\psi_{n_1}{}^{\beta}(xyzt) = \beta\psi_{n_1}{}^{\beta}(xyzt) \tag{85}$$

and the relation

$$\psi_{n_1}{}^{\beta}(xyzt) = D(\beta)\psi_{n_1}{}^{0}(xyzt), \tag{86}$$

where $D(\beta)$ is given by Eq. (43), is satisfied. The scalar product (83) for different β_1 and β_2 has the form (47). The wave function of the coherent state with $s_z = -1$ has the form

$$\psi_{n_1}^{\beta}(xyzt) = \frac{e^{\pm iEt + ip_z z}}{\sqrt{2E(E-m)}} \begin{bmatrix} \sqrt{2m\omega(n_1+1)}\,|n_1+1,\beta\rangle \\ -p_z|n_1\beta\rangle \\ 0 \\ (E-m)|n_1\beta\rangle \end{bmatrix}. \tag{87}$$

One can also construct coherent states with respect to the second variable α. To do this we introduce the operators A and A^+[11]

$$A = \begin{pmatrix} RaR^{-1} & 0 \\ 0 & CaC^{-1} \end{pmatrix}, \quad A^+ = \begin{pmatrix} Ra^+R^{-1} & 0 \\ 0 & Ca^+C^{-1} \end{pmatrix} \tag{88}$$

and construct the operator

$$D(\alpha) = \exp(\alpha A^+ - \alpha^* A). \tag{89}$$

The coherent state $|\alpha\beta s_z p_z\rangle$ is then given by the formula

$$|\alpha\beta s_z p_z\rangle = D(\alpha)|0\beta s_z p_z\rangle. \tag{90}$$

The problem of interpreting the state (90) as the state which is closest to the classical one meets with the same difficulties as in the case of a spinless particle and requires further discussion.

The representation of coherent states is useful for evaluating different quantities, e.g., the density matrix (this problem will be considered in another paper). The coherent states also enable us to find Bloch solutions for a relativistic electron in a magnetic field (see[16]).

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1. J. Schwinger, Phys. Rev. 91, 728 (1953); 92, 1283 (1953).
2. R. J. Glauber, Phys. Rev. Letters 10, 84 (1963); Phys. Rev. 130, 2529 (1963); 131, 2766 (1963).
3. E. C. G. Sudarshan, Phys. Rev. Letters 10, 277 (1963).
4. L. Mandel and E. Wolf, Rev. Mod. Phys. 37, 231 (1965).
5. E. Schrödinger, Naturwiss. 14, 664 (1926).
6. E. H. Kennard, Zs. Phys. 44, 326 (1927).
7. C. G. Darwin, Proc. Roy. Soc. (London) A117, 258 (1928).
8. L. D. Landau, Zs. Phys. 64, 629 (1930) [Collected Papers of L. D. Landau, Pergamon and Gordon and Breach, 1965, p. 31].
9. I. I. Rabi, Zs. Phys. 49, 507 (1928).
10. M. H. Johnson and B. A. Lippmann, Phys. Rev. 76, 828 (1949). In this paper one can find a complete bibliography for the given problem.
11. L. D. Landau and E. M. Lifshitz, Kvantavaya mekhanika (Quantum Mechanics) Fizmatgiz, 1963 [English translation published by Pergamon Press, 1965].
12. I. A. Malkin and V. I. Man'ko, Preprint, Lebedev Institute, 1968.
13. R. J. Glauber, Quantum Optics and Quantum Electronics (Russ. transl.), Mir, 1966, pp. 91–281.
14. K. E. Cahill, Phys. Rev. 138, B1566 (1965).
15. L. D. Landau and E. M. Lifshitz, Teoriya polya (Theory of fields) Fizmatgiz, 1964 [English translation to be published by Pergamon Press].
16. I. A. Malkin and V. I. Man'ko, Dokl. Akad. Nauk SSSR, in press.

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