SOVIET PHYSICS JETP VOLUME 27, NUMBER 6 DECEMBER, 1968

HYDRODYNAMIC MECHANISM OF ELECTRIC CONDUCTIVITY OF METALS IN A MAGNETIC FIELD

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Submitted February 3, 1968

Zh. Eksp. Teor. Fiz. 54, 1899–1906 (June, 1968)

We consider a hydrodynamic mechanism of the electric conductivity of metals in the presence of a strong magnetic field, parallel to the surface of the sample. The coefficient of electric conductivity and its variation with the magnetic field intensity and on the temperature depend significantly on the ratio of the radius of the Larmor orbit of the electron to the phonon mean free path.

1. It was shown by one of the authors[1] that the electric conductivity of certain metals has a hydrodynamic character at low temperatures. Such an electric conductivity mechanism arises under conditions when the normal collisions of the electrons with the phonons are much more frequent than the collisions in which the total quasimomentum is not conserved, and is possible if certain conditions are satisfied with respect to the Fermi surface of the metal. (For example, it is sufficient that the Fermi surface be closed and the number of electrons be unequal to the number of holes; for more details see[1].)

We denote by l_{ep}, l_{pe}, and l_{pp} the free paths characterizing respectively the normal collisions of the electron with the phonons, of the phonon with the electrons, and of the phonon with phonons. It is important that the inequalities $l_{ep} \gg l_{pe}$ and $l_{pp} \gg l_{pe}$ are satisfied with a large margin at low temperatures[1)]. The electrons in sufficiently bulky and perfect metallic samples therefore behave like an ordinary gas of particles that collide with one another. The interaction between the electrons is via exchange of temperature phonons, and is characterized in first approximation by the free path l_{ep}.

Obviously, normal collisions by themselves do not lead to electric resistance, but can greatly change the result of the action of other scattering mechanisms. In relatively thin samples, the resistance of which is determined by the scattering of the electrons by the boundaries, the electric conductivity has the character of Poiseuille flow of a viscous gas. The state of the electron system is characterized in first approximation by a Fermi distribution function with drift: $f_0(\epsilon - \mathbf{p}\cdot\mathbf{u})$. The ordered-motion velocity u, as a function of the coordinates, satisfies a hydrodynamic equation of the Navier-Stokes type[1]. The results obtained thereby find a ready intuitive interpretation. The electric conductivity coefficient is proportional to the mean path l^{eff} traversed by the electron between two collisions with the boundaries of the sample:

$$\sigma \approx \frac{e^2}{3h^3} S_F l^{\text{eff}}, \tag{1}$$

where S_F is the area of the Fermi surface. (It is assumed that the scattering of the electrons by the boundaries is nearly diffuse.) Inasmuch as the thickness of the sample $d \gg l_{ep}$, the electron inside the metal experiences many normal collisions with other electrons before reaching the boundary. The length of the corresponding Brownian trajectory is $l^{\text{eff}} \approx d^2/l_{ep}$.

2. The present paper is devoted to the theory of a similar mechanism of electric conductivity in a magnetic field. We consider for simplicity a metallic plate and assume that the electric and magnetic fields are parallel to its surface, but in general are not parallel to each other. Intuitive qualitative results are obtained in a sufficiently strong magnetic field, when the Larmor radius r is small compared with the free path of the electron and with the thickness of the sample: $r \ll l_{ep}$, $r \ll d$. The relation between l_{ep} and d can be arbitrary in this case.

Under such conditions, the normal collisions lead to diffusion of the centers of the electron orbits in a plane perpendicular to the magnetic field. This occurs in the following manner. During a time $\tau_{ep} = l_{ep}/v_F$, the electron revolves on a closed orbit, after which it emits a phonon. The phonon traverses a path l_{pe} and is absorbed by another electron.

According to the Brownian-motion formulas, $\tau^{\text{eff}} = l^{\text{eff}}/v_F \approx \tau(d/l)^2$, where τ is the time consumed in one step and l is the length of the step. It is clear that $\tau \approx \tau_{ep}$, where the distance l over which the momentum is transferred as a result of one step is determined by the smaller of the lengths, r or l_{pe}. Thus

$$l^{\text{eff}} \approx \begin{cases} l_{ep}(d/r)^2, & r \gg l_{pe} \\ l_{ep}(d/l_{pe})^2, & r \ll l_{pe} \end{cases}.$$

The dependence of the electric conductivity on the characteristic parameters has in the first case the form $\sigma \sim T^{-5}d^2H^2$, and in the second case $\sigma \sim T^{-3}d^2$ (see formula (1) and footnote[1)]).

It is also clear that if the electric and magnetic fields are not parallel, a transverse hole field, proportional to the electric current and to the magnetic field, is produced in the plate.

3. Proceeding to the calculations, we write down the system of kinetic equations for the distribution functions of the electrons f and of the phonons N. Choosing the z axis perpendicular to the surface of the plate and the x axis along the magnetic field, we get

[1)]As is well known, $l_{ep} \sim (\Theta/T)^5$, $l_{pe} \sim \Theta/T$, and $l_{pp} \sim (\Theta/T)^5$ (Θ—Debye temperature), and the coefficients in these ratios have the same order of magnitude, for typical metals, (see, for example[2]).

$$v_z\frac{\partial f}{\partial z}+e(\mathbf{E}\mathbf{v}+E_zv_z)\frac{\partial f_0}{\partial \varepsilon}+\Omega\left[\frac{\partial f}{\partial \mathbf{p}}\mathbf{p}\right]_x=J_{ep}\{f,N\}+J^U, \quad (2)^*$$

$$s_z\frac{\partial N}{\partial z}=J_{pe}\{f,N\}.$$

Here $\Omega = eH/mc$, E—external electric field, E_z—Hall field, $f_0(\epsilon)$—Fermi distribution function, $\mathbf{v} = \partial\epsilon/\partial\mathbf{p}$, $\mathbf{s} = \partial(\hbar\omega)/\partial\mathbf{q}$, $\epsilon(\mathbf{p})$ and $\hbar\omega$—energies and p and q—momenta of the electrons and phonons respectively, J_{ep}—operator of normal collisions of electrons with phonons, and J_{pe}—for collisions of phonons with electrons; the term J^U describes collisions with loss of quasimomentum (Umklapp processes, scattering by lattice defects, etc.). For simplicity, the laws of dispersion of the electrons and phonons are assumed isotropic.

The system (2), naturally, can be solved only approximately. Since we are interested in the hydrodynamic situation, we make use of the fact that at least one of the parameters, r/d or l_{ep}/d, is much smaller than unity, and we also take into account the fact that J^U is small compared with J_{ep}.

The method of successive approximations leads to the chain of equations (see [1]):

$$eE_zv_z\frac{\partial f_0}{\partial \varepsilon}+\Omega\left[\frac{\partial f^{(0)}}{\partial \mathbf{p}}\mathbf{p}\right]_x=J_{ep}\{f^{(0)},N^{(0)}\}, \quad (3)$$

$$J_{pe}\{f^{(0)},N^{(0)}\}=0. \quad (3')$$

$$v_z\frac{\partial f^{(0)}}{\partial z}+\Omega\left[\frac{\partial f^{(1)}}{\partial \mathbf{p}}\mathbf{p}\right]_x=J_{ep}\{f^{(1)},N^{(1)}\}, \quad (4)$$

$$s_z\frac{\partial N^{(0)}}{\partial z}=J_{pe}\{f^{(1)},N^{(1)}\}, \quad (4')$$

$$v_z\frac{\partial f^{(1)}}{\partial z}+\Omega\left[\frac{\partial f^{(2)}}{\partial \mathbf{p}}\mathbf{p}\right]_x=J_{ep}\{f^{(2)},N^{(2)}\}+J^U, \quad (5)$$

$$s_z\frac{\partial N^{(1)}}{\partial z}=J_{pe}\{f^{(2)},N^{(2)}\}. \quad (5')$$

The solution of the first pair of equations is

$$f^{(0)}=f_0(\varepsilon-\mathbf{pu}),\qquad N^{(0)}=N_0(\hbar\omega-\mathbf{qu}),$$

where N_0 is the Bose distribution function and the drift velocity u(z) is parallel to the surface of the plate. From (2) it also follows that

$$E_z=\frac{H}{c}u_y. \quad (6)$$

Going over to the solution of the second pair of equations, we note that the operator J_{pe} does not contain integrations with respect to the phonon momenta. It is therefore convenient to solve (4′) with respect to $N^{(1)}(q, z)$ and substitute the result in (4). We seek a solution of the obtained equation in the form

$$f^{(1)}=-\tau_0\left(\frac{T}{\Theta}\right)^3\varepsilon_F\frac{\partial f_0}{\partial \varepsilon}\left\{[n_yn_z\varphi_1-(n_y^2-n_z^2)\psi_1]\frac{\partial u_y}{\partial z}+[n_xn_z\varphi_2-n_xn_y\psi_2]\frac{\partial u_x}{\partial z}\right\}. \quad (7)$$

Here φ_i, ψ_i—sought functions of the dimensionless variable $\xi = [\epsilon(p) - \epsilon_F]/T$, $\mathbf{n} = \mathbf{p}/p$, and τ_0 coincides in order of magnitude with the electron-phonon path time at the Debye temperature.

The angular dependence of $f^{(1)}$ is determined by the form of the term

$$v_z\frac{\partial f^{(0)}}{\partial z}=-2\varepsilon\frac{\partial f_0}{\partial \varepsilon}\left(n_zn_y\frac{\partial u_y}{\partial z}+n_zn_x\frac{\partial u_x}{\partial z}\right),$$

and also by the fact that the application of the operator $[\mathbf{p}\times\partial/\partial\mathbf{p}]_x$ causes the expressions n_yn_z and n_xn_z to go over respectively into $n_y^2 - n_z^2$ and n_zn_y, and vice versa. It is also important that the operators J_{ep} and J_{pe} do not change the form of the angular dependences in the case of isotropic dispersion laws.

Substituting (7) in (4) and equating terms with identical angular dependence, we obtain four equations for the functions φ and ψ. Rather cumbersome calculations, in which we use the explicit form of the collision integrals, lead to the result

$$\gamma f_0'(\xi)\psi(\xi)=f_0'(\xi)+\int_{-\infty}^{\infty}\left(K_{\xi\xi'}-\left(\frac{T}{\Theta}\right)^2Q_{\xi\xi'}\right)\varphi(\xi')d\xi'. \quad (8)$$

$$-\gamma f_0'(\xi)\varphi(\xi)=\int_{-\infty}^{\infty}\left(K_{\xi\xi'}-\left(\frac{T}{\Theta}\right)^2Q_{\xi\xi'}\right)\psi(\xi')d\xi'. \quad (8')$$

Here

$$\gamma=\Omega\tau_0\left(\frac{\Theta}{T}\right)^3,\quad f_0'(\xi)=\frac{d}{d\xi}[e^{\xi}+1]^{-1}.$$

The indices of the functions φ and ψ have been omitted, since the equations for φ_1, ψ_1 and φ_2, ψ_2 coincide, apart from numerical coefficients of the order of unity. $\hat{K}$ and $\hat{Q}$ are complicated linear integral operators (see [1]), the explicit form of which is of no significance in what follows. We note only that

$$K_{\xi\xi'}=K_{\xi'\xi},\quad \int_{-\infty}^{\infty}K_{\xi\xi'}d\xi'=0,\quad Q_{\xi\xi'}\geqslant 0. \quad (9)$$

Excluding ψ from (8) and (8′), we get

$$\left[\hat{K}-\left(\frac{T}{\Theta}\right)^2\hat{Q}\right]\frac{1}{f_0'}\left[\hat{K}-\left(\frac{T}{\Theta}\right)^2\hat{Q}\right]\varphi+\gamma^2f_0'\varphi=\left(\frac{T}{\Theta}\right)^2\hat{Q}\cdot 1. \quad (10)$$

Let us obtain the solution of this equation in the limiting cases $\gamma \ll 1$ and $\gamma \gg 1$.

When $\gamma \ll 1$, it is natural to seek φ in the form of an expansion in powers of the small parameters $(T/\Theta)^2$ and γ^2. The method of successive approximations yields[2)]

$$\hat{K}\frac{1}{f_0'}\hat{K}\varphi^{(0)}=0, \quad (11)$$

$$\hat{K}\frac{1}{f_0'}\hat{K}\varphi^{(1)}=\left(\frac{T}{\Theta}\right)^2\left(\hat{K}\frac{1}{f_0'}\hat{Q}+\hat{Q}\frac{1}{f_0'}\hat{K}\right)\varphi^{(0)}, \quad (11')$$

$$\hat{K}\frac{1}{f_0'}\hat{K}\varphi^{(2)}=\left(\frac{T}{\Theta}\right)^2\left(\hat{K}\frac{1}{f_0'}\hat{Q}+\hat{Q}\frac{1}{f_0'}\hat{K}\right)\varphi^{(1)}+\left(\frac{T}{\Theta}\right)^2\hat{Q}\cdot 1-\left(\frac{T}{\Theta}\right)^4\hat{Q}\frac{1}{f_0'}\hat{Q}\varphi^{(0)}-\gamma^2f_0'\varphi^{(0)}. \quad (11'')$$

As shown in the appendix, the equation $\hat{K}\varphi = 0$ has a unique solution φ = const. It is seen from (11) that $\hat{K}\varphi^{(0)} = f_0'C$. Inasmuch as the kernel $K_{\xi\xi'}$ is symmetrical, this equation has a solution only when C = 0. Thus, $K\varphi^{(0)} = 0$ and consequently $\varphi^{(0)}$ = const.

Now (11′) takes the form

$$\hat{K}\frac{1}{f_0'}\left[\hat{K}\varphi^{(1)}-\left(\frac{T}{\Theta}\right)^2\varphi^{(0)}\hat{Q}\cdot 1\right]=0$$

meaning that

$$\hat{K}\varphi^{(1)}=\left(\frac{T}{\Theta}\right)^2\varphi^{(0)}\hat{Q}\cdot 1+f_0'C.$$

Determining the constant C from the condition that this equation have a solution, and substituting $\hat{K}\varphi^{(1)}$ in (11″),

*$\left[\frac{\partial f}{\partial \mathbf{p}}\mathbf{p}\right]=\frac{\partial f}{\partial \mathbf{p}}\times\mathbf{p}$

2)If we inclue the terms $(T/\Theta)^2Q\cdot 1$ and $\gamma^2 f_0'\varphi^{(0)}$ in (11′), then it can be readily shown that (11″) cannot be solved when $\gamma \lesssim (T/\Theta)^2$.

we get

$$\hat{K}\frac{1}{f_0'}\hat{K}\varphi^{(2)} = \left(\frac{T}{\Theta}\right)^2 K\frac{1}{f_0'}\hat{Q}\varphi^{(1)} + \left(\frac{T}{\Theta}\right)^2 \hat{Q}\cdot 1 + \left(\frac{T}{\Theta}\right)^4 \varphi^{(0)}\langle\hat{Q}\cdot 1\rangle - \gamma^2 f_0'\varphi^{(0)}. \tag{12}$$

The angle brackets denote here integration with respect to ξ from $-\infty$ to ∞. The condition for the solvability of (12) yields ($\langle f_0'\rangle = -1$)

$$\varphi^{(0)} = -\frac{(T/\Theta)^2\langle\hat{Q}\cdot 1\rangle}{(T/\Theta)^4\langle\hat{Q}\cdot 1\rangle^2 + \gamma^2}. \tag{13}$$

When $\gamma \gg 1$, the quantity $\varphi^{(0)}$ can already be obtained from the zeroth-approximation equation and is given by

$$\varphi^{(0)} = -\frac{1}{\gamma^2}\left(\frac{T}{\Theta}\right)^2\frac{\hat{Q}\cdot 1}{f_0'}. \tag{14}$$

Thus, we have found the function $f^{(1)}$, and the function $N^{(1)}$ can be simply expressed in its terms from Eq. (4′).

Let us multiply, finally, (5) by p and (5′) by q, integrate, and add. The terms containing $f^{(2)}$ and $N^{(2)}$ then drop out, by virtue of the conservation of the total momentum in normal collisions and by virtue of the fact that the current component perpendicular to the surface vanishes. The result can be represented in the form

$$\frac{e}{m}\mathbf{E} = -\nu\frac{\partial^2\mathbf{u}}{\partial z^2} + \frac{\mathbf{u}}{\tau^U}, \tag{15}$$

$$\nu \approx \tau_0 v_F^2(\Theta/T)^5\langle\hat{Q}\cdot 1\rangle^{-1}\eta, \tag{16}$$

$$\eta = \frac{(T/\Theta)^4\langle\hat{Q}\cdot 1\rangle^2}{(T/\Theta)^4\langle\hat{Q}\cdot 1\rangle^2 + \gamma^2}, \quad \gamma \ll 1, \tag{17}$$

$$\eta = \left(\frac{T}{\Theta}\right)^4\frac{\langle\hat{Q}\cdot 1\rangle^2}{\gamma^2} + \left(\frac{T}{\Theta}\right)^8\langle\hat{Q}\cdot 1\rangle\,\zeta(2), \quad \gamma \gg 1. \tag{17′}$$

Here τ^U is the effective free path time of the electron relative to collisions with loss of quasimomentum, ν has the meaning of the kinematic viscosity of the electron-phonon gas, and ν is the Riemann ζ function. We note that the second term in (17′) is the result of the term $s_z\partial N^{(1)}/\partial z$ and is connected with the finite free path length of the phonons. The solution of (15) should satisfy the condition $\mathbf{u} = 0$ on the boundaries of the sample, corresponding to diffuse scattering of the electrons.

It is easy to verify that the density of the electron current is $j = neu$, where $n = (8/3)\pi(p_F/h)^3$ is the electron density. In calculating the resistance, this expression should be averaged over the cross section of the sample. Representing the coefficient of electric conductivity σ in the usual form (see (1)), we obtain for l^{eff} the following expression:

$$l^{\text{eff}} = l^U\left\{1 - \frac{\text{th}\,w}{w}\right\}, \tag{18}$$

where $w = (1/2)d(\nu\tau^U)^{-1/2}$, $l^U = \tau^U v_F$.

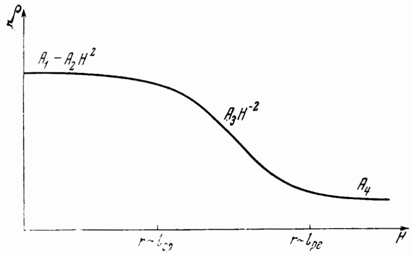
The case $w \gg 1$ corresponds to a bulky sample ($l^{\text{eff}} \approx l^U$). On the other hand, in the case when $w \ll 1$, we have

$$l^{\text{eff}} \approx \frac{d^2}{l_{ep}}\left[1 + \left(\frac{l_{ep}}{r}\right)^2\right], \quad r \gg l_{ep}, \tag{19}$$

$$l^{\text{eff}} \approx (d/r)^2 l_{ep}, \quad l_{ep} \gg r \gg l_{pe}, \tag{19′}$$

$$l^{\text{eff}} \approx (d/l_{pe})^2 l_{ep}, \quad r \ll l_{pe}. \tag{19″}$$

Here $l_{ep} = l_0\langle Q\cdot 1\rangle^{-1}(\Theta/T)^5$ coincides with the electron phonon free path which enters in the Bloch theory of electric conductivity, and $l_{pe} = l_0[\langle Q\cdot 1\rangle\zeta(2)]^{-1/2}\Theta/T$, $l_0 = \tau_0 v_F$. We note that these are precisely the results predicted earlier on the basis of intuitive physical considerations (with the exception, of course, of the correction term in (19)).



Using relation (6), we can also easily find the Hall emf

$$\Phi = \int_0^d E_z\,dz = \frac{H}{c}\int_0^d u_y\,dz = \frac{d}{r}l^{\text{eff}}E_y. \tag{20}$$

We note that Φ is proportional to the cube of the sample thickness.

4. We have assumed earlier for simplicity that the dispersion laws of the electrons and phonons are isotropic. However, it is clear from physical considerations (see Sec. 2) that the results (19)–(19″) are valid in order of magnitude for arbitrary closed Fermi surfaces and for an arbitrary phonon dispersion.

In the case of open Fermi surfaces the situation becomes more complicated. It is necessary first to take into account the Umklapp processes in electron-phonon collisions. As can be readily verified from (3) and (3′), the drift arises only in directions that are perpendicular to all the Umklapp processes and parallel to the surface of the sample. If such directions exist and the vector **E** is not orthogonal to them, then the hydrodynamic mechanism is possible.

On the other hand, in the presence of trajectories that are open or strongly elongated in the direction towards the boundaries, there is a momentum diffusion mechanism. It is easy to understand that in this case the magnetic field has no influence at all on the qualitative results, and $l^{\text{eff}} \approx d$ when $d \lesssim l_{ep}$ or $l^{\text{eff}} \approx d^2/l_{ep}$ when $d \gg l_{ep}$ (of course, if the contribution of the open or strongly elongated trajectories is not small). To illustrate the obtained results, the figure shows a plot of $\rho(H)$. We see from the figure that ρ decreases with increasing H in the region $r \ll d$, whereas in the case of the ordinary electric conductivity mechanism ρ is either constant in this region, or increases with the field (see, for example,[3]). Notice should also be taken of the strong dependence of the coefficients A on the temperature. Using formulas (19), we readily see that $A_1 \sim T^{-5}$, $A_2 \sim T^{-15}$, $A_3 \sim T^5$, and $A_4 \sim T^3$, all the coefficients being proportional to d^{-2}.

Let us write out, finally, the limitations on the thickness of the sample in the cases (19), (19′), and (19″), respectively:

$$l_{ep} \ll d \ll l_{ep}(l^U/l_{ep})^{1/2},$$
$$r \ll d \ll r(l^U/l_{ep})^{1/2}, \quad l_{pe} \ll d \ll l_{pe}(l^U/l_{ep})^{1/2}.$$

It is seen from these inequalities that in order for the hydrodynamic mechanism of electric conductivity to exist the condition $l^U \gg l_{ep}$ is necessary to an equal degree in all cases, and therefore the requirements with respect to the purity of the sample and the temperature do not change in the magnetic field. The possibility of performing the measurements on very thin samples, however, offers certain advantages, since in a sufficiently strong magnetic field the cases (19) and (19″) are possible even when $d \ll l_{ep}$. We note also the possibility of directly determining the free path of the phonons on the electrons l_{pe} (cf. (19′)). To be sure, it is not clear whether the condition $r \ll l_{pe}$ is realized for typical metals in experimentally attainable magnetic fields.

APPENDIX

We shall prove that the equation

$$\hat{K}\varphi = 0$$

has a unique solution φ = const.

Apart from inessential numerical coefficients of the order of unity, the operator $\hat{K}$ is given by[1]

$$K_{\xi\xi'} = \delta(\xi - \xi') \int_{-\infty}^{\infty} R_{\xi\xi''}\, d\xi'' - R_{\xi\xi'} + S_{\xi\xi'},$$

where

$$R_{\xi\xi'} = (\xi - \xi')^2 f_0(\xi) f_0(\xi') \, |e^{-\xi} - e^{-\xi'}|^{-1},$$

$$S_{\xi\xi'} = \exp(\xi + \xi') f_0(\xi) f_0(\xi') \int_{-\infty}^{\infty} |z| f_0(\xi + z) \times [e^z f_0(\xi' + z) - f_0'(\xi' - z)]\, dz.$$

We note that the kernels R and S are symmetrical and

$$R_{\xi\xi'} \geqslant 0, \quad \int_{-\infty}^{\infty} S_{\xi\xi'}\, d\xi' = 0$$

Let us consider the expression $\langle \varphi \hat{K} \varphi \rangle$. It is easy to show that it can be reduced to the form

$$\langle \varphi \hat{K} \varphi \rangle = \int\int_{-\infty}^{\infty} K_{\xi\xi'} \varphi(\xi) \varphi(\xi')\, d\xi\, d\xi'$$
$$= \frac{1}{2} \left\{ \int\int_{-\infty}^{\infty} R_{\xi\xi'} [\varphi(\xi) - \varphi(\xi')]^2\, d\xi\, d\xi' + \int_{-\infty}^{\infty} dz\, [T(z) - T(-z)]^2 \right\},$$

where

$$T(z) = |z|^{1/2} e^{-z/2} \int_{-\infty}^{\infty} \varphi(x) e^x f_0(x) f_0(x + z)\, dx.$$

It follows therefore that $\langle \varphi \hat{K} \varphi \rangle$ vanishes only when φ = const. This proves the statement made above.

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Translated by J. G. Adashko
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