NONLOCALIZABLE THEORIES AND THE "ZERO-CHARGE" PROBLEM

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A connection is established between the behavior of the interaction at high energy and the requirement of localizability of the theory (according to A. Jaffe[7], Meĭman[8], or Guttinger[9,16]). The familiar Lehmann–Simanzyk–Zimmermann theorem concerning the decrease of the vertex function is generalized to include the case of localizable theories. The vanishing of the renormalized charge in the known models or in the approximate solutions of the local quantum field theory equations turns out to be due to violation of the necessary conditions established by this theorem. The possibility of nonlocalizable theories in the framework of the dispersion approach (with conservation of the analytical and spectral properties) is discussed. The overcoming of the "zero charge" difficulty is connected with a transition to the nonlocalizable theories. This transition turns out to be equivalent to a redefinition of the T-product by means of infinite series of derivatives of the δ-function, such that the increase of matrix elements at infinity in momentum space is exponential. The relation between nonlocalizability and nonlocality is considered. Nonlocalizable solutions are found for the simple models for which there is a "zero charge" problem in the local theory. The degree of nonlocality of the solutions is established, as well as their relation to the renormalized series. It is found, in particular, that the renormalized series of perturbation theory may be an asymptotic expansion of the exact solution.

1. INTRODUCTION

ONE of the main difficulties of the renormalizable Lagrangian quantum field theory is the problem of the vanishing of the renormalized charge in the approximate solution (the so-called summation of the principal terms) of the equations of quantum field theory[1] or in the exact solutions of the models[2]. So far, the question of overcoming the difficulties of the "zero charge" has been connected with the following three principal possibilities.

1. It is possible that the "zero charge" in the approximate solutions of the exact equations is the consequence of the incorrectness of the approximation: the expansion parameter $g^2 \ln(p^2/m^2)$ ceases to be small in an appreciable region of large p^2. Moreover, the exact solution can be nonanalytic in the charge[3]. The "zero charge" in the models can then be connected with the insufficient completeness of the models (the absence, for example, of crossing symmetry).

2. It is necessary to choose a different representation of the commutation relations; for example, it is necessary to redefine the vacuum, since the initial vacuum is unstable (the best known example is provided by superconductivity theory). Then a situation of "broken symmetry" can arise—the new vacuum will be, for example, translationally noninvariant[3].

3. It is possible that the "zero charge" is a consequence of the local character of the interactions in the existing theories, while the real Hamiltonians of the interaction should be principally nonlocal[5]. This is related to attempts of introducing a curved momentum space that eliminates the ultraviolet divergences[6].

It has now become clear that all these possibilities are conceived within the framework of the so-called "strictly localized field theories" (according to Jaffe[7]), or else within the framework of several more general theories (localized according to Meĭman[8] and Guttinger[9]). We shall call such theories simply localizable.

The concept of localizability is one of the latest most important accomplishments of the axiomatic approach to the construction of a theory of quantized fields. It is known that the field operators $A(x)$ are by themselves not operators in the Hilbert space of states, and represent generalized operator functions over a certain space of smooth fundamental functions $\{f(x)\}$; in other words, the meaning of operators is possessed by

$$A(f) = \int A(x) f(x)\, dx.$$

Until recently, the spaces of the fundamental functions $\{f(x)\}$ were chosen essentially from considerations of convenience, or by analogy with the results of the Lagrangian perturbation theory. It was recognized only recently that the question of the choice of the fundamental functions and, accordingly, of the permissible classes of generalized functions, should be solved by starting from the basic principles of the quantum field theory. It has become clear that the postulates of spectrality and primarily locality (microcausality) of the theory impose on the assumed classes of generalized functions a powerful limitation—the requirement of localizability[7-9]. Theories satisfying this requirement will be called localizable.

The requirement of localizability denotes that the concept of a point in the coordinate x-space is meaningful in the theory. This is necessary for the fomulation of the locality (microcausality) principle of the theory. Only in a localizable theory does such an important concept of quantum field theory as the T-product of local operators have a rigorous meaning. Obviously, any local (microcausal) theory is of necessity localizable, but the converse is not true, a theory can be localizable and nonlocal.

Theories in which the encountered generalized functions do not satisfy the localizability theory will be called nonlocalizable theories. Nonlocalizable theories are essentially nonlocal, since the concept of a pointlike event becomes meaningless in them. The requirement of localizability denotes, in particular the Green's functions in momentum space increase at infinity with respect to each of the momenta more slowly than any linear exponential:

$$|G_n(p_1, \ldots, p_n)| < \exp \varepsilon \|p_i\|, \quad \|p_i\| = \sqrt{(p_i^0)^2 + (\mathbf{p}_i)^2} \to \infty,$$

where $\epsilon > 0$ and is arbitrarily small. The known spectral representations for the Green's functions lead to analogous limitations on the growth of the latter in the complex momentum plane, too.

We shall propose below one more possibility of overcoming the "zero charge" difficulty. Its fundamental difference from the already mentioned possibilities is connected with the fact that we go beyond the framework of the localizable theories to nonlocalizable theories, where an arbitrary growth at infinity with respect to the momenta is admissible. This possibility is based on the so-called dispersion approach, i.e., on an analysis of the exact analytic and spectral properties of the theory. It is known that "zero charge" is closely connected with the violation of the latter: unphysical poles connected with "ghost" states with negative norm and imaginary mass appear in the Green's functions. The presence or the absence of unphysical singularities is directly connected, in turn, with the behavior of the interaction at high energies. In its purest form, this connection appears in the analysis of the single-particle Green's function of a boson. Lehmann, Symanzik, and Zimmermann have shown[10] that in any field theory in which the postulates of definite metric, relativistic invariance, and spectrality are satisfied and in which non-subtractional dispersion relations are valid for the single-particle Green's functions of a scalar boson $D^c(p^2)$, there should be satisfied the condition

$$\frac{1}{\pi}\int_a^\infty dp^2 \frac{F(p^2)}{(p^2-\mu^2)^2} \leqslant 1. \tag{1}$$

Here μ is the boson mass, $a > \mu^2$ the lower limit of the region where the Kallen-Lehmann spectral function $\rho(p^2)$ differs from zero, and $F(p^2)$ the reduced spectral function, determined from the equality[10]

$$\rho(p^2) \equiv \operatorname{Im} D^c(p^2) = |D^c(p^2)|^2 F(p^2), \quad p^2 \geqslant a. \tag{2}$$

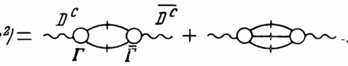
The function $F(p^2)$ is the sum of the positive contributions of n-particle intermediate states:

$$F(p^2) = \sum_n F_n(p^2), \quad F_n(p^2) \geqslant 0. \tag{3}$$

The function $F_n(p^2)$ is proportional to the product of the square of the modulus of the corresponding n-particle vertex by the phase volume of n particles with total four-momentum p_μ. In the case of pseudoscalar mesodynamics, for example,

$$F_2(p^2) = \frac{\theta(p^2-4m^2)}{8\pi} p^2 \sqrt{1-\frac{4m^2}{p^2}} |\Gamma_5(p^2)|^2, \tag{4}$$

where $\Gamma_5(p^2)$—ordinary vertex function and m—nucleon mass. Formulas (2) and (3) can be easily explained by means of the graphical representation of the spectral function

$\rho(p^2) =$ 

The condition (1) denotes, in particular, that the reduced spectral function $F(p^2)$, which is seen to be closely connected with the vertex functions and characterizes to a certain degree the interaction of high energies, should satisfy definite limitations on its growth.

Violation of the condition (1) leads to the appearance in $D^c(p^2)$ of false poles corresponding to the vanishing of the renormalized charge or violation of the definiteness of the metric. However, the dispersion approach allows us to establish a deeper connection, namely a connection between the behavior of the interaction at high energies and the requirement of the localizability of the theory. We shall show, first, that condition (1) is more general. Namely, if the interaction (in terms of the function $F(p^2)$) decreases insufficiently rapidly at high energies then, for the sake of internal consistency, the condition (1) must be satisfied in all the localizable theories. Second, we shall show that condition (1) is not necessary if one goes beyond the framework of the localizable theories. It is precisely with the latter circumstance that the proposed possibility of eliminating the "zero charge" difficulty is connected.

2. "ZERO CHARGE" AS THE CONSEQUENCE OF THE NON-DECREASING INTERACTION AT HIGH ENERGIES IN LOCALIZABLE THEORIES

Let us consider, within the framework of the localizable theory, the single-particle Green's function of a boson, which for simplicity is assumed to be scalar with minimum mass μ, corresponding to a Heisenberg local field $\varphi(x)$:

$$D^c(x-y) = i\langle 0|T(\varphi(x)\varphi(y))|0\rangle.$$

Its Fourier transform $D^c(p^2)$ in localizable field theory[7,9] is the upper bound, on the real axis, of the analytic function $D^c(z)$, which possesses the following properties:

A. $D^c(z)$ is analytic in the complex plane with a cut along the real axis from $a > \mu^2$ to $+\infty$, and has a pole at the point $z = \mu^2$ with a residue equal to -1.

B. $D^c(p^2) = \lim D^c(z)$ ($z \to p^2 + i\epsilon$) is real when $p^2 < a$ and $p^2 \neq \mu^2$, and is complex conjugate on opposite edges of the cut:

$$D^c(p^2 - i\varepsilon) = \overline{D^c(p^2)}, \quad p^2 \geqslant a.$$

C. The jump of $D^c(z)$ on the cut $p^2 \geq a$ yields

$$[D^c(p^2+i\varepsilon) - D^c(p^2-i\varepsilon)]/2i = \operatorname{Im} D^c(p^2) = \rho(p^2) \geqslant 0,$$

where $\rho(p^2)$ is the Kallen-Lehmann spectral function.

D. From the requirement of rigorous localizability follows a limitation on the growth of $D^c(z)$ on the large circle:

$$|D^c(z)| < \exp \varepsilon \sqrt{|z|}, \quad |z| \to \infty,$$

where $\epsilon > 0$ is arbitrarily small.

We assume now that the function $\rho(p^2)$, with the exception of a set of points of zero measure[1)], can be represented in the form (2):

$$\rho(p^2) = |D^c(p^2)|^2 F(p^2), \quad p^2 \geqslant a,$$

where $F(p^2) \geq 0$ is the reduced spectral function. The possibility of the representation (2) imposes on $D^c(p^2)$, as a generalized function, additional limitations, namely, $D^c(p^2)$ must be defined almost everywhere as an ordinary function. We can now formulate the following theorem.

Theorem. In a localizable theory, if

$$F(p^2) \geqslant 1/p^2 f(p^2), \quad p^2 \to \infty, \tag{5}$$

where $f(p^2) \to 0$ as $p^2 \to \infty$, it is necessary to satisfy the condition (1):

$$\frac{1}{\pi}\int_a^\infty dp^2 \frac{F(p^2)}{(p^2-\mu^2)^2} \leqslant 1.$$

In fact, by virtue of property C of the function $D^c(p^2)$, the representation (2), and the conditions of theorem (5), we have the chain of inequalities

$$|D^c(p^2)| \geqslant \operatorname{Im} D^c(p^2) = |D^c(p^2)|^2 F(p^2) \geqslant |D^c(p^2)|^2 / p^2 f(p^2),$$

whence $|D^c(p^2)|/p^2 \leq f(p^2) \to 0$ as $p^2 \to \infty$. It follows from the property B that the function $D^c(z)/z$ decreases on both edges of the cut as $|z| \to \infty$. By virtue of the property D of the function $D^c(z)$, the function $D^c(z)/z$ satisfies the Phragmen-Lindelof theorem[11], according to which $D^c(z)/z \to 0$ also on the large circle. Consequently, we can write for $D^c(z)/z$ the non-subtraction dispersion relations:

$$D^c(z) = D^c(0) + \frac{z}{\mu^2}\frac{1}{\mu^2 - z} + \frac{z}{\pi}\int_a^\infty dp^2 \frac{\rho(p^2)}{p^2(p^2-z)}, \tag{6}$$

where $D^c(0)$ is a real constant. By virtue of (6), $D^c(z)$ is an R-function: $\operatorname{Im} D^c(z) = \lambda(z) \operatorname{Im} z$, where $\lambda(z) > 0$. There is a known general representation for the R function satisfying (6) and (2) (see, for example,[12,13]):

$$-[D^c(z)]^{-1} = A(z-\mu^2) + \frac{z-\mu^2}{\pi}\int_a^\infty dp^2 \frac{F(p^2)}{(p^2-\mu^2)(p^2-z)} + R(z). \tag{7}$$

Here $A \geq 0$, and

$$R(z) = (z-\mu^2)\sum_n \frac{R_n}{\omega_n(\omega_n+\mu^2-z)} + \frac{z-\mu^2}{\pi}\int_a^\infty \frac{d\sigma(p^2)}{(p^2-\mu^2)(p^2-z)}, \tag{8}$$

where the arbitrary constant $R_n \geq 0$ and $\omega_n > 0$, with the exception perhaps of one of the ω_0, and with $\Sigma(R_n/\omega_n^2) < \infty$; $\sigma(p^2)$ is a non-decreasing continuous function, the derivative of which vanishes almost everywhere, and with this

$$\int^\infty \frac{d\sigma(p^2)}{(p^2)^2} < \infty.$$

The function R(z) reflects the so-called CDD arbitrariness (Castillejo—Dalitz—Dyson) in the solution (6) and (2). Obviously, the necessary condition for the representation (7), and consequently the necessary condition for the existence of a solution of (6) and (2) is

$$\int^\infty dp^2 \frac{F(p^2)}{(p^2)^2} < \infty. \tag{9}$$

So far we disregarded the requirement that the residue of $D^c(z)$ at the point $z = \mu^2$ be equal to -1. This requirement leads to the additional condition

$$A + \frac{1}{\pi}\int_a^\infty dp^2 \frac{F(p^2)}{(p^2-\mu^2)^2} + \sum_n \frac{R_n}{\omega_n^2} + \frac{1}{\pi}\int_a^\infty \frac{d\sigma(p^2)}{(p^2-\mu^2)^2} = 1. \tag{10}$$

The condition (10) is necessary and sufficient for the existence of the solution of (6) and (2). By virtue of the non-negativity of the individual terms in (10), we should have

$$\frac{1}{\pi}\int_a^\infty dp^2 \frac{F(p^2)}{(p^2-\mu^2)^2} \leqslant 1. \tag{11}$$

thus concluding the proof of the theorem.

Substituting the expression for A from (10) in (7), we obtain finally for $D^c(z)$

$$[D^c(z)]^{-1} = (\mu^2 - z)\left[1 + \frac{z-\mu^2}{\pi}\int_a^\infty dp^2 \frac{F(p^2)}{(p^2-\mu^2)^2(p^2-z)} + \tilde{R}(z)\right],$$

where

$$\tilde{R}(z) = (z-\mu^2)\sum_n \frac{R_n}{\omega_n^2(\omega_n+\mu^2-z)} + \frac{z-\mu^2}{\pi}\int_a \frac{d\sigma(p^2)}{(p^2-\mu^2)^2(p^2-z)}. \tag{12}$$

In order for the expression in the right side of (11) to be meaningful, it is sufficient to have convergence of the integral

$$\int^\infty dp^2 F(p^2)/(p^2)^3.$$

However, the condition (1) is here, in the form (11) for $D^c(z)$, the condition that $D^c(z)$ have no false poles that lead to the vanishing of the renormalized charge.

The vanishing of the renormalized charge in the known approximate solutions of the exact equations of local quantum field theory or in the exact solutions of the models turns out to be the consequence of the violation of the necessary requirements established by this theorem. Namely, in typical "zero charge" cases

$$F(p^2) \cong F_2(p^2) \approx p^2|\Gamma(p^2)|^2 \text{ when } p^2 \to \infty,$$

where $\Gamma(p^2)$ is the vertex function, with $\Gamma(p^2) \to g$, the renormalized charge, when $p^2 \to \infty$. Condition (5) of the theorem is obviously satisfied, but when $g \neq 0$ the integral in the left side of (1) diverges logarithmically, so that even the necessary condition (9) is not satisfied. The only possibility of satisfying (1) is to put $g = 0$, i.e., we have here a "zero charge."

In proving the theorem we leaned essentially on the condition D for $D^c(z)$, which follows directly from the requirement of localizability of the theory. In this sense we can state that the "zero charge" is a consequence of the requirements of the localizability of the theory if the interaction does not decrease at high energies (in the sense of the condition (5)).

3. DEPARTURE FROM THE FRAMEWORK OF THE LOCALIZABLE THEORIES IN THE DISPERSION APPROACH

The last remark of Sec. 2 indicates simultaneously a way out of the "zero charge" situation when $F(p^2)$ satisfies the condition (5) but not (1). If we forego the

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1) When $p^2 > \mu^2$, the function $D^c(p^2)$ can contain poles, and accordingly $\rho(p^2)$ can contain δ functions. They should be separated from the general $\rho(p^2)$. Allowance for them does not influence the main result. Moreover, the inequality (1) can become only stronger, since in that case, certain R_n in the condition (10) presented below are known to be different from zero.

limitation D on the growth of $D^c(z)$ in the complex plane, then the theorem ceases to be valid. By lifting the limitation D, we inevitably go beyond the framework of the localizable theories and go over to nonlocalizable ones. Of course, in this case many concepts of the local theory become meaningless, for example, the concept of the T-product, and consequently also those analytic properties of the Green's function which follow from it. In particular, it is apparently no longer possible to prove, by starting from the general principles of the theory, that the Green's functions are the limiting values on the real axis of analytic functions.

However, we can ultimately adhere to the purely axiomatic point of view of the dispersion approach (which is known in S-matrix theory as the "principle of maximum analyticity") and postulate definite analytic properties of the Green's function, retaining them invariant compared with the localizable theories in any finite part of the complex plane, but without limiting the degree of their growth on the large circle.

The problem of the dispersion approach now consists of the following:

1) Finding general representations for the functions satisfying the postulated analytic and spectral properties.

2) Analyzing and physically interpreting the arbitrariness that arises in the theory.

3) Establishing the relation of the obtained solutions to the localizability and locality of the theory and to the renormalized perturbation-theory series.

It may turn out here that there exist two classes of solutions. The first includes solutions corresponding to the localizable theory, while the second corresponds to the nonlocalizable theories with different degrees of nonlocality. For example, the nonlocalizable solution (14) presented below for $D^c(z)$ can be written also in the case when condition (1) is satisfied; thus, the nonlocalizable solution exists here together with localizable ones (corresponding to $\varphi(z) \equiv 1$). An analogous situation can arise also for the scattering amplitude (this possibility must apparently be borne in mind in the S-matrix method with "principle of maximum analyticity"). In particular, the same dispersion relations for, say, the forward scattering amplitude can be written with an infinite number of subtractions, corresponding to the nonlocalizable theory, but in such a way that this deviation from locality has no effect in the presently attainable region of energies. The problem of the choice between the localizable and nonlocalizable solutions reduces in this case to the question of the character of the cross sections at high energies (in the case of nonlocalizable solutions, the cross section may, for example, decrease exponentially or oscillate rapidly), and should, as always, be decided experimentally.

Let us consider, within the framework of the proposed approach, the boson Green's function $D^c(z)$. Let $D^c(z)$ satisfy the conditions A, B, and C but not D, and let $F(p^2)$ satisfy the condition (5) but not (1) [2)]. Since (1) is not satisfied. there exists no localizable solution for $D^c(z)$. However, since the limitation D has been lifted, we can point immediately to an entire class of such functions which increase on the large circles more rapidly than $\exp\sqrt{|z|}$ (although this class does not exhaust possibly all the solutions). Namely, let

$$D^c(z) = \varphi(z)\Delta^c(z), \tag{13}$$

where $\Delta^c(z) \to 0$ as $|z| \to \infty$, and $\varphi(z)$ is an entire function satisfying the following conditions:

a) $\varphi(z) \equiv F((z-\mu^2)/\Lambda^2)$, $f(0) = 1$—entire function of order of growth $\geq 1/2$ of nonzero type;

b) $\varphi(p^2)$ is real on the real axis:

$$\varphi(p^2) \geqslant 0 \text{ if } p^2 \geqslant a;$$

c) $\varphi(p^2) \to 0$ as $p^2 \to \infty$, so that

$$\frac{1}{\pi}\int_a^\infty dp^2 \frac{F(p^2)\varphi(p^2)}{(p^2-\mu^2)^2} \leqslant 1.$$

Then we obtain for $\Delta^c(z)$ a problem which is perfectly analogous to the problem for $D^c(z)$ in the case of the localizable theory, except that $F(p^2)$ is replaced by $F(p^2)\varphi(p^2)$. We finally have for $D^c(z)$

$$D^c(z) = f\left(\frac{z-\mu^2}{\Lambda^2}\right)(\mu^2-z)^{-1}$$

$$\times\left[1 + \frac{z-\mu^2}{\pi}\int_a^\infty dp^2 \frac{F(p^2) f((p^2-\mu^2)/\Lambda^2)}{(p^2-\mu^2)^2(p^2-z)} + \tilde{R}(z)\right]^{-1}, \tag{14}$$

where R(z) is given as before by expression (12). However, unlike the localizable theory (see (11)), the solution contains in this case an additional arbitrariness in the entire function satisfying the conditions a)–c). In view of this, the solution (14) for $D^c(z)$ increases in the complex z plane at least like $\exp\sqrt{|z|}$, and corresponds consequently to the nonlocalizable theories.

On the other hand, it is obvious that it is precisely because of this function that the solution has no false pole and does not experience the "zero charge" difficulty, even if $F(p^2)$ increases when $p^2 \to \infty$. Thus, the transition to the nonlocalizable theories (while retaining the analytic and spectral properties) makes it possible to overcome the "zero charge" difficulty where this difficulty is unavoidable in the case of the localizable theory; in the nonlocalizable theory $F(p^2)$ can increase arbitrarily rapidly.

4. NONLOCALIZABILITY AND NONLOCALITY

It is now necessary to clarify the relation between the obtained nonlocalizable solutions and nonlocality. The connection with the renormalized series will be illustrated below with the aid of a simple model.

Although the nonlocalizable theory is in essence nonlocal, one can speak of the character of nonlocality of any particular variant of the solution. This character is determined by the degree of growth of the Green's functions in the complex momentum plane. In the considered case of the boson Green's function, the character of the nonlocality is determined by the degree of growth of the function $\varphi(z)$. In order to explain this, let us see what corresponds to the solutions (14) obtained for $D^c(z)$, from the point of view of the T-product in x-space. As already noted above, strictly speaking, the concept of the T-product loses its meaning in nonlocalizable theories. We shall therefore take the T-product

2) As already noted above, all the formulas that follow are valid also when condition (1) is satisfied independently of condition (5), so that for $D^c(z)$ there exists also a localizable solution (11).

to mean the coordinate Fourier transform $G_n(x_1, \ldots, x_n)$ of the Green's functions.

It is easy to see that the following representation is valid for $D^c(z)$ in the form (13) and (14)

$$D^c(z) = \varphi(z) \int_0^\infty dp^2 \frac{\rho(p^2)}{\varphi(p^2)(p^2 - z)}. \tag{15}$$

Inasmuch as in this case $\rho(p^2) \to 0$ as $p^2 \to \infty$, the latter can be rewritten in the form

$$D^c(z) = \int_0^\infty dp^2 \frac{\rho(p^2)}{p^2 - z} + \Lambda(z), \tag{16}$$

where

$$\Lambda(z) = \int_0^\infty dp^2 \frac{\varphi(z) - \varphi(p^2)}{\varphi(p^2)(p^2 - z)} \rho(p^2) \tag{17}$$

is an entire function of z of order of growth $\geq 1/2$.

Relations (15) and (16) have the form of the Kallen-Lehmann spectral representations, but with an infinite number of subtractions: $D^c(p^2)$ is redefined with the aid of $\Lambda(p^2)$—an entire function in p^2. But the latter is equivalent in x-space to the redefinition of the T-product with the aid of the "quasilocal term" $\Lambda(x)$, which is an infinite series in the derivatives of the function:

$$\Lambda(x) = \sum_n \frac{c_n}{(2n)!} \Box^n \delta(x), \tag{18}$$

and the coefficients c_n violate the localizability condition, since[8,9]

$$|c_n|^{1/2n} \not\to 0 \text{ when } n \to \infty. \tag{19}$$

By virtue of (19), $\Lambda(x)$ is no longer, strictly speaking, a quasilocal term, i.e., it is not a generalized function concentrated at the point x = 0, as should be the case in the local theory. Roughly speaking, $\Lambda(x)$ is "smeared out" in a finite region of space-time. The character of the nonlocality of the theory (since we are dealing only with the boson Green's function) is determined precisely by the character of the "smearing" of $\Lambda(x)$ [3)], and the latter is determined in turn by the order of the growth of the function $\varphi(z)$[14]. In particular, if $\varphi(z)$ is a function of growth order 1/2, so that

$$\overline{\lim_{n\to\infty}} |c_n|^{1/2n} = 1/\Lambda, \tag{20}$$

an example of such a function being

$$\varphi(z) = \left[\sin\left(\frac{1}{2}\sqrt{\frac{z-\mu^2}{\Lambda^2}}\right) \Big/ \sqrt{\frac{z-\mu^2}{\Lambda^2}}\right]^2,$$

then the violation of the locality is concentrated in the region (see[14])

$$|x^2| \leq l^2 = 1/\Lambda^2. \tag{21}$$

The quantity $l = 1/\Lambda$ thus plays the role of the nonlocality parameter and can be called an "elementary length." This can be explained in simpler fashion also in the general case; it is obvious that when $l = 1/\Lambda \to 0$, by virtue of the condition a) on $\varphi(z)$, we return formally in (14) to the case of the localizable theory. It is easy to see, however, that at the given $F(p^2)$, which does not satisfy (1), the condition c) that there be no false pole imposes a definite lower limit on l (if $F(p^2)$ satisfies (1), then $l = 0$ is permissible). Then the higher the degree of growth $F(p^2)$ as $p^2 \to \infty$, the smaller the lower limit of the permissible l.

In conclusion, we note the following curious fact. Since $\rho(p^2) \to 0$ as $p^2 \to \infty$ in the considered class of nonlocalizable solutions (and does not increase, as is customarily assumed[16]), it follows that

$$\langle 0|[\varphi(x)\varphi(y)]|0\rangle = \int D(x-y, \varkappa^2)\rho(\varkappa^2)\, d\varkappa^2 = 0$$

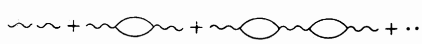
when $x \sim y$, although the theory is nonlocalizable!

Thus, the "zero charge" difficulty can be overcome by redefining the T-product with the aid of quasilocal terms containing δ-function derivatives of arbitrarily high order. The latter leads to an exponential growth of the matrix elements at infinity and to the departure from the framework of the localizable theories. In this sense, the renormalizable local theories with "zero charge" are more non-renormalizable and more nonlocal than is customarily assumed for the nonrenormalizable theories[16].

5. MODELS

We now illustrate the foregoing general considerations by means of very simple models.

1. Let us consider, for example, the model of chain summation

 (22)

for the meson Green's function $D^c(p^2)$ in the case of pseudoscalar mesodynamics with pseudoscalar coupling. In the dispersion formulation, the model reduces to the following series of approximations (see (3) and (4)):

$$F(p^2) = F_2(p^2) = \theta(p^2 - 4m^2) g^2 \frac{p^2}{8\pi} \sqrt{1 - 4m^2/p^2}, \tag{23}$$

where g is the renormalized charge and m the nucleon mass. Condition (5) is obviously satisfied. Under the localizability requirement, the formal solution can be obtained by the R-function method. It yields (see (11))

$$D^c(z) = (\mu^2 - z)^{-1} \left[1 + \frac{g^2}{8\pi^2}(z - \mu^2) \int_{4m^2}^\infty dp^2 \frac{p^2\sqrt{1 - 4m^2/p^2}}{(p^2 - \mu^2)^2 (p^2 - z)}\right]^{-1} \tag{24}$$

(we omit the CDD arbitrariness as being inessential), which coincides with the sum of the geometrical progression (22). However, since the condition (1) is patently violated, (24) has a false pole at $p^2 < \mu^2$. At smaller values of g^2, the position of this pole is given with logarithmic accuracy by the formula

$$p^2_{\text{pole}} = -m^2 \exp(8\pi^2/g^2) \tag{25}$$

(we assume here that $(4m^2 - \mu^2)/4m^2 \sim 1$).

The requirement that there be no false pole leads to the condition $g^2 = 0$—typical "zero charge."

If we forego the localizability condition (omitting the CDD arbitrariness), we obtain

$$D^c(z) = (\mu^2 - z)^{-1} f\left(\frac{z - \mu^2}{\Lambda^2}\right)\left[1 + \frac{g^2}{8\pi^2}(z - \mu^2) \times \int_{4m^2}^\infty dp^2 \frac{p^2\sqrt{1 - 4m^2/p^2}\, f[(p^2 - \mu^2)/\Lambda^2]}{(p^2 - \mu^2)^2 (p^2 - z)}\right]^{-1} \tag{26}$$

which is a solution that differs from the ordinary sum of

3)The latter can be explained, for example, by generalizing Bogolyubov's microcausality condition to nonlocalizable theories [15].

the geometric progression (24) in that it contains the entire function $f[(z-\mu^2)/\Lambda^2]$ satisfying the conditions a)–c). The condition c) of absence of a false pole,

$$\frac{g^2}{8\pi^2}\int_{4m^2}^{\infty} dp^2 \frac{p^2\sqrt{1-4m^2/p^2}}{(p^2-\mu^2)^2} f\left(\frac{p^2-\mu^2}{\Lambda^2}\right) \leqslant 1, \tag{27}$$

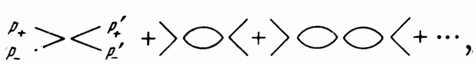
imposes a limitation on Λ^2 at a given g^2, and vice versa (the fact that Λ^2 cannot be arbitrarily large was already noted above). In particular, when $g^2 \ll 1$ and $(4m^2-\mu^2)/4m^2 \sim 1$ [4)] we have with logarithmic accuracy

$$l^2 = \frac{1}{\Lambda^2} \geqslant \frac{1}{\Lambda^2_{max}(g^2)} = \frac{1}{m^2}\exp\left(-\frac{8\pi^2}{g^2}\right). \tag{28}$$

It is curious that $\Lambda^2_{\max} = -p^2_{\text{pole}}$ in (25). Choosing $\Lambda^2 = \Lambda^2_{\max}(g^2)$, we obtain for $D^c(z)$ the "minimal nonlocal solution." This solution, obviously, has an essential singularity in the charge and can be expanded in an asymptotic perturbation-theory series, containing all the summable chains (22), so that from the point of view of the asymptotic expansion in g^2, we return to the case of local theory. The latter, of course, is connected with the fact that the corresponding elementary length $l^2 = l^2_{\min}(g^2) = 1/\Lambda^2_{\max}(g^2)$ vanishes identically in the expansion in g^2. The region of applicability of the asymptotic expansion, and accordingly, of the locality of the theory, is obviously $|p^2| \ll m^2\exp(8\pi^2/g^2)$, as is usually expected.

From the point of view of the latter remarks, the analogous situation takes place in the implementation of the "Redmondization" procedure[3] for $D^c(p^2)$. It is necessary, however, to emphasize the fundamental difference between the proposed method and "Redmondization." First, the present method denotes a transition to nonlocalizable theories, whereas "Redmondization" still leaves us in the class of localizable theories. Second, solutions of the type (14) are exact solutions of the problem, whereas "Redmondization" is equivalent to a redefinition of the vertex function, which corresponds, possibly, to a transition to a nonlocal Hamiltonian[18] (but within the framework of the localizable theory).

2. Finally, as already noted above, the need for going outside the framework of the localizable theories can arise also when scattering amplitudes are considered. For example, by the method proposed above for the boson Green's function, we can sum the annihilation chain



corresponding to the scattering amplitude

$$[\bar{u}(p_+)\gamma_5 u(p_-)][\bar{u}(p_-')\gamma_5 u(p_+')]T(s,t),$$
$$s=(p_++p_-)^2, \quad t=(p_+-p_+')^2 \tag{29}$$

in the four-Fermion interaction

$$\mathcal{L}_{B3} \sim Gj(x)\cdot j(x), \quad j(x)=\bar{\psi}(x)\gamma_5\psi(x). \tag{30}$$

The dispersion formulation of the problem for the invariant function $T(s,t) \equiv T(s)$ consists in this case of the following: find the function $T(z)$ ($T(s) = \lim_{\epsilon\to 0} T(s+i\epsilon)$, $T(s-i\epsilon) = \overline{T(s+i\epsilon)}$), which is analytic in the complex plane with a cut from $4m^2$ to $+\infty$, having on the cut the jump

$$\frac{T(s+i\varepsilon)-T(s-i\varepsilon)}{2i} = \operatorname{Im} T(s) = \frac{s}{8\pi}\sqrt{1-\frac{4m^2}{s}}\,|T(s)|^2 \tag{31}$$

and the limiting value

$$T(4m^2) = G. \tag{32}$$

By virtue of (31) we have

$$|T(s)| < \frac{8\pi}{s\sqrt{1-4m^2/s}} \to 0 \quad \text{as} \quad s\to+\infty. \tag{33}$$

If we remain in the class of localizable theories ($|T(z)| < \exp\epsilon\sqrt{|z|}$), then the following representation holds true for $T(z)$:

$$T(z) = G + \frac{z-4m^2}{\pi}\int_{4m^2}^{\infty} ds\frac{\alpha(s)|T(s)|^2}{(s-4m^2)(s-z)}, \quad \alpha(s)=\frac{s}{8\pi}\sqrt{1-\frac{4m^2}{s}}. \tag{34}$$

Just as in the case of the boson Green's function, Eq. (34) leads to the R-function problem, which has no solution, inasmuch as

$$\int^{\infty}\frac{\alpha(s)}{s^2}ds = \infty,$$

i.e., the necessary condition (9) is not satisfied. On the other hand, if we admit of an exponential growth of $T(z)$ in the complex plane, then the problem has a solution of the type

$$T(z) = Gf\left(\frac{z-4m^2}{\Lambda^2}\right) \times\left[1-G\frac{z-4m^2}{\pi}\int_{4m^2}^{\infty} ds\frac{\alpha(s)f((s-4m^2)/\Lambda^2)}{(s-4m^2)(s-z)}\right]^{-1} \tag{35}$$

(the CDD arbitrariness is immaterial), where $f((z-4m^2)/\Lambda^2)$ is an entire function of growth order $\geq 1/2$, positive, and decreasing along the positive semi-axis, and $f(0)=1$.

A distinction must be made here between two cases: $G>0$ and $G<0$. When $G>0$ no false pole can appear, and the only requirement on $f((z-4m^2)/\Lambda^2)$ is

$$\int^{\infty} f\left(\frac{s-4m^2}{\Lambda^2}\right)\frac{ds}{s} < \infty.$$

Accordingly, the nonlocality parameter $l = 1/\Lambda$ can be arbitrarily small.

When $G<0$, the conditions for the absence of a false pole will be

$$1+\frac{G}{\pi}\int_{4m^2}^{\infty}\frac{\alpha(s)f((s-4m^2)/\Lambda^2)}{s-4m^2}ds \geqslant 0. \tag{36}$$

When $G \ll 1/m^2$, we should have, accurate to G^2,

$$l^2 = \frac{1}{\Lambda^2} \geqslant \frac{|G|}{8\pi^2}\int_0^{\infty} f(x)\,dx. \tag{37}$$

Accordingly, the minimal elementary length is

$$l_{min}(G) \sim \sqrt{|G|}, \tag{38}$$

as is customarily expected in the theory of weak interactions.

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4)It is obvious that when $\mu^2 \to 4m^2$ we should have either $\Lambda^2 \to 0$ or $g^2 \to 0$. This case, from the point of view of limitations on g^2 in local theory, was discussed in [17].

cussions, fruitful remarks, and support.

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