"CLASSICAL" AND QUANTUM S-MATRIX IN THE BORN-INFELD TWO-DIMENSIONAL MODEL OF THE FIELD

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Submitted December 21, 1967

Zh. Eksp. Teor. Fiz. 54, 1806–1815 (June, 1968)

We consider the problem of finding the S matrix for a nonlinear scalar Born-Infeld field in two-dimensional space x, t. On the basis of an exact solution of the classical equation of this field we obtain for the problem of the scattering of two plane waves the "classical" S-matrix, which transforms the incident waves into outgoing waves. We show that in a definite approximation with respect to the "nonlinearity" parameter g, the quantum S-matrix coincides with the "classical" one. In this approximation, the S-matrix leads only to elastic scattering processes.

1. INTRODUCTION

A scalar nonlinear field of the Born-Infeld type [1,2] has in two-dimensional space x, t the following Lagrange function:

$$\mathcal{L} = \varkappa^2\{1 - \sqrt{1 + \varkappa^{-2}(\varphi_x^2 - \varphi_t^2)}\}, \tag{1}$$

Here κ is the characteristic constant of the nonlinear field, which plays the role of an absolute scale of the field gradient $\varphi_x = \partial\varphi/\partial x$, $\varphi_t = \partial\varphi/\partial t$. If $x \to \infty$, then the Lagrangian (1) goes over into the Lagrangian of the linear field $\mathcal{L}_0 = (\varphi_t^2 - \varphi_x^2)/2$, which obeys the d'Alambert equation $\varphi_{tt} - \varphi_{xx} = 0$. If we introduce the conjugate field momentum $\pi(x, t) = \delta\mathcal{L}/\delta\varphi_t$, then we can write the Hamiltonian function of this field:

$$\mathcal{H} = g^{-2}\{\sqrt{(1 + g^2\varphi_x^2)(1 + g^2\pi^2)} - 1\}. \tag{2}$$

We have introduced here a constant g, which is the reciprocal of κ, and which will be convenient in what follows ($g^2 = \kappa^{-2}$), and can be regarded as the nonlinearity constant. As $g^2 \to 0$, the Hamiltonian (2) goes over into the Hamiltonian of the linear field $\mathcal{H}_0 = (\pi^2 + \varphi_x^2)/2$. The field equation that follows from (1) or (2) pertains to the class of quasilinear equations of the hyperbolic type [3]

$$(1 - g^2\varphi_t^2)\varphi_{xx} + g^2\varphi_x\varphi_t\varphi_{xt} - (1 + g^2\varphi_x^2)\varphi_{tt} = 0. \tag{3}$$

In [4] we solved the problem of scattering of two plane waves in this theory. Namely, we found the solution $\varphi(u, v)$ of Eq. (3), with $u = x - t$ and $v = x + t$; this solution satisfies the following asymptotic conditions:

$$\lim_{v\to-\infty} \varphi(u, v) = \psi_1(u), \qquad \lim_{u\to\infty} \varphi(u, v) = \psi_2(v). \tag{4}$$

Here $\psi_1(u)$ is a plane wave of arbitrary form, moving in the positive x direction, and $\psi_2(v)$ is a plane wave moving in the opposite x direction. The solution was obtained in the form

$$\varphi(u, v) = \psi_1(u + \mu(u, v)) + \psi_2(v + \nu(u, v)),$$

where

$$\mu(u,v) = g^2 \int_{-\infty}^{v} dy \psi_2'^2(y + \nu(u,v)),$$

$$\nu(u,v) = -g^2 \int_{u}^{\infty} dy \psi_1'^2(y + \mu(u,v)) \tag{5}$$

(the primes of $\psi_{1,2}$ denote their derivatives with respect to the argument).

It is possible to obtain from (4) and (5) the expressions into which the two plane waves ψ_1 and ψ_2 go over ($t = (v - u)/2$ and in both cases of (4) we have $t \to -\infty$). We obtain the scattered waves by letting $u \to -\infty$ and $v \to \infty$ (in this case $t \to \infty$):

$$\lim_{v\to\infty} \varphi(u, v) = \psi_1(u + g^2P_2), \qquad \lim_{u\to-\infty} \varphi(u, v) = \psi_2(v - g^2P_1), \tag{6}$$

where

$$P_1 = \int_{-\infty}^{\infty} \psi_1'^2(y)\,dy, \quad P_2 = \int_{-\infty}^{\infty} \psi_2'^2(y)\,dy. \tag{7}$$

Thus, as $p = -\infty$, the two plane waves $\psi_1(u)$ and $\psi_2(v)$ go over as a result of scattering into two plane waves of the same form, but with shifted arguments. The quantities P_1 and P_2, which determine the shifts, equal respectively the energies of the first and second waves.

2. QUANTUM ANALYSIS OF THE SCATTERING PROCESS

Let us turn to the quantum formulation of the process of the scattering of waves in this model. It is very difficult to obtain the operators of the scattered waves $\hat{\varphi}_{out}$ by solving the Heisenberg equation for the field operator, since it is not even clear how to write down the nonlinear operator equation that follows from the Lagrangian (1), inasmuch as the nonlinear combinations of the noncommuting operators $\hat{\varphi}(x, t)$ may turn out to be indetermined operator equations at one point x, t. We therefore cannot transfer Eq. (3) to quantum theory automatically [1)].

It is our task to show that even in the asymptotic region of the scattered waves (6) the solution of the quantum equations for the system with the Hamiltonian (2) differs from the classical solution (6), if we interpret there the classical quantities as quantum field

[1)]In an earlier paper by the author [5] it was assumed, without justification, that Eq. (3) retains its form also in quantum theory. It is easy to note, however, that even the solution of this equation by iteration with respect to the constant g^2 depends strongly on the sequence of the arrangement of the operators φ_x, φ_t, φ_{xx} in Eq. (3). The question of the correct sequence of the operators remains open.

operators $\hat{\varphi}(x, t)$. For linear systems, as a rule, the solution of the classical equation of motion makes it possible to obtain the solution of the operator equation[6].

Let us introduce, following Yang, Feldman[7], and Kallen[8], the operator of the incident waves $\hat{\varphi}_{in}(u, v)$ satisfying the free D'Alambert equation (g = 0)

$$\hat{\varphi}_{in\,tt} - \hat{\varphi}_{in\,xx} = 0 \quad \text{or} \quad \frac{\partial^2 \hat{\varphi}_{in}}{\partial u\, \partial v} = 0. \tag{8}$$

From (8) there follows a Fourier representation for the operator $\hat{\varphi}_{in}$:

$$\begin{aligned}\hat{\varphi}_{in} &= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \frac{dk}{\sqrt{2|k|}} [a^+(k) e^{-ikx+i|k|t} + a^-(k) e^{ikx-i|k|t}] \\ &= \frac{1}{\sqrt{2\pi}} \int_0^{\infty} \frac{dk}{\sqrt{2k}} [a^+(k) e^{-iku} + a^-(k) e^{iku}] \\ &+ \frac{1}{\sqrt{2\pi}} \int_0^{\infty} \frac{dk}{\sqrt{2k}} [a^+(-k) e^{ikv} + a^-(-k) e^{-ikv}],\end{aligned} \tag{9}$$

where, as usual, the Bose operators $a^{\pm}$ satisfy the permutations

$$[a^+(k), a^+(p)] = [a^-(k), a^-(p)] = 0, [a^-(k), a^+(p)] = \delta(k-p). \tag{10}$$

We denote, in analogy with the classical problem (4), the operator of the incident wave in the positive x direction by

$$\hat{\psi}_1(u) = \frac{1}{\sqrt{2\pi}} \int_0^{\infty} \frac{dk}{\sqrt{2k}} [a^+(k) e^{-iku} + a^-(k) e^{iku}] \tag{11}$$

and the operator of the wave going in the opposite direction, by

$$\hat{\psi}_2(v) = \frac{1}{\sqrt{2\pi}} \int_0^{\infty} \frac{dk}{\sqrt{2k}} [a^+(-k) e^{ikv} + a^-(-k) e^{-ikv}]. \tag{12}$$

From (9) we have

$$\hat{\varphi}_{in}(u, v) = \hat{\psi}_1(u) + \hat{\psi}_2(v).$$

From the representations (10) follow commutation relations for $\hat{\psi}_1(u)$ and $\hat{\psi}_2(v)$. We note, first, that inasmuch as the operator $\hat{\psi}_1(u)$ contains $a^{\pm}(k)$ only with positive values of k, and the operator $\hat{\psi}_2(v)$ contains $a^{\pm}(-k)$ only with negative values k, it follows that

$$[\hat{\psi}_1(u), \hat{\psi}_2(v)] = 0.$$

Further,

$$\begin{aligned}[\hat{\psi}_1(u), \hat{\psi}_1(u')] &= \frac{i}{2\pi} \int_0^{\infty} \frac{dk}{k} \sin k(u-u') = \frac{i}{4} \varepsilon(u-u'), \\ [\hat{\psi}_2(v), \hat{\psi}_2(v')] &= -\frac{i}{2\pi} \int_0^{\infty} \frac{dk}{k} \sin k(v-v') = -\frac{i}{4} \varepsilon(v-v'),\end{aligned} \tag{13}$$

where

$$\varepsilon(x) = \begin{cases} 1, & x > 0 \\ -1, & x < 0. \end{cases}$$

We need also to know the commutators for the derivatives $\varphi'_{1,2}$, which are obtained by differentiating the relations (13):

$$[\hat{\psi}_1(u), \hat{\psi}_1'(u')] = -{}^1\!/_2 i\delta(u-u'). [\hat{\psi}_2(v), \hat{\psi}_2'(v')] = {}^1\!/_2 i\delta(v-v'). \tag{14}$$

We now assume that the operator $\hat{\varphi}_{out}$ of the scattered waves has the same form that follows from the solution of the classical equation (3), which is now taken to mean the Heisenberg equation for the field operators $\hat{\varphi}$. We denote it by $\hat{\varphi}^{cl}_{out}$; according to (6), we have

$$\hat{\varphi}^{cl}_{out}(u, v) = \hat{\psi}_1(u + g^2 \hat{P}_2) + \hat{\psi}_2(v - g^2 \hat{P}_1). \tag{15}$$

The meaning of the operators $\hat{\psi}_1(u + g^2\hat{P}_2)$ and $\hat{\psi}_2(v - g^2\hat{P}_1)$, the argument of which contains other operators P_2 and P_1, becomes clear if it is noted that the operator $\hat{P}_1$ commuted with $\hat{\psi}_2$, and $\hat{P}_2$ commutes with $\hat{\psi}_1$. Indeed, from (11), (12), and (14) respectively we have

$$\begin{aligned}\hat{P}_1 &= \int_{-\infty}^{\infty} \hat{\psi}_1'^2(u)\, du = \int_0^{\infty} k a^+(k) a^-(k)\, dk, \\ \hat{P}_2 &= \int_{-\infty}^{\infty} \hat{\psi}_2'^2(v)\, dv = \int_0^{\infty} k a^+(-k) a^-(-k)\, dk, \\ [\hat{P}_1, \hat{P}_1] &= [\hat{P}_2, \hat{P}_2] = [\hat{P}_1, \hat{\psi}_2] = [\hat{P}_2, \hat{\psi}_1] = 0, \\ [\hat{P}_1, \hat{\psi}_1(u)] &= i\hat{\psi}_1'(u); \quad [\hat{P}_2, \hat{\psi}_2(v)] = -i\hat{\psi}_2'(v).\end{aligned} \tag{16}$$

Therefore the operators in (15) can be represented in the form of Taylor series in $\hat{P}_1$ and $\hat{P}_2$:

$$\begin{aligned}\hat{\psi}_1(u + g^2\hat{P}_2) &= \sum_{n=0}^{\infty} \frac{(g^2\hat{P}_2)^n}{n!} \hat{\psi}_1^{(n)}(u), \\ \hat{\psi}_2(v - g^2\hat{P}_1) &= \sum_{n=0}^{\infty} \frac{(-g^2\hat{P}_1)^n}{n!} \hat{\psi}_2^{(n)}(v).\end{aligned} \tag{17}$$

Further, the proposed operator $\hat{\varphi}^{cl}_{out}$, as should be the case in the Yang-Feldman theory, satisfies the free equation (8) and has a Fourier representation

$$\begin{aligned}\hat{\varphi}^{cl}_{out}(u, v) &= \frac{1}{\sqrt{2\pi}} \int_0^{\infty} \frac{dk}{\sqrt{2k}} \{A^+(k, g^2) e^{-iku} + A^-(k, g^2) e^{iku}\} \\ &+ \frac{1}{\sqrt{2\pi}} \int_0^{\infty} \frac{dk}{\sqrt{2k}} \{e^{ikv} A^+(-k, g^2) + A^-(-k, g^2) e^{-ikv}\},\end{aligned} \tag{18}$$

where the operators $A^{\pm}(\pm k, g^2)$ are defined in accordance with (15), (12), and (11):

$$\begin{aligned}A^{\pm}(k, g^2) &= a^{\pm}(k) \exp\{\mp ikg^2 P_2\}, \\ A^{\pm}(-k, g^2) &= a^{\pm}(-k) \exp\{\mp ikg^2 P_1\}.\end{aligned} \tag{19}$$

It can be shown that the $A^{\pm}$ satisfy the same commutation relations (10) as $a^{\pm}$. This will follow further from the fact of the existence of a unitary transformation $S^+_{cl} a^{\pm} S_{cl} = A^{\pm}$, i.e., an operator relating $\hat{\varphi}_{in}$ with $\hat{\varphi}^{cl}_{out}$.

The operator S_{cl}, which realizes the transformation

$$S_{cl}^+ \hat{\varphi}_{in}(u, v) S_{cl} = \hat{\varphi}^{cl}_{out}(u, v),$$

which, with allowance for (12) and (15), breaks up into the two operators

$$\begin{aligned}S^+_{cl} \hat{\psi}_1(u) S_{cl} &= \hat{\psi}_1(u + g^2\hat{P}_2), \\ S^+_{cl} \hat{\psi}_2(v) S_{cl} &= \hat{\psi}_2(v - g^2\hat{P}_1),\end{aligned} \tag{20}$$

is of the form

$$S_{cl} = \exp\{ig^2 \hat{P}_1 \hat{P}_2\}. \tag{21}$$

We shall show that S_{cl} satisfies the equations in (20). We note first that inasmuch as the operators $\hat{P}_1$ and $\hat{P}_2$ commute (see (16)), the interpretation of the exponential with the operator $\hat{P}_1\hat{P}_2$ in the argument entails no difficulty. Expression (21) should be regarded as a sum of the ordinary product of the operator $\hat{P}_1\hat{P}_2$

$$\exp\{ig^2\hat{P}_1\hat{P}_2\} = \sum_{n=0}^{\infty}\frac{(ig^2)^n}{n!}(\hat{P}_1\hat{P}_2)^n.$$

To prove (20) with the operator S_{cl}, we differentiate $S_{cl}^+\hat{\psi}_1(u)S_{cl}$ and $S_{cl}^+\hat{\psi}_2(v)S$ with respect to g^2; with allowance for the commutation relations (16) we get

$$\begin{aligned}\partial(S_{cl}^+\hat{\psi}_1(u)S_{cl})/\partial g^2 &= S_{cl}^+ i\hat{P}_2[\hat{\psi}_1(u), \hat{P}_1]S_{cl}\\ &= \hat{P}_2 S_{cl}^+\hat{\psi}_1'(u)S_{cl} = \hat{P}_2\partial(S_{cl}^+\hat{\psi}_1(u)S_{cl})/\partial u;\\ \partial(S_{cl}^+\hat{\psi}_2(v)S_{cl})/\partial g^2 &= S_{cl}^+ i\hat{P}_1[\hat{\psi}_2(v), \hat{P}_2]S_{cl}\\ &= -\hat{P}_1 S_{cl}^+\psi_2'(v)S_{cl} = -\hat{P}_1\partial(S_{cl}^+\hat{\psi}_2(v)S_{cl})/\partial v.\end{aligned} \tag{22}$$

Solving these equations with initial conditions

$$(S_{cl}^+\hat{\psi}_1(u)S_{cl})_{g^2=0} = \hat{\psi}_1(u);\ (S_{cl}^+\hat{\psi}_2(v)S_{cl})_{g^2=0} = \hat{\psi}_2(v),$$

we get

$$S_{cl}^+\hat{\psi}_1(u)S_{cl} = \hat{\psi}_1(u+g^2\hat{P}_2);\ S_{cl}^+\hat{\psi}_2(v)S_{cl} = \hat{\psi}_2(v-g^2\hat{P}_1),$$

q.e.d.

The operator S_{cl} is determined by (20) uniquely, apart from a phase factor, since the system of operators $\hat{\varphi}_{in}$ and $\hat{\varphi}_{out}^{cl}$, which are solutions of the d'Alambert equation, constitutes an irreducible set of operators[19] in the two-dimensional space x, t, satisfying the commutation relations (10). Thus, an expression for S_{cl} follows uniquely from the classical solutions.

In momentum space, we can also easily prove for the operators the equalities

$$A^{\pm}(\pm k, g^2) = S_{cl}^{+}a^{\pm}(\pm k)S_{cl}. \tag{23}$$

It is merely necessary to recognize that when $k > 0$

$$[a^{\pm}(k), \hat{P}_1] = \mp ka^{\pm}(k),$$
$$[a^{\pm}(-k), \hat{P}_2] = \mp ka^{\pm}(-k),$$

and then we get from these operators the equalities (19) for $A^{\pm}(\pm k, g^2)$. It follows directly from (23) that the $A^{\pm}$ satisfy the same commutation relations as the $a^{\pm}$. It is further easy to note that the operator of the number of incident particles

$$\hat{N}_{in} = \int_0^{\infty} dk\, a^+(k)a^-(k) + \int_0^{\infty} dk\, a^+(-k)a^-(-k)$$

commutes with S_{cl}, and therefore S_{cl} leads only to scattering processes in which the number of particles remains unchanged. From this we get the equality of the operators of the number of incident and scattering particles:

$$\hat{N}_{in} = \hat{N}_{out}^{cl} = \int_0^{\infty} dk\, A^+(k, g^2)A^-(k, g^2) + \int_0^{\infty} dk\, A^+(-k, g^2)A^-(-k, g^2).$$

3. S-MATRIX IN THE INTERACTION REPRESENTATION

We now turn to the standard procedure for obtaining the S-matrix in the interaction representation. To this end we separate from the Hamiltonian of our system $\mathcal{H}$ (2) the free-field Hamiltonian, from which follows the d'Alembert equation (8). As already noted, when $g^2 = 0$ our Hamiltonian (2) goes over into

$$\mathcal{H}_0 = {}^1/_2[\varphi_x^2 + \pi^2]. \tag{24}$$

Thus, we break up $\mathcal{H}$ into two parts $\mathcal{H} = (\mathcal{H} - \mathcal{H}_0) + \mathcal{H}_0$, where the difference $(\mathcal{H} - \mathcal{H}_0)$ is denoted by $\mathcal{H}_{int}$ and is represented in the form of an expansion in powers of g^2. In the interaction representation we have, in accordance with (12)

$$\begin{aligned}\hat{\varphi}_x(x, t) &= \hat{\psi}_1'(u) + \hat{\psi}_2'(v),\\ \pi(x, t) &= \hat{\psi}_2'(v) - \hat{\psi}_1'(u),\\ \mathcal{H}_0 &= \psi_1'^2(u) + \psi_2'^2(v).\end{aligned} \tag{25}$$

The equation for the S-matrix ($\lim_{t\to\infty} S(t) = S$) is

$$i\frac{\partial S(t)}{\partial t} = \int dx\,\mathcal{H}_{int}(x, t)S(t), \tag{26}$$

where

$$\begin{aligned}\mathcal{H}_{int}(x, t) &= \exp\left\{it\int dx\,\mathcal{H}_0(x)\right\}(\mathcal{H}(x) - \mathcal{H}_0(x))\exp\left\{-it\int dx\,\mathcal{H}_0(x)\right\}\\ &= g^{-2}{:}\{[1 + g^2(\psi_1'(u) + \psi_2'^2(v))]^{1/2}[1 + g^2(\psi_1'(u) - \psi_2'(v))^2]^{1/2} - 1\}{:}\\ &\quad - \mathcal{H}_0(x, t).\end{aligned} \tag{27}$$

This complicated operator expression for $\mathcal{H}_{int}(x, t)$ must be understood as an expansion, in powers of g^2, of the normal products of the operators $\psi_1'^2(u)$ and $\psi_2'^2(v)$ (: :—sign of normal product of the operators).

This expansion is of the form

$$\mathcal{H}_{int}(u, v) = -2g^2{:}\psi_1'^2(u)\psi_2'^2(v){:}$$
$$+ 2\sum_{n=1}^{\infty}(-g^2)^{n+1}\sum_{m=0}^{n}A_{m,n}{:}\psi_1'^{2(m+1)}(u)\psi_2'^{2(n+1-m)}(v){:}, \tag{28}$$

where

$$A_{m,n} = \frac{n!(n+1)!}{m!(m+1)!(n-m)!(n+1-m)!}.$$

From (26) follows an expression for the S-matrix as the T-exponent

$$S = T\exp\left\{-i\int_{-\infty}^{\infty}dt\int_{-\infty}^{\infty}dx\,\mathcal{H}_{int}(x, t)\right\}. \tag{29}$$

In order to be able to compare the S-matrix (29) with the expression for S_{cl} obtained by us in the preceding section on the basis of the assumption concerning the form of the operator $\hat{\varphi}_{out}$, it is necessary to go over in (29) from the T-product of the operators $\hat{\psi}_1$ and $\hat{\psi}_2$ to the ordinary products, inasmuch as S_{cl} is a sum of ordinary products of the operators $\hat{P}_1\hat{P}_2$. This procedure is algebraically equivalent to the Wick theorem for the transition from the T to the N product, except that in place of the chronological contractions of the operators in Wick's theorem it is necessary to use in our case retarded contractions. This is seen from the example of the two operators $\hat{\psi}_1'(u)$ and $\hat{\psi}_2'(v)$:

$$\begin{aligned}T(\hat{\psi}_1'(u)\hat{\psi}_1'(u')) &= \hat{\psi}_1'(u)\hat{\psi}_1'(u') + D_1^{ret}(u'-u),\\ T(\hat{\psi}_2'(v)\hat{\psi}_2'(v')) &= \hat{\psi}_2'(v)\hat{\psi}_2'(v') + D_2^{ret}(v'-v),\end{aligned} \tag{30}$$

where

$$\begin{aligned}D_1^{ret}(u'-u) &= \theta(t'-t)[\hat{\psi}_1'(u'), \hat{\psi}_1'(u)],\\ D_2^{ret}(v'-v) &= \theta(t'-t)[\hat{\psi}_2'(v'), \hat{\psi}_2'(v)],\\ \theta(t'-t) &= \begin{cases}1, & t' > t\\ 0, & t' \leqslant t\end{cases}.\end{aligned} \tag{31}$$

Using the Hori method[10] we effect the transition from the T-product of the operators $\hat{\psi}_1'$ and $\hat{\psi}_2'$ in (29) to the ordinary product with the aid of the operator exp $\Delta^{\rm ret}$:

$$T\exp\left\{-i\int dt\int dx\,\mathscr{H}_{int}(x,t)\right\}=e^{\Delta^{ret}}\exp\left\{-i\int dt\,dx\,\mathscr{H}_{int}\right\}. \quad (32)$$

The symbol $\Delta^{\rm ret}$ denotes in the case of our operators $\hat{\psi}_1'(u)$ and $\hat{\psi}_2'(v)$

$$\Delta^{ret}=\int dt_{1,2}\int dx_{1,2}\left\{D_1^{ret}(u_2-u_1)\frac{\delta^2}{\delta\hat{\psi}_1'(u_1)\delta\hat{\psi}_1'(u_2)}\right. \quad (33)$$

$$\left.+D_2^{ret}(v_2-v_1)\frac{\delta^2}{\delta\hat{\psi}_2'(v_1)\delta\hat{\psi}_2'(v_2)}\right\}.$$

For further calculations it is convenient to go over to the momentum representation of the operators $\hat{\psi}_1'$ and $\hat{\psi}_2'$ and of the functions $D_{1,2}^{\rm ret}$:

$$\hat{\psi}_1'(u)=\frac{1}{\sqrt{2\pi}}\int dp\,e^{-ipu}\,\hat{\alpha}(p),$$

$$\hat{\psi}_2'(v)=\frac{1}{\sqrt{2\pi}}\int dp\,e^{-ipv}\,\hat{\beta}(p). \quad (34)$$

Taking (11) and (12) into account, we obtain for the operators $\hat{\alpha}(p)$ and $\hat{\beta}(p)$ the following expressions

$$\hat{\alpha}(p)=\sqrt{\frac{|p|}{2}}\frac{1}{i}[\theta(p)a^+(p)-\theta(-p)a^-(-p)],$$

$$\hat{\beta}(p)=\sqrt{\frac{|p|}{2}}i[\theta(-p)a^+(p)-\theta(p)a^-(-p)]. \quad (35)$$

The functions $D_{1,2}^{\rm ret}$ have the following Fourier representation:

$$D_1^{ret}(u_2-u_1)=\frac{\theta(t_2-t_1)}{2\pi}\int dp\,p\exp\{ip(u_2-u_1)\},$$

$$D_2^{ret}(v_2-v_1)=-\frac{\theta(t_2-t_1)}{2\pi}\int dp\,p\exp\{ip(v_2-v_1)\}. \quad (36)$$

In formula (32) it is now necessary to substitute $\Delta^{\rm ret}$ and $\int dtdx\,\mathscr{H}_{\rm int}(x,t)$ in the following form:

$$\Delta^{ret}=\int dt_{1,2}\int dp_{1,2}\left\{\Delta_1(t_1p_1;t_2p_2)\frac{\delta^2}{\delta\hat{\alpha}_{t_1}(p_1)\delta\hat{\alpha}_{t_2}(p_2)}\right.$$

$$\left.+\Delta_2(t_1p_1;t_2p_2)\frac{\delta^2}{\delta\hat{\beta}_{t_1}(p_1)\delta\hat{\beta}_{t_2}(p_2)}\right\}, \quad (37)$$

where

$$\Delta_{1,2}(t_1p_1;t_2p_2)=-\frac{\theta(t_1-t_2)}{2}p_{1,2}\delta(p_1+p_2),$$

$$\int dt\int dx\,\mathscr{H}_{int}(x,t)=-\frac{g^2}{\pi}\int dt\sum_{n=0}^{\infty}\left(-\frac{g^2}{2\pi}\right)^n\sum_{m=0}^{n}A_{m,n}$$

$$\times\int dp_1\ldots dp_{2(m+1)}dq_1\ldots dq_{2(n+1-m)}\exp\left\{2it\sum_{j=1}^{2(m+1)}p_j\right\}$$

$$\times\delta\left(\sum_1^{2(m+1)}p_j+\sum_1^{2(n+1-m)}q_j\right):\hat{\alpha}_t(p_1)\ldots\hat{\alpha}_t(p_{2(m+1)})\hat{\beta}_t(q_1)\ldots\hat{\beta}_t(q_{2(n+1-m)}): \quad (38)$$

The operators $\hat{\alpha}_t$ and $\hat{\beta}_t$ are labeled in the expressions (37) and (38) with the index t; this is connected with the fact that although these operators do not depend on the time explicitly, t is an "ordering" product in the T-product of the operators in (32) (see [11]). For the total Hamiltonian $\mathscr{H}_{\rm int}(x,t)$, defined in (38), it is impossible to carry out the operation (32) of the transition from the T-product to the ordinary product without an expansion in powers of the parameters g^2.

It is our task to show that if we confine ourselves to the first term of (38), which is proportional to g^2, then the operation (32) can be performed, accurate to g^2 in the argument of the exponential, without a series expansion, and the resultant expression coincides exactly with $S_{\rm cl}$ obtained in Sec. 2.

Let us consider the expression

$$S_1=T\exp\left\{\frac{ig^2}{\pi}\int_{-\infty}^{\infty}dt\int_{-\infty}^{\infty}dp_{1,2}dq_{1,2}\exp\{2it(p_1+p_2)\}\delta(p_1+p_2+q_1+q_2)\right.$$

$$\left.\times:\hat{\alpha}_t(p_1)\hat{\alpha}_t(p_2)\hat{\beta}_t(q_1)\hat{\beta}_t(q_2):\right\}=e^{\Delta^{ret}}\exp\left\{\int dt\,H_1(t)\right\}, \quad (39)$$

$H_1(t)$ is the first term in the expansion (38). We shall perform all the operations connected with the variational differentiation, accurate to g^2 in the argument of the exponential, as a result of which we obtain

$$S_1=\exp\left\{ig^2\int_{-\infty}^{\infty}dp_{1,2}dq_{1,2}\delta(p_1+p_2)\right.$$

$$\left.\times\delta(q_1+q_2):\hat{\alpha}(p_1)\hat{\alpha}(p_2)\hat{\beta}(q_1)\hat{\beta}(q_2):\right\}. \quad (40)$$

Substituting here the expressions for the operators $\hat{\alpha}(p)$ and $\hat{\beta}(q)$ in terms of $a^{\pm}$ from (35), we verify that S_1 coincides with $S_{\rm cl}$. Indeed, in the argument of the exponential (40) we have

$$\int_{-\infty}^{\infty}dp\,dq:\hat{\alpha}(p)\hat{\alpha}(-p)::\hat{\beta}(q)\hat{\beta}(-q):=\int_{-\infty}^{\infty}dp\,dq\frac{|p||q|}{4}$$

$$\times\{:[\theta(p)a^+(p)-\theta(-p)a^-(-p)][\theta(-p)a^+(-p)-\theta(p)a^-(p)]:$$

$$\times:[\theta(-q)a^+(q)-\theta(q)a^-(-q)][\theta(q)a^+(-q)-\theta(-q)a^-(q)]:\}$$

$$=\int_0^{\infty}dp\,pa^+(p)a^-(p)\int_0^{\infty}dq\,qa^+(-q)a^-(-q)=\hat{P}_1\hat{P}_2. \quad (41)$$

Thus, our statement is proven: $S_1=\exp\{ig^2\hat{P}_1\hat{P}_2\}$. Consequently, our approximation for finding the S-matrix leads to $S_{\rm cl}$, which, as already noted above, gives only elastic scattering processes.

The exact S-matrix, which is determined by the total Hamiltonian $H_{\rm int}(t)$, contains terms that lead also to inelastic processes. This follows from the form of the S-matrix in the normal form, obtained from perturbation theory up to order g^4 inclusive:

$$S=1+ig^2\hat{P}_1\hat{P}_2+\frac{ig^4}{2}:\left\{\hat{P}_1^2\hat{P}_2^2+\hat{P}_1^2\int_{-\infty}^{\infty}dq|q|\hat{\beta}(q)\hat{\beta}(-q)\right.$$

$$+\hat{P}_2^2\int_{-\infty}^{\infty}dp|p|\hat{\alpha}(p)\hat{\alpha}(-p)+\int_{-\infty}dp|p|\hat{\alpha}(p)\hat{\alpha}(-p)\int_{-\infty}^{\infty}dq|q|\hat{\beta}(q)\hat{\beta}(-q)$$

$$+(\hat{P}_1+\hat{P}_2)\frac{3}{i\pi^2\varepsilon}\left(\int_0^{\infty}dp\,p\right)^2+\frac{1}{2i\pi^3\varepsilon^2}\left(\int_0^{\infty}dp\,p\right)^3$$

$$+\int dp_1\ldots dp_4[\hat{\alpha}(p_1)\hat{\alpha}(p_2)\hat{\alpha}(p_3)\hat{\alpha}(p_4)+\hat{\beta}(p_1)\hat{\beta}(p_2)\hat{\beta}(p_3)\hat{\beta}(p_4)]$$

$$\left.\times\delta\left(\sum_1 p_i\right)\frac{1}{i\pi\varepsilon}\int_0 dp\,p\right\}:.$$

The last term leads to the possibility of inelastic processes even in this order of perturbation theory. The parameter ϵ is due to the factor $\exp(-|t|)$, which is introduced in $\mathscr{H}_{\rm int}$ in accordance with the adiabatic hypothesis and which appears in the vacuum matrix elements of the S-matrix. In our case, the vacuum is defined with respect to particles of sort α or β (with positive or negative momentum p). The infinite integrals $\int_0^\infty dpp$ are due to the causal contractions of the operators on going from the T to the N product.

It now becomes clear that the exact operator of the scattered waves is not equal to $\hat{\varphi}^{cl}_{out}$, since they are obtained from $\hat{\varphi}_{in}$ with the aid of different S-matrices, and they coincide only in the approximation made above, in which there are no inelastic processes.

In conclusion, the author thanks D. I. Blokhintsev and N. A. Chernikov for continuous interest in the work and interesting discussions.

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Translated by J. G. Adashko
209