POSSIBILITY OF A PARAFIELD REPRESENTATION OF INTERNAL DEGREES OF FREEDOM, LIKE ISOSPIN AND STRANGENESS

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The possibility of introducing internal degrees of freedom, like isospin and strangeness, on the basis of the paracommutation relations of Green and Volkov is demonstrated. For this purpose we propose to consider all irreducible (separable) representations of the parafermion commutation relations of second and third order and to interpret the various internal states of the paraparticles as states which either belong to different irreducible representations of these relations, or differ in the symmetry of the indices of the state vectors. For the case of second order parastatistics an explicit transformation is exhibited in matrix form, mapping the parafield into a system of two fields having ''isospin'' symmetry. An electromagnetic interaction violating this latter symmetry is introduced in the framework of parafield theory. For the case of third parastatistics the corresponding classification of particle states agrees with the classification in terms of ''strangeness'' and ''isomultiplets.''

1. INTRODUCTION

It is a well-known fact that such a remarkable natural phenomenon as the existence of isospin and SU(3) symmetries of the hadrons was discovered in a purely empirical manner, and that the description of these ''internal'' symmetries within the framework of present day theory is phenomenological. In this connection any attempt at a more consistent introduction of internal symmetries is of interest. In the present paper we shall point out that such a possibility opens up if one founds field theory on the generalized quantization method proposed by Green [1] and Volkov [2], namely on the so-called paracommutation relations (PCR). The considerations which follow will refer to a free field which can be expanded in terms of the positive- and negative-frequency solutions of the corresponding field equation (Dirac, Klein-Gordon, etc.):

$$\psi(x)=\sum_k\{b_{k-}\varphi_k^{(-)}(x)+b_{k+}^*\varphi_k^{(+)}(x)\}. \tag{1}$$

The state k is determined, e.g., by the momentum and helicity and the operators b_{k-} and b_{k+} refer to particles and antiparticles, respectively. In the sequel the subscript ± indicating this distinction will be included in the state label k. The totality of all the PCR can be written in the following symbolic form:

$$[b_r,[b_s,b_t]_\varepsilon]_-=2\{b_r,b_s\}_{-\varepsilon}b_t+2\varepsilon\{b_r,b_t\}_{-\varepsilon}b_s, \tag{2}$$

where each operator b may be replaced by its adjoint b*. The Volkov symbols [2] in the right-hand side of (2) have the following numerical values

$$\{b_r,b_s^*\}_{-\varepsilon}=\delta_{rs},\quad \{b_r,b_s\}_{-\varepsilon}=\{b_r^*,b_s^*\}_{-\varepsilon}=0, \tag{3}$$

and the signature ϵ indicates that in the left-hand side of (2) one must take

for ϵ = the anticommutator for integral spin fields,
- the commutator for half-odd-integral spin fields.

Depending on the last relation the commutation relations (2) are called paraboson commutation relations (PBR) or parafermion commutation relations (PFR). The number operator in the state k has the form

$$N_k={}^1/_2[b_k^*,b_k]_- - {}^1/_2\varepsilon p \tag{4}$$

and as a consequence of (2) it possesses all the required properties:

$$[b_s,N_r]_-=\delta_{rs}b_r,\quad [b_s^*,N_r]_-=-\delta_{rs}b_r^*. \tag{5}$$

The free-field Hamiltonian is defined by

$$H=\sum_k|k^0|(N_{k-}+N_{k+}). \tag{6}$$

The connection of the PCR with generalized statistics of identical particles is established via the supplementary condition that the maximal number of parafermions in a symmetric state or of parabosons in an antisymmetric state is bounded. This number has been designated as the order of the appropriate parastatistics. It was shown in [3] that the selection of the order together with the general relations (2) valid in any order can be replaced by fields obeying commutation relations only and including both these requirements. Thus, the usual binary relations for fermions and bosons transform (2) into an identity. For second order parastatistics the relations are

$$b_rb_sb_t-\varepsilon b_tb_sb_r=2\{b_r,b_s\}_{-\varepsilon}b_t+2\{b_s,b_t\}_{-\varepsilon}b_r. \tag{7}$$

The relations (2) admit solutions in the form of the so-called Green Ansatz [1]:

$$b_r=\sum_{A=1}^p a_r^A, \tag{8}$$

the components of which satisfy the commutation relations

$$[a_r^A,a_s^{B*}]_{-\varepsilon\varepsilon_{AB}}=\delta_{AB}\delta_{rs},\quad [a_r^A,a_s^B]_{-\varepsilon\varepsilon_{AB}}=[a_r^{A*},a_s^{B*}]_{-\varepsilon\varepsilon_{AB}}=0, \tag{9}$$

where $\epsilon_{AB}=2\delta_{AB}-1$. The number p determines the order of the parastatistics. For instance, in the case of parafermions ($\epsilon = -$) it is easy to, derive that $(b_r)_{p+1} = 0$. Greenberg and Messiah[4] have studied the Fock representation of the PCR and have shown that for free fields such relations cannot lead to anything else than the spectrum of parastatistics with orders p = 1, 2, ... This also implies that a parafield can always be uniquely (up to unitary equivalence) described by the Green Ansatz. These facts about paraquantization should suffice for the considerations which follow. A complete bibliography can be found in[4] or in the review article[5].

In the present paper we consider a Dirac spinor field quantized according to second and third order PFR. In both cases the considerations are similar. First we indicate a method for constructing all irreducible (separable) representations of the PFR, in distinction from the preceding papers on the subject, which have considered only the fock representation. This method consists in separating from the Fock space of the representation of the algebra $\mathcal{A}$ of the operators a^A and satisfying the relations (9) the irreducible representations of the algebra $\mathcal{B}$ of the operators b, satisfying the PCR (2). Then, in the representations of the algebra $\mathcal{A}$ determined in this manner we consider vectors referring to the same number f of particles and the same set of one-particle states $k_1, \ldots, k_f$. An internal state of the system of f particles is then defined either through the fact that the vectors belong to, different irreducible representations of the PFR (which will be denoted by the index α) or, for vectors belonging to the same representation, through the permutation symmetry of the indices of the one-particle states (which will be denoted by the index ν). Thus a vector of a definite internal state of an f-particle system will be represented in the form $|f, \alpha, \nu\rangle$.

It is shown further that such a classification of states for the case of second order PFR coincides with the classification of states of two ordinary fermion fields possessing an isospin-like symmetry. An explicit matrix transformation of the parafield to the two ordinary fields is introduced. A resolution of the Galindo-Yndurain paradox[6] is given. The properties of the interaction of the Green-Volkov parafield with an ordinary electromagnetic field which violates the "isospin" symmetry are investigated. For the case of third order PFR only one-, two-and three-particle states are considered. The classification of states indicated above corresponds in this case to a classification of "isomultiplets" in terms of their "strangeness." To states with an "isomultiplet" correspond vectors of equivalent representations appearing in the representations of the algebra $\mathcal{B}$ are separated from the representations of the large algebra $\mathcal{A}$. In this sense a separation of the states of particles of different "strangeness" occurs at the outset in the classification itself. Finally, in both cases the analogy of the irreducible representations of the PFR and the algebra of higher spins[2,3,7-9] is stressed. In conclusion of the introductory part we note that the possibility of describing a system of particles of two different kinds in terms of one parafermi field has also been discussed in refs.[1,9] and in the recent papers[10,11].

2. IRREDUCIBLE REPRESENTATIONS OF SECOND ORDER PFR AND "ISOSPIN"

2.1. We construct all irreducible representations of the Green-Volkov algebra (7) assuming first that the parafield is represented by the Green Ansatz:

$$b_r = a_r^1 + a_r^2, \tag{10}$$

and then generalizing the result in such a manner as to make it independent of this assumption. In the sequel we shall consider everywhere the parafermion case ($\epsilon = -$), although we shall retain the symbol for possible generalizations to the parabose case ($\epsilon = +$).

The Fock representation of the algebra $\mathcal{A}$ is constructed in the usual manner: one requires the uniqueness of the vacuum state $|0\rangle$ such that

$$a_r^A|0\rangle = 0 \qquad \text{for all } r \text{ and } A = 1,2 \tag{11}$$

and all vectors of the representation are obtained through the action of all possible polynomials $\mathcal{P}(a^*)$ of the operators a_r^{A*} on the state $|0\rangle$. Such a representation is irreducible with respect to the algebra $\mathcal{A}$, but is reducible with respect to the algebra $\mathcal{B}$. In order to separate from it the irreducible representations of the algebra $\mathcal{B}$ we introduce the auxiliary operator

$$\tilde{b}_r = a_r^1 - a_r^2; \tag{12}$$

As a consequence of (9) we have[1)]:

$$\begin{gathered}\tilde{b}_r b_s = \varepsilon\tilde{b}_s b_r, \quad b_r\tilde{b}_s = \varepsilon b_s\tilde{b}_r,\\ \tilde{b}_r b_s^* = \varepsilon\tilde{b}_s^* b_r, \quad b_r\tilde{b}_s^* = \varepsilon b_s^*\tilde{b}_r,\\ b_r b_s = \varepsilon\tilde{b}_s\tilde{b}_r, \quad b_r b_s^* = \varepsilon\tilde{b}_s^*\tilde{b}_r + 2\delta_{rs},\\ \tilde{b}_r\tilde{b}_s^* = \varepsilon b_s^* b_r + 2\delta_{rs}.\end{gathered} \tag{13}$$

The permutations of the operators b_r and $\tilde{b}_s$ are accompanied by a substitution of these operators by one another.

The results of considering all the irreducible representations of the Green-Volkov algebra (7) is formulated as the following:

Theorem. Each irreducible (separable) representation of the Green-Volkov algebra (7) is obtained by means of the action of all possible polynomials $\mathcal{P}(b^*)$ in the operators b^* on the preceding vectors

$$\Phi_{r_1\ldots r_f} = \tilde{b}_{r_1}^* b_{r_2}^* \tilde{b}_{r_3}^* \ldots |0\rangle, \tag{14}$$

defined by the properties

$$b_r\Phi_{r_1\ldots r_f} = 0, \tag{15}$$

$$b_r^*\Phi_{r_1\ldots r_f} = \varepsilon b_{r_1}^*\Phi_{rr_2\ldots r_f}, \tag{16}$$

$$b_r b_s^*\Phi_{r_1\ldots r_f} = 2\delta_{rs}\Phi_{r_1\ldots r_f} + 2\varepsilon\delta_{rr_1}\Phi_{sr_2\ldots r_f} + \ldots + 2\varepsilon\delta_{rr_f}\Phi_{r_1\ldots r_{f-1}s}, \tag{17}$$

$$\Phi_{r_1\ldots r_i\ldots r_k\ldots r_f} = \varepsilon\Phi_{r_1\ldots r_k\ldots r_i\ldots r_f}. \tag{18}$$

In particular, for f = 0 this defines the usual Fock representation with a unique vector Φ_0 satisfying (15). For this vector (17) takes the form

1)The replacement of the trilinear relations of Green-Volkov by these relations has also been considered in the recent preprint [12].

$$b_r b_s^* \Phi_0 = 2\delta_{rs}\Phi_0. \quad (19)$$

The conditions (15)–(18) determine the irreducible representation uniquely (up to unitary equivalence) and the representation is characterised by the number of indices f of the "preceding" vectors.

The irreducibility of the representations obtained in this manner is proved in the same manner as this was done for the Fock representation[4] utilizing the Haag-Schroer lemma[13]: a representation of an algebra is irreducible if it contains a cyclic vector and the algebra under consideration contains the projection onto that vector. Any one of the "preceeding" vectors can play the role of cyclic vector for the representation. The projection operator has the form

$$\Lambda_{r_1\ldots r_f} = \frac{\sin[\pi(N_{r_1}-1)]}{\pi(N_{r_1}-1)}\cdots\frac{\sin[\pi(N_{r_f}-1)]}{\pi(N_{r_f}-1)}\prod_{s\neq r_1,\ldots,r_f}{}' \frac{\sin(\pi N_s)}{\pi N_s}, \quad (20)$$

since the action of the particle number operator (4) on a "preceding" vector yields

$$N_r \Phi_{r_1\ldots r_f} = \delta_{rr_1}\Phi_{rr_2\ldots r_f} + \ldots + \delta_{rr_f}\Phi_{r_1\ldots r_{f-1}r}. \quad (21)$$

Note that the constant in the definition of the operator (4) is selected in such a manner that

$$N_r|0\rangle = 0. \quad (22)$$

At the same time we remark that the vector $\Phi_{r_1\cdots r_f}$ describes a state of f particles, despite the fact that it satisfies the condition (15).

It remains to verify that in this manner we have decomposed the whole large Fock representation of the algebra $\mathcal{A}$ into irreducible representations of the algebra $\mathcal{B}$. With the help of (13) one can convince oneself that each successive representation is orthogonal to all preceding ones and consequently cannot be contained in them. Further, the number of vectors containing f fixed indices and referring to different representations:

$$\Phi_{r_1\ldots r_f},\quad b_{r_1}^*\Phi_{r_2\ldots r_f},\quad b_{r_1}^*b_{r_2}^*\Phi_{r_3\ldots r_f},\quad b_{r_2}^*b_{r_1}^*\Phi_{r_3\ldots r_f},\ldots, b_{r_1}^*\ldots b_{r_f}^*|0\rangle, \quad (23)$$

is 2^f as a consequence of (7), (16), and (18). This number is the same as the number of linearly independent vectors

$$a_{r_1}^{1*}a_{r_2}^{1*}\ldots a_{r_f}^{1*}|0\rangle,\quad a_{r_1}^{2*}a_{r_2}^{1*}\ldots a_{r_f}^{1*}|0\rangle,\ldots, a_{r_1}^{2*}\ldots a_{r_f}^{2*}|0\rangle.$$

2.2. The fact that the indices r assume an infinite sequence of values was inessential to our considerations. The reasoning is valid if this sequence is finite: r = 1, 2, ..., M. In the case $\epsilon = -$ the representations obtained in this manner will be irreducible representations of the Duffin-Kemmer algebra in a euclidean space of even dimension 2M. Indeed, let us introduce the operators

$$\beta_{2r-1} = {}^1\!/_2 i(b_r - b_r^*),\quad \beta_{2r} = {}^1\!/_2(b_r + b_r^*),\quad r = 1,\ldots,M. \quad (24)$$

It is easy to verify that they satisfy the Duffin-Kemmer algebra:

$$\beta_k\beta_l\beta_m - \varepsilon\beta_m\beta_l\beta_k = \delta_{kl}\beta_m + \delta_{ml}\beta_k, \quad k, l, m = 1, 2, \ldots, 2M. \quad (25)$$

To the Green Ansatz(10) corresponds the Duffin decomposition of the operators β into the operators of the spinor representations:

$$\beta_m = {}^1\!/_2(\gamma_m^1 + \gamma_m^2),\quad \gamma_{2r-1}^A = i(a_r^A - a_r^{A*}),\quad \gamma_{2r}^A = a_r^A + a_r^{A*},\quad A = 1,2. \quad (26)$$

As a consequence of (18) the number of "preceding" vectors of each representation is $_MC_f$, and the total dimensionality of this representation is $_{2M+1}C_{M-f}$ as a consequence of (7), (16) (18)—results which were obtained earlier in[9]. The isomorphism of second order PFR with the Duffin-Kemmer algebra was noted by Volkov[2] and the irreducible representations obtained above were studied on this basis by Chernikov[9]. In[3,7,8] the general connection between parafermi quantization of any order and the rotation group in a higher-dimensional space was pointed out (respectively of the parabose quantization with a symplectic group[3]).

2.3. According to what was said in the introduction we are now able to give the following definition of an "internal" state of a system of f particles. A state of f particles is defined by its belonging to a certain irreducible representation of the PFR, to be denoted by the index α, and described accordingly by one of the vectors (23), which in addition is symmetrized with respect to the indices $r_1, \ldots, r_f$ according to a definite Young pattern (ν). The state vector can now be represented in the form $|f, \alpha, \nu\rangle$. We can further introduce the operators

$$I_1 = \sum_k \{{}^1\!/_4[\tilde{b}_{k-}^*, b_{k-}]_- - {}^1\!/_4[b_{k+}^*, \tilde{b}_{k+}]_-\}, \quad (27)$$

$$I_2 = \sum_k \{{}^1\!/_4 i[\tilde{b}_{k-}^*, b_{k-}]_+ + {}^1\!/_4 i[b_{k+}^*, \tilde{b}_{k+}]_+\}, \quad (28)$$

$$I_3 = \sum_k \{{}^1\!/_4[b_{k-}^*, b_{k-}]_{-\varepsilon} + {}^1\!/_4[b_{k+}^*, b_{k+}]_{-\varepsilon} + \varepsilon\}(-\varepsilon), \quad (29)$$

$$Y = \sum_k \{{}^1\!/_2[b_{k-}^*, b_{k-}]_\varepsilon - {}^1\!/_2[b_{k+}^*, b_{k+}]_\varepsilon\}, \quad (30)$$

which, owing to (13), satisfy the algebra

$$[I_i, I_j]_- = i\varepsilon_{ijk}I_k,\quad [Y, I_i]_- = 0, \quad (31)$$

of course, coinciding with the "isospin" algebra; the operator Y plays the role of an additive quantum number ("hypercharge"). The basis vectors of an irreducible representation can now be classified according to the eigenvalues of Y, I_3 and I^2: $|f, \alpha, \nu\rangle = |Y, I^2, I_3\rangle$.

2.4. Now we wish to show that to each vector of an irreducible representation of the PFR one can associate in a one-to-one manner a state vector of two ordinary fermion fields which are subject to "isospin" symmetry. The commutation relations of these fields have the normal form:

$$\begin{gathered}\{p_r, p_s^*\}_{-\varepsilon} = \delta_{rs},\quad \{p_r, p_s\}_{-\varepsilon} = \{p_r^*, p_s^*\}_{-\varepsilon} = 0,\\ \{n_r, n_s^*\}_{-\varepsilon} = \delta_{rs},\quad \{n_r, n_s\}_{-\varepsilon} = \{n_r^*, n_s^*\}_{-\varepsilon} = 0,\\ \{p_r, n_s^*\}_{-\varepsilon} = \{p_r, n_s\}_{-\varepsilon} = \{p_r^*, n_s^*\}_{-\varepsilon} = 0.\end{gathered} \quad (32)$$

We introduce the algebra of Pauli matrices ($\sigma_\pm = \sigma_1 \pm i\sigma_2$, σ_3). The linear combinations ($\otimes$ denotes the direct product)

$$\begin{aligned} b_r &= \frac{1}{\sqrt{2}}(\sigma_-\otimes p_r + \sigma_+\otimes n_r),\\ b_r^* &= \frac{1}{\sqrt{2}}(\sigma_+\otimes p_r^* + \sigma_-\otimes n_r^*), \end{aligned} \quad (33)$$

satisfy the Green-Volkov relations (7), as can be seen by direct computation. The auxiliary operator defined by the relations (13) has the form

$$\tilde{b}_r = \frac{1}{\sqrt{2}}(\sigma_+ \otimes p_r + \sigma_- \otimes n_r),$$
$$\tilde{b}_r^* = \frac{1}{\sqrt{2}}(\sigma_- \otimes p_r^* + \sigma_+ \otimes n_r^*). \quad (34)$$

It should be stressed that the operators b_r, b_r^*, $\tilde{b}_r$, $\tilde{b}_r^*$ now operate in the space of the direct product of the algebra of Pauli matrices and the algebra of fermion fields (32). The operators σ and p, n commute of course. Each representation vector has the structure of a product of two vectors referring to the representations of these two algebras. In particular, the vacuum vector has the form

$$\Phi_0 = \xi_0 \otimes |0\rangle. \quad (35)$$

The σ matrices operate on the vector ξ_0 and we select this vector in such a manner that

$$\sigma_-\xi_0 = 0, \quad \sigma_-\sigma_+\xi_0 = 4\xi_0,$$
$$\sigma_3\xi_0 = -\xi_0, \quad \sigma_3\sigma_+\xi_0 = \sigma_+\xi_0. \quad (36)$$

The "isospin" operators (27)–(29) and the "hypercharge" (30) can be represented in the usual form:

$$I_i = \sum_k \left\{ c_{k-}^* \frac{1}{2} T_i c_{k-} - c_{k+}^* \frac{1}{2} T_i c_{k+} \right\}$$
$$= \frac{1}{2} \int d^3x \left[\bar{\Psi}(x) \gamma^0, \frac{1}{2} T_i \Psi(x) \right]_-, \quad (37)$$

$$Y = \sum_k \{ c_{k-}^* c_{k-} - c_{k+}^* c_{k+} \} = \frac{1}{2} \int d^3x [\bar{\Psi}(x) \gamma^0, \Psi(x)]_-, \quad (38)$$

where we have introduced the notations

$$T_1 = \tau_1, \quad T_2 = \tau_2 \otimes \sigma_3, \quad T_3 = \tau_3 \otimes \sigma_3; \quad (39)$$

the matrices τ act on the column

$$c = \begin{pmatrix} p \\ n \end{pmatrix}, \quad \Psi(x) = \begin{pmatrix} \psi_p(x) \\ \psi_n(x) \end{pmatrix}; \quad (40)$$

and the σ matrices act, as before, on the vector ξ_0 belonging to another space.

We have the following classification of states: the one-particle states form an isodoublet

$$|Y = 1, I = 1/2, I_3 = +1/2\rangle = \frac{1}{\sqrt{2}} b_k^* \Phi_0 = p_k^* |0\rangle \otimes (1/2\sigma_+\xi_0),$$
$$|Y = 1, I = 1/2, I_3 = -1/2\rangle$$
$$= \frac{1}{\sqrt{2}} \Phi_k = \frac{1}{\sqrt{2}} \tilde{b}_k^* \Phi_0 = n_k^* |0\rangle \otimes (1/2\sigma_+\xi_0);$$

the two-particle states form an isosinglet

$$|Y = 2, I = 0, I_3 = 0\rangle = \frac{1}{2\sqrt{2}} [b_{k_1}^*, b_{k_2}^*]_+ \Phi_0$$
$$= \frac{1}{\sqrt{2}} (n_{k_1}^* p_{k_2}^* - p_{k_1}^* n_{k_2}^*) |0\rangle \otimes \xi_0$$

and an isotriplet

$$|Y = 2, I = 1, I_3 = -1\rangle = 1/2\Phi_{k_1k_2} = 1/2\tilde{b}_{k_1}^* b_{k_2}^* \Phi_0 = p_{k_1}^* p_{k_2}^* |0\rangle \otimes \xi_0,$$
$$|Y = 2, I = 1, I_3 = 0\rangle = \frac{1}{2\sqrt{2}} [b_{k_1}^*, b_{k_2}^*]_- \Phi_0$$
$$= \frac{1}{\sqrt{2}} (n_{k_1}^* p_{k_2}^* + p_{k_1}^* n_{k_2}^*) |0\rangle \otimes \xi_0,$$
$$|Y = 2, I = 1, I_3 = +1\rangle = 1/2 b_{k_1}^* \Phi_{k_2}$$
$$= 1/2 b_{k_1}^* \tilde{b}_{k_2}^* \Phi_0 = n_{k_1}^* n_{k_2}^* |0\rangle \otimes \xi_0$$

etc. Thus an arbitrary vector of an irreducible representation of the Green-Volkov relations can indeed be represented as follows: for an even number of particles in the form $\mathscr{P}_{\text{even}}(p^*, n^*)|0\rangle \otimes \Phi_0$, and for an odd number of particles in the form $\mathscr{P}_{\text{odd}}(p^*, n^*)|0\rangle \otimes (\sigma_+\xi_0)$ where the evenness or oddness of polynomials refers to the total number of p^* and n^* operators in them. The operator I_3 which according to (39) contains the matrix σ_3 has, owing to (36), the eigenvalues $(N_n - N_p)/2$ and $(N_p - N_n)/2$, respectively, in the two cases. However, in view of the orthogonality of the vectors ξ_0 and $\sigma_+\xi_0$ transitions between states with even and odd numbers of particles are prohibited. Therefore in the case of an even number of particles we may relabel: $p \rightleftarrows n$. Then the parafield theory of Green-Volkov is finally reduced to a theory of two ordinary fields with "isospin" projections $+1/2$ for the p and $-1/2$ for the n. It should be noted that an analogous "matrix" representation of fields subject to trilinear commutation relations has been used by Scharfstein[14].

2.5. In connection with the above one can now give a solution to the Galindo-Yndurain paradox[6]. The paradox consists in the assertion that the vectors $b_{k_1}^* \ldots b_{k_f}^* \Phi_0$ do not form a representation of the permutation group of the operators b^* (cf. also[4,15]). Considering, for example, in the case $\epsilon = -$ the vector $b_k^* b_k^* b_s^* \Phi_0$, then owing to (7) a permutation of the second and third operators b_k^* and b_s^* lead to a vanishing of this vector. The solution consists in the vectors not forming a representation of the permutation group, since they refer to different internal states of the particles n and p, and permutations of the states $k_1, \ldots, k_f$ must be accompanied by simultaneous permutations of these internal states. In the indicated example we have

$$\frac{1}{\sqrt{2}} b_k^* b_k^* b_s^* \Phi_0 = p_k^* n_k^* p_s^* |0\rangle \otimes (\sigma_+\xi_0),$$

therefore the permutation of only the indices k and s of the second and third operators should obviously lead to zero. The vanishing of some of the vectors $b_{k_1}^* \ldots b_{k_f}^* \Phi_0$ corresponds to the fact that to each state of the system of n and p particles corresponds only one such vector.

2.6. We now discuss the possibility of introducing an electromagnetic interaction of the Green-Volkov parafield, interaction which violates the "isosymmetry." Usually the interaction of the Green-Volkov parafermion field with the electromagnetic field was introduced in the form[2,16-18]

$$H_{\text{em}} = 1/2 e A_\mu(x) [\bar{\psi}(x), \gamma^\mu \psi(x)]_- = e A_\mu(x) : \bar{\Psi}(x) \gamma^\mu \Psi(x) : \quad (41)$$

Such an interaction corresponds to the assumption that the n and p fields have identical charges and therefore it does not violate the symmetry of these fields. The appearance of the factor 2 in the photoproduction cross section of a pair of paraparticles[17,18] is related to this fact. However, one may define the interaction differently

$$H_{\text{em}} = 1/2 e A_\mu(x) \{ \bar{\psi}(x) \gamma^\mu \psi(x) - \langle \bar{\psi}(x) \gamma^\mu \psi(x) \rangle_0 \}$$
$$= e A_\mu(x) : \bar{\Psi}(x) (1/2 + 1/2 T_3) \gamma^\mu \Psi(x) : \quad (42)$$

Then only one of the fields p or n will be charged and the symmetry between them will be violated. With this definition no redundant factors appear in the expressions of the cross sections.

Since the interaction (41) contains the commutator current of the parafield, it is invariant with respect to the usual definition of charge conjugation (cf. e.g.,[19]):

$$U_c\psi(x)U_c^{-1} = \eta_c C\bar{\psi}^T(x), \quad \bar{U}_c\psi(x)U_c^{-1} = -\bar{\eta}_c\psi^T(x)C^+. \quad (43)$$

But in the case of the interaction (42) there is no invariance with respect to the transformation (43). However the interaction is invariant with respect to the "combined" conjugation:

$$U_c\psi(x)U_c^{-1} = \eta_c C\tilde{\bar{\psi}}^T(x), \quad U_c\bar{\psi}(x)U_c^{-1} = -\bar{\eta}_c\tilde{\psi}^T(x)C^+, \quad (44)$$

when the charge conjugation is accompanied by a transition from the field $\psi(x)$ to $\tilde{\psi}(x)$ (the operators $\tilde{b}^*$ and $\tilde{b}$) and vice versa. But then the whole preceding theory is invariant under the substitution $\psi \rightleftarrows \tilde{\psi}$, since the operators $\tilde{b}$ and b satisfy the Green-Volkov relations (7) and the corresponding irreducible representations are obtained from the old ones by the substitution $\tilde{b} \rightleftarrows b$.

We note that if we wish to consider transitions between various internal states ("weak" decays), i.e., transitions among irreducible representations of the PFR, we must introduce into the interaction currents containing both the field $\psi(x)$ and the field $\tilde{\psi}(x)$. Thus we are forced to go outside the framework of the theory of parafields proper, appealing to the field $\tilde{\psi}(x)$ now not only as an auxiliary in the construction of irreducible representations of the field, but as a field which enters the interaction Hamiltonian.

3. THE IRREDUCIBLE REPRESENTATIONS OF A THIRD ORDER PARAFERMION FIELD AND "STRANGENESS"

3.1. A third order parafield is described by the Green Ansatz

$$\psi(x) = \sum_{A=1}^{3}\psi^A(x), \quad b_r = \sum_{A=1}^{3} a_r^A. \quad (45)$$

The construction of the large representation space of the algebra $\mathcal{A}$ of the components of the Green Ansatz is again carried out in the standard fashion. The separation from this space of the irreducible representations of the algebra $\mathcal{B}$ is again done by constructing in it "preceding" (lower order) vectors, satisfying the conditions

$$b_r f = 0 \qquad \text{for all } r, \quad (46)$$

and then letting all possible polynomials $\mathcal{P}(b^*)$ act on these vectors. We shall write out below only the first few among them.

The Fock representation is constructed from the vector $|0\rangle$, the only vector of this representation satisfying the condition (46). The vector $|0\rangle$ satisfies

$$b_r b_s^*|0\rangle = 3\delta_{rs}|0\rangle. \quad (47)$$

The following two "preceding" vectors have the form

$$f_r' = (a_r^{1*} - a_r^{2*})|0\rangle, \quad f_r'' = (a_r^{2*} - a_r^{3*})|0\rangle. \quad (48)$$

They both satisfy the relation (46) and

$$b_r b_s^* f_t = 3\delta_{rs} f_t - 2\delta_{rt} f_s, \quad (50)$$

$$b_r^* b_s^* f_t + b_r^* b_t^* f_s + b_s^* b_r^* f_t + b_s^* b_t^* f_r + b_t^* b_r^* f_s + b_t^* b_s^* f_r = 0. \quad (49)$$

Forming orthogonal combinations of the vectors (48) (e.g., their sum and difference) Φ_r' and Φ_r'' we obtain two equivalent orthogonal irreducible representations of the algebra $\mathcal{B}$. We note that owing to (46) and (49) the action of the particle number operator (4) (p = 3) on these "preceding vectors" yields

$$N_r\Phi_s^{(i)} = \delta_{rs}\Phi_r^{(i)}, \quad i = 1, 2. \quad (51)$$

We see that the vectors $\Phi_r^{(i)}$ describe the one-particle states.

The following three preceding vectors have the form

$$f_{rs}' = (a_r^{1*} - a_r^{2*})(a_s^{1*} + a_s^{2*})|0\rangle,$$
$$f_{rs}'' = (a_r^{2*} - a_r^{3*})(a_s^{2*} + a_s^{3*})|0\rangle,$$
$$f_{rs}''' = (a_r^{1*} - a_r^{3*})(a_s^{1*} + a_s^{3*})|0\rangle \quad (52)$$

and satisfy the relations

$$b_r b_s^* f_{tu} = 3\delta_{rs} f_{tu} - 2\delta_{rt} f_{su} - 2\delta_{ru} f_{ts}, \quad (53)$$

$$f_{rs} = -f_{sr}. \quad (54)$$

Taking three orthogonal combinations of these vectors $\Phi_{rs}^{(i)}$ (i = 1, 2, 3) and acting upon these with all possible polynomials $\mathcal{P}(b^*)$ we obtain three equivalent orthogonal irreducible representations of the algebra $\mathcal{B}$. In view of (46) and (53) the action of the particle number operator (4) yields:

$$N_r\Phi_{st} = \delta_{rs}\Phi_{st} + \delta_{rt}\Phi_{st}, \quad (55)$$

and consequently the "preceding" vectors $\Phi_{rs}^{(i)}$ (i = 1, 2, 3) describe two-particle states.

The following four vectors have the form

$$f_{rst}^{\mathrm{I}} = (a_r^{1*} - a_r^{2*})(a_s^{1*} + a_s^{2*})(a_t^{1*} - a_t^{2*})|0\rangle,$$
$$f_{rst}^{\mathrm{II}} = (a_r^{1*} - a_r^{3*})(a_s^{1*} + a_s^{3*})(a_t^{1*} - a_t^{3*})|0\rangle,$$
$$f_{rst}^{\mathrm{III}} = (a_r^{2*} - a_r^{3*})(a_s^{2*} + a_s^{3*})(a_t^{2*} - a_t^{3*})|0\rangle,$$
$$f_{rst}^{\mathrm{IV}} = \{-(a_r^{1*} - a_r^{2*})(a_s^{1*} - a_s^{3*})(a_t^{1*} + a_t^{3*}) + (a_r^{1*} - a_r^{3*}) \cdot [(a_s^{1*} - a_s^{2*})(a_t^{1*} + a_t^{2*}) + (a_s^{2*} + a_s^{3*})(a_t^{2*} - a_t^{3*})]\}|0\rangle. \quad (56)$$

They satisfy the conditions (46) and are antisymmetric in the indices r, s, t:

$$f_{rst} = -f_{srt} = -f_{rts}, \quad (57)$$

and

$$b_u b_v^* f_{rst} = 3\delta_{uv} f_{rst} - 2\delta_{ur} f_{vst} - 2\delta_{us} f_{rvt} - 2\delta_{ut} f_{rsv}. \quad (58)$$

Orthogonalizing the vectors (56) we obtain four orthogonal combinations $\Phi_{rst}^{(i)}$ (i = 1, 2, 3, 4) and further we construct four orthogonal irreducible representations. In analogy with the preceding it is easy to see that the $\Phi_{rst}^{(i)}$ describe three particle states. We stop this discussion here, because in the sequel we consider only states with particle number not exceeding three.

3.2. The internal state of a system of f particles will now be determined by a state vector $|f, \alpha, \nu, \beta\rangle$ where: 1) the index α indicates that the vector belongs to a certain group of equivalent representations, 2) the label ν indicates the symmetry (Young pattern) of the vector with respect to permutations of the one-particle states $k_1, \ldots, k_f$, 3) the label β denotes that the vector belongs to a definite representation from the group α of equivalent representations. Vectors characterized by the criteria 1) and 2) will be called essentially distinct. As shall be seen below, the states described by these vectors can be associated in the usual classification in terms of SU(3) with states of different strangeness. For vectors which differ only in the criterion 3) one can form any orthogonal combinations without violating the irreducibility of the representation. In other words, within the space of equivalent representations one can effect arbitrary rotations, and therefore to vectors of these representations one can associate states with different isospin states for equal values of strangeness.

We obtain the following classification.

The one-particle states form the isosinglet $(1/\sqrt{3})b_r^*|0\rangle$ and the isodoublet $\Phi_r^{(i)}$ $(i = 1, 2)$.

The states of a particle-antiparticle system form: two isosinglets $(1/2\sqrt{6})\,[b_{r-}^*, b_{s+}^*]_+|0\rangle$ and $(1/2\sqrt{3})\,[b_{r-}^*, b_{r+}^*]_-|0\rangle$, two isodoublets $(1/\sqrt{2}$ $(1/\sqrt{2})(b_{r-}^*\Phi_{s+}^{(i)} + b_{s+}^*\Phi_{r-}^{(i)})(i = 1, 2)$ and $(1/\sqrt{10})\,(b_{r-}^*\Phi_{s+}^{(i)} - b_{s+}^*\Phi_{r-}^{(i)}$ $(i = 1, 2)$, one isotriplet $\Phi_{r-,s+}^{(i)}$ $(i = 1, 2, 3)$.

In classifying the three-particle states the consideration of symmetry properties gets somewhat more involved. For an arbitrary function f_{xyz} one can form six combinations belonging to different irreducible representations of the permutation group of the indices x, y, z. In our case one must indicate the Green relations (2) $(\epsilon = -)$:

$$b_r^*b_s^*b_t^* - b_r^*b_t^*b_s^* - b_s^*b_t^*b_r^* + b_t^*b_s^*b_r^* = 0 \text{ и}$$
$$b_s^*b_r^*b_t^* - b_s^*b_t^*b_r^* - b_r^*b_t^*b_s^* + b_t^*b_r^*b_s^* = 0, \qquad (59)$$

and also the relation (50) and the antisymmetry properties of the preceding vectors with respect to their indices.

The only symmetric vector forms a singlet:

$$(1/2[6(1 + \delta_{rs} + \delta_{ts} + \delta_{rt} + 2\delta_{st}\delta_{rt})]^{1/2})\,(b_r^*b_s^*b_t^* + b_t^*b_s^*b_r^*)\,|0\rangle.$$

The antisymmetric vectors form a decuplet, consisting of:

1) an isosinglet $(1/6\sqrt{5})\,(2b_r^*b_s^*b_t^* - 2b_r^*b_t^*b_s^* - b_s^*b_r^*b_t^* + b_t^*b_r^*b_s^*)\,|0\rangle$,

2) an isodoublet $(1/\sqrt{15})\,(b_r^*b_s^*\Phi_t^{(i)} + b_t^*b_r^*\Phi_s^{(i)} + b_s^*b_t^*\Phi_r^{(i)})$ $(i = 1, 2)$,

3) an isotriplet $(1/\sqrt{21})\,(b_r^*\Phi_{st}^{(i)} - b_s^*\Phi_{rt}^{(i)} - b_t^*\Phi_{sr}^{(i)})\,|\,(i = 1, 2, 3)$,

4) an isoquartet $\Phi_{rst}^{(i)}$ $(i = 1, 2, 3, 4)$.

The vectors of mixed symmetry, symmetric in r and s form an octet:

5) one isosinglet $(1/2\sqrt{6(\ldots)})\,(b_s^*b_r^*b_t^* - b_t^*b_s^*b_r^*)\,|0\rangle$,

6) two isodoublets, $(1/[2(1 \mp 4/\sqrt{21})(\ldots)]^{1/2})[(1/2\sqrt{3})\,(b_r^*b_s^*\Phi_t^{(i)} + b_s^*b_r^*\Phi_t^{(i)}) \pm (1/\sqrt{7})\,(b_r^*b_t^*\Phi_s^{(i)} + b_s^*b_t^*\Phi_r^{(i)})]$ $(i = 1, 2)$,

7) one isotriplet $(1/\sqrt{(\ldots)})\,(b_r^*\Phi_{st}^{(i)} + b_s^*\Phi_{rt}^{(i)})$ $(i = 1, 2, 3)$,

here $(\ldots) = (2 - \delta_{rt} + 2\delta_{rs} - \delta_{st} - 2\delta_{st}\delta_{rt})$.

The vectors of mixed symmetry which are antisymmetric in r and s also form an octet:

8) one isosinglet $(1/6\sqrt{2(\ldots)})\,(2b_r^*b_t^*b_s^* - b_s^*b_r^*b_t^* - b_t^*b_s^*b_r^*)\,|0\rangle$,

9) two isodoublets $(1/[2(1 \mp 4/\sqrt{21})(\ldots)]^{1/2})[(1/6)\,(b_r^*b_t^*\Phi_s^{(i)} + b_t^*b_r^*\Phi_s^{(i)} - b_t^*b_s^*\Phi_r^{(i)} - b_s^*b_t^*\Phi_r^{(i)}) \pm (1/\sqrt{21})\,(b_r^*b_s^*\Phi_t^{(i)} + b_t^*b_s^*\Phi_r^{(i)} - b_s^*b_r^*\Phi_t^{(i)} - b_t^*b_r^*\Phi_s^{(i)})]$ $(i = 1, 2)$,

10) one isotriplet $(1/\sqrt{3(\ldots)})\,(2b_t^*\Phi_{rs}^{(i)} + b_r^*\Phi_{ts}^{(i)} - b_s^*\Phi_{tr}^{(i)})$ $(i = 1, 2, 3)$,

here $(\ldots) = (2 + \delta_{rt} - 2\delta_{rs} + \delta_{st} - 2\delta_{st}\delta_{rt})$.

3.3. Thus for the indicated classification of paraparticle states the same multiplets have arisen as in the classification according to the composite SU(3) model of physical particles, where all particles are built up out of a triplet of quarks. An interesting consequence of the fact that the paraparticle s states belong simultaneously to irreducible representations of the PFR and to irreducible representations of the internal symmetry group SU(3) is the fact that the latter are from the outset classified into "isomultiplets," which differ in their "strangeness." Thus, in the triplet of one-particle states one of the states $(1/\sqrt{3})b_r^*|0\rangle$ is distinguished from the other two $\Phi_r^{(i)}$ $(i = 1, 2)$, referring to equivalent representations, corresponding to the fact that the λ-quark is distinguished from the p- and n-quarks.

As in the case of second order paraquantization one can define the hermitian operators

$$\beta_m = \gamma_m^{(1)} + \gamma_m^{(2)} + \gamma_m^{(3)}, \qquad (60)$$

where the operators γ_m^A are defined by (26) for A = 1, 2, 3. Then the construction of irreducible representations of the PFR corresponds to separating representations of definite "spin" in a multidimensional space of even dimension. Thus, in a four-dimensional space to a spin 3/2 representation and two equivalent spin ½ representations obtained by adding in two different ways three spin-½ states into a half-integral spin, correspond representations formed on "preceding" vectors $|0\rangle$, $\Phi_r^{(1)}$ and $\Phi_r^{(2)}$. In this sense the distinguished character of the λ-quark with respect to the p- and n-quarks corresponds the difference between the representation of "spin" 3/2 from two equivalent "spin" ½ representations.

4. CONCLUSION

The preceding considerations show that in constructing all the irreducible representations (in a separable Hilbert space) of the PFR of second and third order, internal symmetries like isospin and strangeness can be introduced in a consistent way. On the other hand it shows that these schemes do not involve any generalizations of the statistics of elementary particles, except the taking into account of internal degrees of freedom with the ordinary statistics. Indeed, in the PFR scheme of second order a matrix transformation associates explicitly to every paraparticle state vector a state of ordinary fermions having an "isospin" degree of freedom. Thus, if we consider as it is usually done only one irreducible representation of the PFR containing the unique vector Φ_0, $b_k\Phi_0 = 0$, the states of this representation correspond, according to the results of Subsection 2.4 (up to normalization) to the following fermion states:

$$b^*_{k_1}\dots b^*_{k_f}\Phi_0 \approx \begin{cases} \overset{*}{n}_{k_1}\overset{*}{p}_{k_2}\overset{*}{n}_{k_3}\overset{*}{p}_{k_4}\dots \overset{*}{n}_{k_{f-1}}p^*_{k_f}|0\rangle\otimes\xi_0 & \text{for even f} \\ \overset{*}{p}_{k_1}\overset{*}{n}_{k_2}\overset{*}{p}_{k_3}\overset{*}{n}_{k_4}\dots \overset{*}{n}_{k_{f-1}}\overset{*}{p}_{k_f}|0\rangle\otimes\sigma_+\xi_0 & \text{for odd f} \end{cases}$$

Thus, the usual interpretation of the PFR reduces to the selection from among all possible fermion states possessing an "isospin" degree of freedom of only those for which $I_3 = 0$ or $+\frac{1}{2}$. It was also possible to introduce, in the framework of the parafield theory an electromagnetic interaction which violates "isospin" symmetry. For this it was necessary to modify the definition of charge conjugation which leaves this interaction invariant.

For the case of third order PFR it was also possible to carry out a classification of the paraparticles and to show that it coincides with the classification according to the usual SU(3) symmetry, in the sense of the existence of the same multiplets in both schemes, and of a correspondence between "equivalent" states of paraparticles with "isospin" states. However, in the present paper the "isospin" and "strangeness" operators have not been obtained explicitly, and this has prevented us from exhibiting explicitly the paraparticle states in terms of vectors describing fermions which can be in three internal states (p, n, λ). But in this case also one can assert that the third order PFR scheme is equivalent to the ordinary Fermi statistics in the presence of "unitary" internal symmetry. The usual restriction to only irreducible representations of the PFR with the abovementioned properties corresponds in this case to the separation of only certain among all the internal states of the fermion system. Therefore the tempting assumption[20] that the quark fields are subject to third order PFR seems to be equivalent to the assumption that they exhibit an internal degree of freedom. The same refers to the proposal[3] to consider the electron and muon as differing in the quantization rules of the appropriate fields.

An interesting consequence of the fact that the state vectors of paraparticles form both irreducible representations of the PFR and of the internal symmetry group SU(3) is the fact that in the latter the states of different "strangeness" are distinguished from the outset. At the same time it remains an open question whether it is possible to introduce an interaction which violates the "unitary" symmetry. This is related to the fact that in the case of second order PFR the currents taken in the form of commutators or anticommutators are both local, whereas in the case of the third order PFR only the first one is local. Therefore, whereas in the first case both currents entered into the interaction (42) in the form of their sum, in the second case both terms can not occur in one interaction. The anticommutator current can here appear only into the interaction with a paraboson field of the third order[4].

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