APPLICATION OF THE SMOOTH-PERTURBATION METHOD TO THE SOLUTION OF GENERAL EQUATIONS OF MULTIPLE WAVE-SCATTERING THEORY

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Submitted December 8, 1967

Zh. Eksp. Teor. Fiz. **54**, 1775–1784 (June, 1968)

We consider the effect of a plane monochromatic wave (a particle beam in quantum mechanics, an acoustic or electromagnetic wave) incident on a half-space filled with a scattering medium having large-scale fluctuations of the effective scattering potential. The investigation is based on general equations of the Dyson type for the mean field and an equation of the Bethe-Salpeter type for the field correlation function. Simplified, purely differential equations for the mean field and field correlation function are derived in the approximation of weak nonlocality of the mass-operator and intensity-operator kernels. The equation for the field correlation function thus obtained is solved by the Rytov smooth-perturbation method. A connection is established between the obtained solution and the results of application of the smooth-perturbation theory and the Kalashnikov-Ryazanov method to the nonaveraged wave equation, and also its relation to the radiation-transfer equation in the small-angle approximation. The conditions of applicability of the smooth-perturbation method are formulated on the basis of a calculation of the complex phase of the second approximation. Restrictions on the parameters of the problem, which ensure that the method is applicable at distances on the order of or larger than the extinction length, are indicated.

A number of recent papers devoted to the theory of multiple scattering of waves employ the Dyson and Bethe-Salpeter equations for the solution of general and concrete problems. The name of these equations and the method of their derivation are borrowed from quantum field theory. In the theory of multiple wave scattering, these equations were introduced in the papers of Foldy[1], Bouret[2], Tatarskiĭ and Gertsenshteĭn[3], Furutsu[4], Tatarskiĭ[5], Frisch[6-8], and Finkel'berg[9].

These papers contain, besides an investigation of the main problem of the asymptotic form of the kernels of the equations, also the solution of the Dyson equation for a statistically homogeneous unbounded and isotropic scattering medium. Ryzhov, Tamoĭkin, and Tatarskiĭ[10], using the results of Finkel'berg[11], employed Dyson's equation to calculate the tensor of the effective dielectric polarization of a statistically homogeneous and isotropic medium. Ryzhov[12] considered an analogous problem for an anisotropic medium. Tatarskiĭ[13,14] used the Bethe-Salpeter equation in the so-called ladder approximation for the calculation of the correlation function of the field in a statistically homogeneous unbounded medium with small-scale fluctuations of the refractive index. In our paper[15], and also in a paper written jointly with Finkel'berg[16], the Dyson and Bethe-Salpeter equations are the basis of a statistical derivation of a radiation transport equation.

In the present paper we use equations of the Dyson and Bethe-Salpeter type to consider the problem of the incidence of waves on a scattering medium occupying a bounded or unbounded region of space. The paper consists of two parts. In the first we derive equations that are much simpler than the initial ones and are valid in the case of weak nonlocality of the mass operator and of the intensity operator. The second part of the paper is devoted to an approximate solution of the obtained equations by Ryzhov method of smooth perturbations.

1. Let the scattering medium occupy the right half-space $z > 0$ in a coordinate frame (x, y, z) the z axis of which is perpendicular to the separation boundary, which coincides with the xy plane. The incident field $\psi_0(\mathbf{r})$ is chosen in the form of a plane monochromatic wave, $\psi_0(z) = \exp(ik_0z)$, incident on the medium from the left half-space $z < 0$ normally to the separation boundary. This may be a de Broglie wave describing the monoenergetic flux of particles in quantum mechanics, a monochromatic acoustic wave, or an electromagnetic wave[1)]. We are interested in the average field $\langle\psi(\mathbf{r})\rangle$ and its correlation function $\langle\psi(\mathbf{r}_1)\overline{\psi}(\mathbf{r}_2)\rangle$. Here $\mathbf{r}$, $\mathbf{r}_1$, and $\mathbf{r}_2$ are points in three-dimensional space, the superior bar indicates the complex conjugate, and the angle brackets denote averaging over the ensemble of realizations of the scattering medium.

The non-averaged field $\psi(\mathbf{r})$ satisfies the wave equation

$$[\Delta + k_0^2 - V(\mathbf{r})]\psi(\mathbf{r}) = 0, \qquad (1)$$

where k_0 is the wave number of the free space and $V(\mathbf{r})$ is the effective potential of the scattering medium. For a continuous medium, the effective potential is $V(\mathbf{r}) = -k_0^2\mu(\mathbf{r})$, where $\mu(\mathbf{r})$ is the fluctuating part of the square of the refractive index. If the medium is discrete, then the effective potential is equal to the sum of the potentials of the individual scatterers, the position of which is random. In order to satisfy the radiation condition, we shall assume that the wave number k_0 of free space has a small positive imaginary part $\epsilon > 0$ ($k_0 \to k_0 + i\epsilon$).

Our solution of the problem is based on the general equations for the average field and its correlation function. The equations are of the form

1)As is well known (see, for example, [5]) in the case of large scale fluctuations of the scattering medium the system of Maxwell's equation reduces to three independent scalar equations.

$$\langle\psi(\mathbf{r})\rangle = \psi_0(\mathbf{r}) + \int G_0(\mathbf{r}-\rho)\, d^3\rho M(\rho, \mathbf{r}')\, d^3\mathbf{r}' \langle\psi(\mathbf{r}')\rangle, \tag{2}$$

$$\langle\psi(\mathbf{r}_1)\overline{\psi}(\mathbf{r}_2)\rangle = \langle\psi(\mathbf{r}_1)\rangle\langle\overline{\psi}(\mathbf{r}_2)\rangle + \int \langle G(\mathbf{r}_1, \rho_1)\rangle \times \langle\overline{G}(\mathbf{r}_2, \rho_2)\rangle\, d^3\rho_1 d^3\rho_2 K(\rho_1, \mathbf{r}_1'; \rho_2, \mathbf{r}_2')\, d^3\mathbf{r}_1'\, d^3\mathbf{r}_2' \langle\psi(\mathbf{r}_1')\overline{\psi}(\mathbf{r}_2')\rangle. \tag{3}$$

Here

$$G_0(\mathbf{r}-\mathbf{r}') = \exp(ik_0|\mathbf{r}-\mathbf{r}'|) / (-4\pi|\mathbf{r}-\mathbf{r}'|)$$

is the retarded Green's function of free space, and $\langle G(\mathbf{r}, \mathbf{r}')\rangle$ is the average Green's function of the scattering medium, satisfying Eq. (2) in the presence in space at the point $\mathbf{r}'$ of a pointlike source, producing the incident field $\psi_0(\mathbf{r}) = G_0(\mathbf{r}'\mathbf{r}')$. In this case Eqs. (2) and (3) go over into the Dyson and Bethe-Salpeter equations. The kernels M and K of the equations are customarily called the mass operator and the intensity operator.

In the literature there are three known approximations for the mass operator M and for the intensity operator K. These are: 1) the Foldy approximation, the Bourret approximation and the ladder approximation, and 3) the Finkel'berg approximation.

In the Foldy approximation, which pertains to a discrete medium consisting of uncorrelated scatterers, the kernels M and K are expressed in terms of the scattering operator of an isolated scatterer and its bilinear combination.

In the Bourret approximation (for the mass operator) and the ladder approximation (for the intensity operator) pertain to a continuous medium the square of the refractive index of which fluctuates in accordance with a normal law. In this approximation

$$M(\mathbf{r}, \mathbf{r}') = B(\mathbf{r}, \mathbf{r}')G_0(\mathbf{r}-\mathbf{r}'), \tag{4}$$

$$K(\mathbf{r}_1, \mathbf{r}_1'; \mathbf{r}_2, \mathbf{r}_2') = B(\mathbf{r}_1, \mathbf{r}_2)\delta(\mathbf{r}_1-\mathbf{r}_1')\delta(\mathbf{r}_2-\mathbf{r}_2'), \tag{5}$$

where $B(\mathbf{r}_1, \mathbf{r}_2) = \langle V(\mathbf{r}_1)V(\mathbf{r}_2)\rangle$ is the correlation function of the fluctuations of the effective potential ($\langle V\rangle = 0$).

The Finkel'berg approximation generalizes the aforementioned approximations to the case when the correlation between the scatterers or the deviations of the fluctuations of the square of the refractive index from the normal law are significant.

A characteristic feature of the foregoing approximations is that the corresponding kernels M and K have finite nonlocality radii l_M and l_K. In order to explain this circumstance, we introduce the following definition.

We represent the kernels M and K in the form $M(\mathbf{r}, \mathbf{r}') = \mathcal{M}(\mathbf{r}, \mathbf{r}-\mathbf{r}')$ and $K(\mathbf{r}_1, \mathbf{r}_1'; \mathbf{r}_2, \mathbf{r}_2') = \mathcal{K}(\mathbf{r}_1, \mathbf{r}_1-\mathbf{r}_1'; \mathbf{r}_2, \mathbf{r}_2-\mathbf{r}_2')$, separating explicitly the dependence on the argument differences $\mathbf{r}-\mathbf{r}'$, $\mathbf{r}_1-\mathbf{r}_1'$, and $\mathbf{r}_2-\mathbf{r}_2'$. We shall say that the kernels M and K have finite effective nonlocality radii l_M and l_K if the functions $\mathcal{M}$ and $\mathcal{K}$ tend sufficiently rapidly to zero when the foregoing differences of the argument exceed l_M and l_K in absolute value, i.e., $|\mathbf{r}-\mathbf{r}'| \gg l_M$ and $|\mathbf{r}_1-\mathbf{r}_1'|$, $|\mathbf{r}_2-\mathbf{r}_2'| \gg l_K$.

We can conclude from the foregoing qualitative definition that in the Foldy approximation the nonlocality radii of the kernels M and K are of the order of the radius of the scatterers. Analogously, in the Bourret approximation the nonlocality radius of the kernel M is of the order of the correlation radius of the potential fluctuations. In the ladder approximation the kernel K has a zero nonlocality radius. In the Finkel'berg approximation, both nonlocality radii are of the order of the correlation radius of the group of scatterers or the aggregate of values of the potential in a number of points.

Our definition of the nonlocality radii calls for a refinement. In this connection we propose that the mass operator and the intensity operator have the following Fourier transforms with respect to the difference arguments:

$$\widetilde{\mathcal{M}}(\mathbf{r}, \mathbf{k}) = \int \mathcal{M}(\mathbf{r}, \mathbf{r}')\exp(-i\mathbf{k}\mathbf{r}')\, d^3\mathbf{r}', \tag{6}$$

$$\widetilde{\mathcal{K}}(\mathbf{r}_1, \mathbf{k}; \mathbf{r}_2, \mathbf{k}') = \int \mathcal{K}(\mathbf{r}_1, \mathbf{r}_1'; \mathbf{r}_2, \mathbf{r}_2')\exp(-i\mathbf{k}\mathbf{r}_1')\exp(i\mathbf{k}'\mathbf{r}_2')\, d^3\mathbf{r}_1' d^3\mathbf{r}_2', \tag{7}$$

which can be differentiated a sufficient number of times with respect to the wave vectors $\mathbf{k}$ and $\mathbf{k}'$. The nonlocality radii l_M and l_K can now naturally be defined with the aid of a logarithmic differentiation of the Fourier transforms (6) and (7) with respect to the wave vectors $\mathbf{k}$ and $\mathbf{k}'$ on the "energy" shell $k = k' = k_0$.

The intensity operator of a statistically homogeneous medium depends on three differences of the arguments. Two of them were already mentioned. The third is the difference of the first arguments $\mathbf{r}_1 - \mathbf{r}_2$. It also corresponds to a certain spatial scale, which we denote by l. In the ladder approximation, it coincides with the correlation radius of the potential fluctuations.

Let us return to the initial equations (2) and (3) for the average field and its correlation function. We shall solve these equations, making with respect to the kernels M and K only the general assumption that they have finite effective nonlocality radii l_M and l_K. In the investigation of the equations for the purpose of reducing the number of manipulations, it is frequently convenient to use the symbolic operator language, and also the concept of the tensor or direct product[17] of functions and operators, which we denote by the symbol $\otimes$.

We turn first to the equation for the average field (2). We apply to it the operator $\Delta + k_0^2$, going over into an integro-differential equation

$$(\Delta + k_0^2 - M)\langle\psi\rangle = 0. \tag{8}$$

We seek the average field in the form of the product

$$\langle\psi(\mathbf{r})\rangle = \exp\{ik_0 z\}\langle u(\mathbf{r})\rangle, \tag{9}$$

separating the factor equal to the incident field. This manner of expressing the field corresponds to the notion of a wave beam propagating in the direction of the z axis. The function $u(\mathbf{r})$ is called the ray amplitude.

Substituting the sought solution (9) in (8), we arrive on the basis of (6) formally to a differential equation of infinitely large order

$$(\Delta + k_0^2 - \widetilde{\mathcal{M}})\langle\psi\rangle = O(l). \tag{10}$$

Here $\widetilde{\mathcal{M}}(\mathbf{r}, k_0)$ is the Fourier transform of the mass operator with respect to the difference argument, calculated on the energy shell $k = k_0$. In the right side of the equation are the nonlocal terms due to the nonlocality of the mass operator. We denoted them by $O(l)$. They are equal to

$$O(l) = \exp(ik_0z)[\exp(-i\nabla_{\mathbf{k}_0}\nabla) - 1]\tilde{\mathcal{M}}(\mathbf{r}, \mathbf{k}_0)\langle u(\mathbf{r})\rangle = \exp(ik_0z)\{-i[\nabla_{\mathbf{k}}\tilde{\mathcal{M}}(\mathbf{r}, \mathbf{k})]_{\mathbf{k}=\mathbf{k}_0}\cdot\nabla\langle u(\mathbf{r})\rangle - \ldots\}. \quad (11)$$

The exponential operator $\exp(-i\nabla_{k_0}\nabla)$ calls for some explanation. As usual, it is represented by a Taylor series. In the space of the wave vectors, the gradient operator ∇_{k_0}, act only on the Fourier transform of the mass operator $\tilde{\mathcal{M}}$. The gradient operator in the coordinate space, ∇, to the contrary, acts only on the average ray amplitude $\langle u\rangle$. The index k_0 labeling the gradient operator in the wave-vector space denotes that the corresponding derivatives calculated on the energy shell $k = k_0$.

It is easy to see that the nonlocal terms (11) constitute an expansion in powers of the ratio of the nonlocality radius of the mass operator to the inhomogeneity scale of the average ray amplitude. Assuming this ratio to be small and omitting from (10) all the nonlocal terms, we arrive at the simplified equation for the average field

$$(\Delta + k_0^2 - \tilde{\mathcal{M}})\langle\psi\rangle = 0. \quad (12)$$

It has the form of a Helmholtz wave equation with effective wave number k_{eff}, which is determined by the relation $k_{\text{eff}}^2 = k_0^2 - \tilde{\mathcal{M}}(\mathbf{r}, \mathbf{k}_0)$.

Let us analyze the conditions for the applicability of the simplified equation (12). It is natural to require, by the way of such a condition, that the nonlocal terms $O(l)$ be small compared with $\tilde{\mathcal{M}}\langle\psi\rangle$. Retaining in (11) only the first-order spatial derivatives, we arrive at the inequality

$$\nabla_{\mathbf{k}_0}\tilde{\mathcal{M}}\cdot\nabla\langle u\rangle \ll \tilde{\mathcal{M}}\langle u\rangle. \quad (13)$$

Just as all the succeeding inequalities, this inequality should be taken to mean that the modulus of the left hand side is small compared with the modulus of the right hand side.

Let us specify more concretely the condition (13) as applied to the statistically homogeneous and isotropic scattering half space $z > 0$. In this case the Fourier transform with respect to the difference argument $\tilde{\mathcal{M}}(\mathbf{r}, \mathbf{k})$ is constant inside the medium, with the exception of a narrow boundary strip, and equals $\mathcal{M}(\mathbf{r}, \mathbf{k}) = \tilde{M}_0(\mathbf{k}, \mathbf{k}) \equiv \tilde{M}_0(k^2)$, where $\tilde{M}_0(\mathbf{k}, \mathbf{k}')$ is the Fourier transform of the specific mass operator[16] characterizing the scattering properties of the individual inhomogeneity of the medium. We call attention to the fact that, by virtue of the mass statistical isotropy of the medium, the Fourier transform of the mass operator with respect to the different arguments depends only on the square of the wave vector.

The simplified equation (12) is easy to solve[2] for a homogeneous scattering half-space. The average field inside the medium has the form of a plane wave $\langle\psi(z)\rangle = T\exp(ik_{\text{eff}}z)$ with effective wave number k_{eff}. By T we denote the amplitude refraction coefficient, which is determined from the boundary condition of the continuity of the average field and its normal derivative. Separating the average ray amplitude from the average field, in accordance with (9), and substituting it in (13), we get

$$d\tilde{M}_0/dk_0^2 \ll 1. \quad (14)$$

The physical meaning of the inequality (14) is discussed in the cited papers[9,16]. Together with the inequality $\tilde{M}_0/k_0^2 \ll 1$, it denotes neglect of the spatial dispersion of the wave.

Let us formulate the condition (14) in terms of the nonlocality radius of the mass operator and the inhomogeneity scale of the average ray amplitude. To this end, the mass operator is taken in the Bourret approximation (4), and the correlation function of the fluctuations of the effective potential is taken in the form of the exponential $B(|\mathbf{r}|) = k_0^4\sigma^2\exp(-|\mathbf{r}|/l)$. We confine ourselves to the most interesting case of large-scale fluctuations of the potential, when $k_0 l \gg 1$. In this case the inequality (14) takes the form

$$l/d \ll 1, \quad (15)$$

where d is the extinction length, the reciprocal of which is $1/d = 2\operatorname{Im} k_{\text{eff}} \sim \sigma^2 k_0^2 l$. The nonlocality scale of the mass operator $l_M = d\ln\tilde{M}_0/dk_0$ turns out to be of the order of the correlation radius of the fluctuations of the potential, $l_M \sim l$. On the other hand, the role of the inhomogeneity scale of the average ray amplitude is played by the extinction length d.

We proceed now to the equation for the field correlation function (3), from which we can also derive the corresponding simplified equation. The reasoning is similar to that used in the consideration of the average field. We first operate on the equation with the tensor product of the operators $(\Delta + k_0^2 - M) \otimes (\Delta + k_0^2 - \overline{M})$, transforming it into an integro-differential equation

$$(\Delta + k_0^2 - M) \otimes (\Delta + k_0^2 - \overline{M})\langle\psi\otimes\overline{\psi}\rangle = K\langle\psi\otimes\overline{\psi}\rangle. \quad (16)$$

The correlation function of the field is sought in the form of a product

$$\langle\psi(\mathbf{r}_1)\overline{\psi}(\mathbf{r}_2)\rangle = \exp[ik_0(z_1 - z_2)]\langle u(\mathbf{r}_1)\overline{u}(\mathbf{r}_2)\rangle, \quad (17)$$

separating the factor equal to the correlation function of the ray amplitude of the field. Substituting the sought solution (17) in (16), we arrive, on the basis of (6) and (7), at a differential equation of infinitely high order with nonlocal terms that are due to the nonlocality of both the mass operator and the intensity operator. Leaving out the nonlocal terms, we arrive at a simplified equation for the correlation function of the field

$$(\Delta + k_0^2 - \tilde{\mathcal{M}}) \otimes (\Delta + k_0^2 - \overline{\tilde{\mathcal{M}}})\langle\psi\otimes\overline{\psi}\rangle = \tilde{\mathcal{K}}\langle\psi\otimes\overline{\psi}\rangle, \quad (18)$$

where $\tilde{\mathcal{K}}$, in analogy with $\tilde{\mathcal{M}}$, is the Fourier transform, calculated on the energy shell, of the intensity operator with respect to the difference arguments. The remainder of the problem is to investigate the conditions of applicability of the obtained equation.

In the case of a continuous scattering medium, the square of the refractive index of which fluctuates in accordance with a normal law, the most significant are the nonlocal terms due to the nonlocality of the mass operator. This can be understood at least from the fact that in this case the nonlocality of the intensity operator becomes manifest only in the second order in the correlation function of the fluctuations of the effective potential of the medium.

Let us consider the characteristic aggregate of the nonlocal terms due to the nonlocality of the mass operator

$$[(\Delta + k_0^2) \otimes I]\{\exp[ik_0(z_1 - z_2)][I \otimes (\exp i\nabla_{\mathbf{k}_0}\nabla - 1)](I \otimes \overline{\tilde{\mathcal{M}}})\langle u \otimes \overline{u}\rangle\}, \quad (19)$$

2)The same can be said with respect to the initial equation (8).

where I denotes the unit operator. By way of the condition under which the given aggregate of nonlocal terms can be neglected, we stipulate that the expression in the curly bracket of (19) be small compared with $(I \otimes \overline{\widetilde{\mathcal{M}}})\langle u \otimes \bar{u}\rangle$. Thus

$$[I \otimes (\exp i\nabla_{\mathbf{k}_0}\nabla - 1)](I \otimes \bar{\tilde{\mathcal{M}}})\langle u \otimes \bar{u}\rangle \ll (I \otimes \bar{\tilde{\mathcal{M}}})\langle u \otimes \bar{u}\rangle. \quad (20)$$

We rewrite the condition (20) again specifically for the case of a statistically homogeneous and isotropic scattering half-space, confining ourselves to an examination of large-scale fluctuations of the effective potential. Getting ahead of ourselves, we note that in the case of large-scale fluctuations of the potential the correlation of the field is characterized by two correlation radii[3)]: one longitudinal with respect to the direction of the incidence of the external field, and one transverse radius. The longitudinal correlation radius is of the order of $k_0 l^2$, and is perhaps larger than the correlation radius l of the effective potential. The transverse radius is of the order of l. On the other hand we know that the non-locality radius of the mass operator is also of the order of l. From this it may appear, at first glance, that the nonlocal terms due to the nonlocality of the mass operator are appreciable and cannot be neglected. However, a more careful analysis shows that this is not the case. The reason for this circumstance is connected with the isotropy of the scattering medium.

Let us turn to inequality (20). We expand in it the exponential operator and retain the spatial derivatives of first and second order. We differentiate the Fourier transform of the mass operator with respect to the components of the wave vector

$$\partial\tilde{\mathcal{M}} / \partial k_\mu = 2k_\mu d\tilde{M}_0 / dk^2, \quad (21)$$

$$\partial^2\mathcal{M} / \partial k_\mu \partial k_\nu = 2\delta_{\mu\nu} d\tilde{M}_0 / dk^2 + 4k_\mu k_\nu d^2\tilde{M}_0 / d(k^2)^2. \quad (22)$$

It follows from (21) that there are no transverse first-order spatial derivatives in the left side of (20). As regards the transverse spatial derivatives of second order, by virtue of (22) they appear only in the form of the combination

$$(d\tilde{M}_0 / dk_0^2)\,\Delta_2^\perp\langle u(\mathbf{r}_1)\,u(\mathbf{r}_2)\rangle, \quad (23)$$

where $\Delta_2^\perp$ denotes the transverse Laplacian acting on the second argument $\mathbf{r}_2$. In order of magnitude, the term (23) equals $(1/k_0 l)\,\tilde{\mathcal{M}}\langle u \otimes \bar{u}\rangle$, and is consequently small compared with the right side of the inequality (20).

In order to obtain more detailed conditions that make it possible to neglect the nonlocal terms, it is necessary to specify more concretely the correlation function of the field. We shall calculate it below, solving the simplified equation (18) by the method of smooth perturbations. Substituting the field correlation function calculated in this manner in the inequality (20), we can verify that at coinciding observation points it reduces to the condition

$$(1 / k_0 l)\,(z_1 / d) \ll 1. \quad (24)$$

It includes besides the known parameter $k_0 l$ also the ratio of the distance z_1 traversed by the wave to the extinction length.

2. Having analyzed the conditions for the applicability of the approximate equation (18) for the correlation function of the field, we proceed to the question of methods of its solution. Unlike the initial equation, this equation is purely differential. It admits, in particular, of the use of the well known method of smooth perturbations (Ritov[18], see also[14,19,20]). We shall construct the solution of Eq. (18) by this method and indicate incidentally a number of interesting consequences.

First, we combine k_0^2 and $\tilde{\mathcal{M}}$ in (18) to form the square of the effective number k_{eff}^2. Further, we use as the small expansion parameter the Fourier transform $\tilde{\mathcal{K}}$ of the intensity operator with respect to the difference argument. As usual, we seek the solution in exponential form $\langle\psi(\mathbf{r}_1)\bar{\psi}(\mathbf{r}_2)\rangle = \exp S(\mathbf{r}_1, \mathbf{r}_2)$, expanding the complex phase S in a series in the small parameter, $S = S_0 + S_1 + S_2 + \ldots$. The zeroth-approximation phase S_0 is assumed to equal $S_0 = ik_{\text{eff}}z_1 - ik_{\text{eff}}z_2$, and is expressed in terms of the effective wave number. Substituting the sought solution in (18), we obtain a closed equation for the first-approximation phase S_1 and a chain of equations for the phases of the higher approximations $S_2, \ldots$. These equations are of the form

$$(R \otimes \bar{R})S_1 = \tilde{\mathcal{K}}, \quad (25)$$

$$(R \otimes \bar{R})S_2 = O(S_1^2), \ldots . \quad (26)$$

In the left side of these equations we introduce the operator $R = \Delta + 2ik_{\text{eff}}\partial/\partial z$. If we replace in it the effective wave number k_{eff} by the wave number of free space, we get the characteristic operator that appears when the non-averaged equation (1) is investigated by the method of smooth perturbations. $O(S_1^2)$ in the right side of (26) denotes the aggregate of terms that depend on the first-approximation phase S_1.

In the first approximation, the field correlation function turns out to be

$$\langle\psi(\mathbf{r}_1)\psi(\mathbf{r}_2)\rangle = \langle\psi(z_1)\rangle\langle\psi(z_2)\rangle \exp S_1(\mathbf{r}_1, \mathbf{r}_2), \quad (27)$$

where the amplitude coefficient of refraction T should be set equal to unity in the expression for the average field. The differential equation (25) for the first-approximation phase can be easily solved by representing the solution in the form of an integral. The integral is calculated approximately by the well known method, in which the effect of the reflected radiation is neglected, and the effect of the scattered radiation is calculated in the Fresnel approximation. As a result, the first-approximation phase turns out to be

$$S_1(\mathbf{r}_1, \mathbf{r}_2) = (4\pi)^{-2}k_0^{-2}z_1 \int d^2\mathbf{a}_\perp \exp[i\mathbf{a}_\perp(\mathbf{r}_1^\perp - \mathbf{r}_2^\perp)] \times \exp[-i(a_\perp^2/2k_0(z_1 - z_2)]\tilde{K}_0(\mathbf{k}_0 + \mathbf{a}_\perp, \mathbf{k}_0; \mathbf{k}_0 + \mathbf{a}_\perp, \mathbf{k}_0), \quad (28)$$

where for concreteness we put $z_1 \leq z_2$. By K_0 we denote the Fourier transform of the specific intensity operator[16] that characterizes the scattering properties of the individual inhomogeneity and which coincides in this case with the Fourier transform of $\mathcal{K}(\mathbf{r}_1, \mathbf{k}_0; \mathbf{r}_2, \mathbf{k}_0)$ with respect to the difference of the first arguments $\mathbf{r}_1 - \mathbf{r}_2$.

Let us analyze the obtained expression (28). In the ladder approximation, when the Fourier transform of a specific intensity operator $\tilde{K}_0$ coincides with the Fourier

[3)]We determine the correlation radii by logarithmic differentiation of the correlation function of the ray amplitude of the field in coinciding observation points.

transform of the correlation function of the effective potential, the first-approximation phase (28) goes over into the mean value of the bilinear combination $\varphi_1(\mathbf{r}_1)\bar{\varphi}_1(\mathbf{r}_2)$ of the complex phase $\varphi_1(\mathbf{r})$, obtained by applying the method of smooth perturbations to the non-averaged wave equation (1). For a discrete medium in the Foldy approximation, when the Fourier transform of the specific intensity operator K_0 is equal to the product of the density of the scatterers by the bilinear combination of the Fourier transform of the scattering operator of the isolated scatterer, the correlation function of the field (27), calculated with the aid of the complex phase (28), coincides with the result of the averaging of the bilinear combination $\psi(\mathbf{r}_1) \cdot \psi(\mathbf{r}_2)$ of the field $\psi(\mathbf{r})$, obtained by applying the method of Kalashnikov and Ryazanov[21] to the non-averaged equation (1).

It is very important to consider the expression (28) from the point of view of the law of energy conservation. In the theory of multiple scattering of waves, the law of energy conservation is formulated in the form of an optical theorem[22], which establishes the connection between the imaginary part of the mass operator and the intensity operator. On the basis of the optical theorem, we obtain immediately from (27) and (28) that the mean square of the modulus of the field has a constant quantity, $\langle|\psi|^2\rangle = 1$.

Connected with the energy conservation law and with the optical theorem is another important property of the complex phase (28). In order to formulate this law, it is necessary to introduce the transverse spectrum (the Wigner function $f_W(z, \mathbf{p}_\perp)$ of the field, defining it as the Fourier transform with respect to the difference of the transverse coordinates $\mathbf{r}_1^\perp - \mathbf{r}_2^\perp$ of the correlation function of the field $\langle\psi(\mathbf{r}_1)\psi(\mathbf{r}_2)\rangle$ in the plane $z_1 = z_2 = z$. With the aid of (27), (28), and the optical theorem it is easy to verify that the transverse spectrum satisfies the radiation-transport equation in the small-angle approximation[23, 24].

Let us discuss the important question of the conditions of applicability of the first approximation of the method of smooth perturbations to the simplified equation (18). To answer this question it is necessary to calculate the complex second-approximation phase $S_2(\mathbf{r}_1, \mathbf{r}_2)$. Such a calculation was carried out in the ladder approximation for coinciding observation points. The condition for the applicability of the first approximation of the method of smooth perturbations is the requirement that the modulus of the complex phase of the second approximation be small compared with unity, $|S_2| \ll 1$. As a result of the calculations, this condition reduces to the inequality

$$z_1 / d \ll (k_0 l^2 / d)^{1/2} (k_0 l)^{1/2}. \quad (29)$$

In the right side of the inequality (29) there appear two dimensionless parameters, $k_0 l^2/d$ and $k_0 l$. The first is equal to the ratio of a quantity on the order of the longitudinal correlation radius of the field to the extinction length. This parameter should be small, $k_0 l^2/d \ll 1$, in view of the fact in calculating the integral for the first-approximation phase S_1 we neglected the effect of the reflected radiation. The second parameter, to the contrary, is large, $k_0 l \gg 1$, since the fluctuations of the effective potential are assumed to be large-scale. In order for the right side of the inequality (29) be nevertheless of the order of or larger than unity, it is necessary to impose the additional requirement

$$k_0 l^2 / d \gtrsim 1 / k_0 l. \quad (30)$$

When (30) is satisfied, the permissible values of the distance v_1 covered by the wave are comparable with the extinction length, or even exceed it. However, in this case very stringent limitations are imposed on the parameters of the problem. Thus, for example, the double inequalities $1/k_0 l \lesssim k_0 l^2/d \ll 1$ or $1/(k_0 l)^4 \lesssim \sigma^2 \ll 1/(k_0 l)^3$ are obtained.

It is of interest to compare the condition for the applicability of the method of smooth perturbations (29) with the condition of applicability (24) of the simplified equation for the correlation function of the field. Rewriting the inequality (29) in the form $(1/k_0 l)(z_1/d) \ll (l/d)^{1/2}$ and taking (15) into account, we conclude immediately that with respect to the distance covered by the wave the condition for the applicability of the simplified equation is much broader than the condition for the applicability of the method of smooth perturbations.

In conclusion, we turn once more to the simplified equations for the average field (12) and its correlation function (18). We have analyzed the conditions of applicability of these equations to the case of large-scale fluctuations of the effective potential of the medium. It is easy to perform a similar analysis in the opposite limit of small-scale fluctuations, when $k_0 l \ll 1$. In this case the condition of applicability of the simplified equation for the average field has as before the form of the inequality (15). The reason lies in the fact that for small-scale fluctuations the nonlocality radius of the mass operator is of the order of $l_M \sim k_0 l^2$, and the inhomogeneity scale of the average amplitude $\langle u(z)\rangle$ is of the order of $k_0 l d$. To the contrary, the condition for the applicability of the simplified equation for the correlation function of the field undergoes an appreciable change. It can be obtained again from the inequality (20), by substituting in it the correlation function of the field, corresponding this time to the small-scale fluctuations of the potential. On the basis of the Foldy model of isotropic pointlike scatterers, investigated in[15], it is easy to conclude that in the case of interest to us the correlation radius of the field correlation function is of the order of the wavelength $2\pi/k_0$ at large optical depth $z \gg d$. It follows therefore that at a large optical depth the inequality (20) reduces simply to the requirement $k_0 l \ll 1$, and is thus satisfied automatically.

The author thanks V. M. Komissarov and V. M. Finkel'berg for discussions contributing to a clarification of the number of problems.

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Translated by J. G. Adashko
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