DENSITY FLUCTUATIONS CORRELATIONS IN A CLASSICAL PERFECT GAS

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We propose a new method to evaluate the density fluctuations correlations and spectra in a classical perfect gas; this method leads to results which are the same as the well-known ones if the gas has infinite in volume or is enclosed between ideally reflecting screens; new results are obtained for the cases when absorption of particles takes place at the screens and also when we take into account the fluctuations in the rate at which gas particles are produced. The general formulae derived by us enable us to calculate the density correlations both for an arbitrary Markovian and also for a non-Markovian character of particle trajectories. As example we evaluate the correlation spectrum for a set of particles when they are generated in the volume and through a diffusion current through the boundary; we establish the high-frequency asymptotic behavior of such spectra which is different from that in earlier published papers[6,7].

1. INTRODUCTION

ALREADY Smoluchowski[1,2] considered the problem of fluctuations in a classical perfect gas, and it has recently aroused new interest mainly in connection with calculations of conductivity fluctuations in semiconductors.[3-7] Most recognition,[7] apparently was given to Lax's method based upon the relation between the single-time $K_p(\mathbf{r}, t, \mathbf{r}', t)$ and the different-time $K_p(\mathbf{r}, t, \mathbf{r}', t')$ density correlations:

$$K_p(\mathbf{r},t,\mathbf{r}',t') = \int\limits_V G(\mathbf{r}',t'|\mathbf{r}'',t) K_p(\mathbf{r}'',t,\mathbf{r},t)\,d\mathbf{r}'', \tag{1}$$

where V is the volume of the system and $G(\mathbf{r}', t' | \mathbf{r}'', t)$ the Green function of the problem which characterizes the average rate of dissipation of the fluctuations.

It is well known[8] that for a uniform, isotropic perfect gas of infinite extension the single-time correlation can be expressed by the formula

$$K_p(\mathbf{r},\ t,\ \mathbf{r}',\ t) = \bar{p}\delta(\mathbf{r}'-\mathbf{r}), \tag{2}$$

where $\bar{p}$ is the average density of the gas. At the same time it is clear that for a gas bounded by "screens" Eq. (2) can not be valid: if the screens are perfectly reflecting and the total number P of particles in the system is rigorously conserved there must occur in Eq. (2) yet one more term which can be determined just from this conservation condition:[8] however, when absorption on the screens takes place, $\bar{p}$ can clearly not be conserved and Eq. (2) needs an additional correction. Moreover, a stationary state of a gas enclosed between absorbing screens is brought about by the presence of the production of its particles and the inevitable fluctuation in the rate of production involves in turn a particular change in the equation for $K_p(\mathbf{r}, t, \mathbf{r}', t)$.

The fluctuations in the quantity P in the presence of absorbing screens were considered in[6,7] under the assumption that Eq. (2) was valid and this leads, as we shall show, to an unjustified conclusion about the form of the frequency dependence of the spectral density of the fluctuations also in the case where Eq. (1), which was obtained in[5] heuristically, is by itself valid.

Yet the function $K_p(\mathbf{r}, t, \mathbf{r}', t')$ can be evaluated directly by a relatively simple method, the essence of which is clear from the following example.

Let it be necessary to evaluate the density dispersion in a volume element v of a perfect gas enclosed in a vessel of volume V with ideally reflecting walls. Each particle in the gas has a probability $\alpha = v/V$ to be inside and a probability $1 - \alpha$ to be outside the volume element v; the dispersion in the gas density produced by one particle is thus determined by the formula

$$\overline{\Delta_1^2} = \left(\frac{1}{v} - \frac{1}{V}\right)^2 \frac{v}{V} + \left(-\frac{1}{V}\right)^2\left(1 - \frac{v}{V}\right) = \frac{1}{v^2}\alpha(1-\alpha). \tag{3}$$

Since the behavior of different particles in a perfect gas is independent, the density dispersion caused by the presence of P particles in the system can be found from the formula

$$\overline{\Delta_\Sigma^2} = P\frac{1}{v^2}\alpha(1-\alpha) \tag{4}$$

or, if $\alpha \ll 1$, by the formula

$$\overline{\Delta_\Sigma^2} = P\frac{1}{v^2}\alpha. \tag{5}$$

Equation (4) corresponds to a binomial distribution and Eq. (5) to a Poisson distribution of the number of particles in a volume element v; there is, however, no necessity whatever to calculate these distributions beforehand in the present problem.

The simplest example considered here, that of calculating the moment of the density distribution produced by all particles of a perfect gas using the moment of the density distribution produced by a single particle, can be generalized to calculate the function $K_p(\mathbf{r}, t, \mathbf{r}', t')$ under various conditions by summing the correlation functions of "elementary" random density fields caused by the motion of separate particles in the gas.

The practical advantage of such a procedure is that the statistical characteristics of an "elementary" random density field can be evaluated by standard methods (e.g., by considering the random walk of a single particle in configuration space in the diffusion approximation and accordingly using the Fokker-Planck equation), while there are no standard equations at all for the resulting random density field.

2. DENSITY FLUCTUATIONS IN A UNIFORM MEDIUM

In the present section we shall demonstrate the method and consider density fluctuations in a monatomic gas enclosed in a vessel with perfectly reflecting walls or infinite dimensions (in the latter case we assume that as $V \to \infty$ we also have $P \to \infty$, while $\bar{p}$ = const.).

We can consider to begin with an "elementary" random field $x_k(\mathbf{r}, t)$, namely the number of particles in a volume element of arbitrarily small magnitude v_r (with its center at $\mathbf{r}$) which arises due to the random walk of a single particle, and introduce the average value $\bar{x}_k(\mathbf{r}, t)$ and the correlation function $K_{x_k}(\mathbf{r}, t, \mathbf{r}', t')$ of such a field using the standard formulae:

$$\bar{x}_k(\mathbf{r}, t) = \int_{-\infty}^{\infty} x f_{1k}(x, \mathbf{r}, t)\, dx, \tag{6}$$

$$\overline{x_k(\mathbf{r}, t) x_k(\mathbf{r}', t')} = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} x x' f_{2k}(x, \mathbf{r}, t, x', \mathbf{r}', t')\, dx\, dx', \tag{7}$$

$$K_{x_k}(\mathbf{r}, t, \mathbf{r}', t') = \overline{x_k(\mathbf{r}, t) x_k(\mathbf{r}', t')} - \bar{x}_k(\mathbf{r}, t)\bar{x}_k(\mathbf{r}', t'), \tag{8}$$

where $f_{ik}(x, \mathbf{r}, t)$ and $f_{2k}(x, \mathbf{r}, t, x', \mathbf{r}', t')$ are the one- and two-dimensional probability densities for the field $x_k(\mathbf{r}, t)$. Since the field $x_k(\mathbf{r}, t)$ has only two possible values, 0 and 1, $\bar{x}_k(\mathbf{r}, t)$ is, in accordance with Eqs. (6) and (7), simply the probability to find the particle at time t inside the element v_r, while $\overline{x_k(\mathbf{r}, t) x_k(\mathbf{r}', t')}$ is the probability that the particle is at time t in v_r and at the time t' in the element $v_{r'}$.

In the following we assume that the random walk of each particle in the gas is described by some well-known kind of random process, e.g., a diffusion process, and that thus the one-dimensional $(\omega_k(\mathbf{r}, t))$, the two-dimensional $(\omega_k(\mathbf{r}, t, \mathbf{r}', t'))$, and the conditional $(\omega_k(\mathbf{r}', t' \mid \mathbf{r}, t))$ probability densities which characterize the position of the particle in space are well-known functions. If v is sufficiently small, we have clearly:

$$\bar{x}_k(\mathbf{r}, t) = \omega_k(\mathbf{r}, t) v, \tag{9}$$

$$\overline{x_k(\mathbf{r}, t) x_k(\mathbf{r}', t')} = \omega_k(\mathbf{r}, t, \mathbf{r}', t') v^2, \tag{10}$$

$$K_{x_k}(\mathbf{r}, t, \mathbf{r}', t') = \omega_k(\mathbf{r}, t, \mathbf{r}', t') v^2 - \omega_k(\mathbf{r}, t)\omega_k(\mathbf{r}', t') v^2. \tag{11}$$

The resulting random field of "the total number of particles in the volume element $v_{r''}$" is moreover determined by the formula

$$n(\mathbf{r}, t) = \sum_{k=1}^{P} x_k(\mathbf{r}, t). \tag{12}$$

Since the statistical characteristics of all "elementary" random fields are the same in the model considered and the behavior of the separate particles is independent, we have

$$\bar{n}(\mathbf{r}, t) = P\bar{x}(\mathbf{r}, t); \tag{13}$$

$$K_n(\mathbf{r}, t, \mathbf{r}', t') = P K_x(\mathbf{r}, t, \mathbf{r}', t'). \tag{14}$$

By definition the macroscopic density $p(\mathbf{r}, t) = v^{-1} n(\mathbf{r}, t)$ and from the formulae given here it follows that

$$\bar{y}(\mathbf{r}, t) = \omega(\mathbf{r}, t), \tag{15}$$

$$K_y(\mathbf{r}, t, \mathbf{r}', t') = \omega(\mathbf{r}, t, \mathbf{r}', t') - \omega(\mathbf{r}, t)\omega(\mathbf{r}', t') \tag{16}$$

($y(\mathbf{r}, t)$ is an elementary random density field),

$$\bar{p}(\mathbf{r}, t) = P\omega(\mathbf{r}, t), \tag{17}$$

$$K_p(\mathbf{r}, t, \mathbf{r}', t') = \bar{p}(\mathbf{r}, t)[\omega(\mathbf{r}', t' \mid \mathbf{r}, t) - \omega(\mathbf{r}', t')], \tag{18}$$

while we can show rigorously, letting $v \to 0$, that the formulae for the correlations in the density (and in the resulting and "elementary" field) are valid also for $t = t'$.

Of course, in our model the state of the gas is assumed to be stationary and therefore $\omega(\mathbf{r}, t) = 1/V$ = const and $\bar{p} = P/V$ = const, and we can rewrite Eq. (18) as follows

$$K_p(\mathbf{r}, t, \mathbf{r}', t') = \bar{p}[\omega(\mathbf{r}', t' \mid \mathbf{r}, t) - 1/V] \tag{19}$$

and for $t = t'$

$$K_p(\mathbf{r}, \mathbf{r}') = \bar{p}\delta(\mathbf{r}' - \mathbf{r}) - \bar{p}/V. \tag{20}$$

If $V \to \infty$ the second term in Eq. (20) vanishes but for any finite V it may be appreciable, e.g., if we evaluate the dispersion $\overline{(\Delta P_s)^2}$ of the number of particles in a volume V_s comparable with the volume of the whole system; since

$$\overline{(\Delta P_s)^2} = \int_{V_s}\int_{V_s} K_p(\mathbf{r}, \mathbf{r}')\, d\mathbf{r}\, d\mathbf{r}', \tag{21}$$

it follows from (20) that

$$\overline{(\Delta P_s)^2} = \bar{p}V_s - \bar{p}V_s \frac{V_s}{V} \tag{22}$$

and neglect of the second term in (20) may lead to the absurd conclusion that the dispersion of the number of particles in the volume V, where it is strictly conserved, is different from zero.

We note that Eq. (20) follows, generally speaking, from Eqs. (118.3) and (118.6) of [8]; it was also obtained in [7] after a preliminary calculation of the single-time probability density for the distribution of particles in the volume element V. In the framework of the model considered here Eq. (1), due to Lax, also turns out to be valid[1)], since substitution of (20) into (1) again leads to Eq. (19).

In [1,2], to characterize the correlations in the fluctuations in the number of colloidal particles in an observed volume element V_s, the quantity

$$\overline{k^2} = \overline{[P_s(t) - P_s(t')]^2} \tag{23}$$

was introduced.

Under stationary conditions ($\overline{(\Delta P_s)^2}$ = const and $\bar{P}_s$ = const) this quantity can be determined also from the equation

$$\overline{k^2} = 2[\overline{(\Delta P_s)^2} - \overline{\Delta P_s(t)\Delta P_s(t')}] \tag{24}$$

and evaluated using (22), (19) and the formula

$$\overline{\Delta P_s(t)\Delta P_s(t')} = \int_{V_s}\int_{V_s} K_p(\mathbf{r}, t, \mathbf{r}', t')\, d\mathbf{r}\, d\mathbf{r}'. \tag{25}$$

Finally we get

[1)]We assume here that the equation determining $\omega(\mathbf{r}', t \mid \mathbf{r}, t)$ is linear and that this function is the same as the Green function of the problem (we shall return to the problem of the limits of applicability of Eq. (1) in Sec. 4); the correlation formulae are written down for the case $t > t$; when $t > t$ these arguments and also $\mathbf{r}$ and $\mathbf{r}'$ change places on the right-hand side of the equations.

$$\overline{k^2} = 2\bar{P}_s W, \tag{26}$$

where

$$W = 1 - \int_{V_s} \int_{V_s} \frac{1}{V_s} \omega(\mathbf{r}', t' | \mathbf{r}, t)\, d\mathbf{r}\, d\mathbf{r}' \tag{27}$$

is the probability that a particle which at time t is somewhere in the observed volume V_S will be outside it at time t'; this result remains valid also for finite values of a volume V bounded by reflecting screens and also in the case where the volume V_S is part of an infinite medium. Equations (26) and (27) are the same as those given in[1,2], but the method used in those papers is suitable only when the volume V is infinite since it is based upon the calculation of different simultaneous probabilities for numbers of particles which are at time t in the volume V_S and which leave the volume V_S in the interval $t' - t$, and so on, while it is assumed in particular that the number of particles entering in the interval $t' - t$ the volume V_S is independent of $P_S(t)$.

3. DENSITY FLUCTUATIONS IN A GAS WHEN PARTICLES ARE PRODUCED AND ABSORBED

In the present section we shall assume that particles are absorbed at least on part of the screens, which bound the volume of the gas, and that they are produced in the whole volume. At the same time we assume that the fluctuating rate of production of particles per unit volume is characterized by known functions, viz., the average rate of generation $\bar{g}(r_0, t_0)$ and the correlation function of the random sources of production $K_g(r_0, t_0, r_0', t_0')$. We note that we actually have in mind here the production of particles by electromagnetic or other "ionizing" radiation with well-known statistics and in those problems the correlation function of the random field of the rate of production is indeed known and it is not necessary to evaluate it additionally as is the case when we introduce fictitious random sources.

The scheme for calculations[2)] for the given model is not greatly different from the one developed in Sec. 2, but is is necessary to take at once into account that the statistical characteristics of the elementary random fields are now no longer the same and depend on the time and the place where the corresponding particles are produced. What we have just said is, e.g., reflected in the fact that Eq. (12) is replaced by the formula

$$n(\mathbf{r}, t) = \sum_{i,j} x(\mathbf{r}, t, \mathbf{r}_{0i}, t_{0j}), \tag{28}$$

where $\mathbf{r}_{0i}$ and t_{0j} indicate the coordinate and time of the production of the appropriate particle; morevoer, we must introduce in Eqs. (15) and (16) a double index i, j:

$$\bar{y}_{i,j}(\mathbf{r}, t) = \omega_{i,j}(\mathbf{r}, t), \tag{15a}$$

$$K_{y_{i,j}}(\mathbf{r}, t, \mathbf{r}', t') = \omega_{i,j}(\mathbf{r}, t, \mathbf{r}', t') - \omega_{i,j}(\mathbf{r}, t)\, \omega_{i,j}(\mathbf{r}', t') \tag{16a}$$

The individuality of the statistical characteristics of the elementary random fields can be expressed more directly, if we take into account the practical meaning of the index i, j:

$$\omega_{i,j}(\mathbf{r}, t_{0j}) = \delta(\mathbf{r} - \mathbf{r}_{0i}). \tag{29}$$

Defining now the one- and two-dimensional probability densities by means of integrals of the probability densities of higher order, i.e., through the formulae

$$\omega_{i,j}(\mathbf{r}, t) = \int_V \omega_{i,j}(\mathbf{r}, t, \mathbf{r}_1, t_{0j})\, d\mathbf{r}_1, \tag{30}$$

$$\omega_{i,j}(\mathbf{r}, t, \mathbf{r}', t') = \int_V \omega_{i,j}(\mathbf{r}, t, \mathbf{r}', t', \mathbf{r}_1, t_{0j})\, d\mathbf{r}_1, \tag{31}$$

we get by standard methods

$$\bar{y}_{i,j}(\mathbf{r}, t) = \omega(\mathbf{r}, t | \mathbf{r}_{0i}, t_{0j}), \tag{32}$$

$$K_{y_{i,j}}(\mathbf{r}, t, \mathbf{r}', t') = \omega(\mathbf{r}', t' | \mathbf{r}, t, \mathbf{r}_{0i}, t_{0j})\, \omega(\mathbf{r}, t | \mathbf{r}_{0i}, t_{0j}) - \omega(\mathbf{r}, t | \mathbf{r}_{0i}, t_{0j})\, \omega(\mathbf{r}', t' | \mathbf{r}_{0i}, t_{0j}). \tag{33}$$

If the random walk of the particles is described by a Markovian random process the following formula holds

$$\omega(\mathbf{r}', t' | \mathbf{r}, t, \mathbf{r}_{0i}, t_{0j}) = \omega(\mathbf{r}', t' | \mathbf{r}, t), \tag{34}$$

i.e., the conditional probability density at time t' for a particle which at time t is at a position $\mathbf{r}$, is independent of the additional information about the position and time of its production. The assumption about the Markovian nature is not at all necessary for the derivation of the subsequent formulae of the present section and we shall assume it to be satisfied only in order to simplify the notation; of course, the use of Eq. (34) is not connected with any further assumptions about the actual form of the Markovian process.

For the time being we shall also assume that the rate of production does not fluctuate; in that case we get the required formulae at once: since the summation of averages and of correlation functions of elementary random fields under easily justified assumptions can be replaced by an integration, we have

$$\bar{p}(\mathbf{r}, t) = \int_V \int_{-\infty}^{t} \bar{g}(\mathbf{r}_0, t_0)\, \omega(\mathbf{r}, t | \mathbf{r}_0, t)\, d\mathbf{r}_0\, dt_0, \tag{35}$$

$$K_p(\mathbf{r}, t, \mathbf{r}', t') = \int_V \int_{-\infty}^{t} \bar{g}(\mathbf{r}_0, t_0)[\omega(\mathbf{r}', t' | \mathbf{r}, t)\, \omega(\mathbf{r}, t | \mathbf{r}_0, t_0) - \omega(\mathbf{r}, t | \mathbf{r}_0, t_0)\, \omega(\mathbf{r}', t' | \mathbf{r}_0, t_0)]\, d\mathbf{r}_0\, dt_0. \tag{36}$$

When we take the fluctuations in the rate of production into account the number of terms on the right-hand side of Eq. (28) itself is a random number. If the single-time and statistical characteristics of all random terms are essentially different, it is impossible to obtain the explicit dependence of the characteristics of the density fluctuations on the statistical characteristics of the fluctuations in the number of terms, i.e., the fluctuations in the rate of production. However, in the physical problem under consideration this difficulty is absent since we can clearly ascribe the same statistical characteristics[3)] to all fields produced inside a single "physically infinitesimally small" interval $\Delta V \Delta t_0$.

2)In the present section we restrict ourselves to calculating only equations for $\bar{p}(\mathbf{r}, t)$ and $K_p(\mathbf{r}, t, \mathbf{r}', t)$, postponing a discussion of their consequences to Sec. 4.

3)We have essentially already used such an assumption in the foregoing, since the replacement of the summation of the statistical characteristics of the elementary random fields by their integration with as weight the statistical average of the rate of production (Eqs. (35) and (36)), which by its definition is a macroscopic quantity, was basically just an assumption about the constancy of the statistical characteristics of the terms in each of the macroscopic intervals for which the function $\bar{g}(\mathbf{r}_0, t_0)$ is defined.

Taking this fact into account and operating directly with the random density fields, we get

$$\bar{p}(\mathbf{r}, t)_{\text{arb}} = \sum_{i=1}^{V/\Delta V} \sum_{p=-\infty}^{t/\Delta t} \alpha(\mathbf{r}_{0i}, t_{0p}) \bar{y}(\mathbf{r}, t, \mathbf{r}_{0i}, t_{0p}), \tag{37}$$

$$\overline{p(\mathbf{r}, t) p(\mathbf{r}', t')}_{\text{arb}} = \sum_{i=1}^{V/\Delta V} \sum_{p=-\infty}^{t/\Delta t} \sum_{j=1}^{V/\Delta V} \sum_{q=-\infty}^{t'/\Delta t} \alpha(\mathbf{r}_{0i}, t_{0p}) \alpha(\mathbf{r}_{0j}, t_{0q})$$
$$\times \bar{y}(\mathbf{r}, t, \mathbf{r}_{0i}, t_{0p}) \bar{y}(\mathbf{r}', t', \mathbf{r}_{0j}, t_{0q}) + \sum_{i=1}^{V/\Delta V} \sum_{p=-\infty}^{t/\Delta t} \alpha(\mathbf{r}_{0i}, t_{0p}) K_y(\mathbf{r}, t, \mathbf{r}', t', \mathbf{r}_{0i}, t_{0p}), \tag{38}$$

where $\alpha(\mathbf{r}_{0i}, t_{0p})$ is the random number of fields which produce particles in the interval $\Delta V_{0i} \Delta t_{0p}$. Performing then a second averaging over the number of terms and writing for sufficiently small intervals

$$\bar{\alpha}(\mathbf{r}_{0i}, t_{0p}) = \bar{g}(\mathbf{r}_{0i}, t_{0p}) \Delta \mathbf{r}_{0i} \Delta t_{0p}, \tag{39}$$

$$\overline{\Delta\alpha(\mathbf{r}_{0i}, t_{0p}) \Delta\alpha(\mathbf{r}_{0j}, t_{0q})} = K_g(\mathbf{r}_{0i}, t_{0p}, \mathbf{r}_{0j}, t_{0q}) \Delta \mathbf{r}_{0i} \Delta t_{0p} \Delta \mathbf{r}_{0j} \Delta t_{0q}, \tag{40}$$

we get after changing to integration

$$\bar{p}(\mathbf{r}, t) = \int_V \int_{-\infty}^{t} \bar{g}(\mathbf{r}_0, t_0) \bar{y}(\mathbf{r}, t, \mathbf{r}_0, t_0)\, d\mathbf{r}_0\, dt_0, \tag{41}$$

$$K_p(\mathbf{r}, t, \mathbf{r}', t') = \int_V \int_{-\infty}^{t} \int_V \int_{-\infty}^{t'} \bar{y}(\mathbf{r}, t, \mathbf{r}_0, t_0) \bar{y}(\mathbf{r}', t', \mathbf{r}_0', t_0')$$
$$\times K_g(\mathbf{r}_0, t_0, \mathbf{r}_0', t_0')\, d\mathbf{r}_0\, dt_0\, d\mathbf{r}_0'\, dt_0'$$
$$+ \int_V \int_{-\infty}^{t} \bar{g}(\mathbf{r}_0, t_0) K_y(\mathbf{r}, t, \mathbf{r}', t', \mathbf{r}_0, t_0)\, d\mathbf{r}_0\, dt_0. \tag{42}$$

Now, the use of Eqs. (32) and (33) and also of Eq. (34) leads to the expressions

$$\bar{p}(\mathbf{r}, t) = \int_V \int_{-\infty}^{t} \bar{g}(\mathbf{r}_0, t_0) \omega(\mathbf{r}, t | \mathbf{r}_0, t_0)\, d\mathbf{r}_0\, dt_0, \tag{43}$$

$$K_p(\mathbf{r}, t, \mathbf{r}', t') = \int_V \int_{-\infty}^{t} \int_V \int_{-\infty}^{t'} \omega(\mathbf{r}, t | \mathbf{r}_0, t_0) \omega(\mathbf{r}', t' | \mathbf{r}_0', t_0')$$
$$\times K_g(\mathbf{r}_0, t_0, \mathbf{r}_0', t_0')\, d\mathbf{r}_0\, dt_0\, d\mathbf{r}_0'\, dt_0' + \int_V \int_{-\infty}^{t} \bar{g}(\mathbf{r}_0, t_0)$$
$$\times [\omega(\mathbf{r}', t' | \mathbf{r}, t) \omega(\mathbf{r}, t | \mathbf{r}_0, t_0) - \omega(\mathbf{r}, t | \mathbf{r}_0, t_0) \omega(\mathbf{r}', t' | \mathbf{r}_0, t_0)]\, d\mathbf{r}_0\, dt_0. \tag{44}$$

These formulae are suitable both for stationary and for non-stationary states if we understand the latter, in particular, to mean states where some single equation is valid consistently for the probability density (ω) while the particle production is non-uniform and settles down, let us say, over a finite time interval from the start of the observation.

4. EXAMPLE OF CALCULATION OF A FLUCTUATION SPECTRUM AND CONCLUSION

We consider first the simplest consequences of Eqs. (43) and (44) and assume to begin with that the medium is either bounded by perfectly reflecting screens or is infinite. A stationary state is then only possible if there is no systematic production of particles; accordingly putting

$$g(\mathbf{r}_0, t_0) = \bar{g}(\mathbf{r}_0, t_0) = P\delta(\mathbf{r}_0 - \mathbf{r}_1)\delta(t_0 - t_1), \tag{45}$$

where $\mathbf{r}_1$ is arbitrary and $t_1 \to -\infty$, we get again the formulae of Sec. 2; in particular, for an infinite medium we get the formula

$$K_p(\mathbf{r}, t, \mathbf{r}', t') = \bar{p}\omega(\mathbf{r}', t' | \mathbf{r}, t), \tag{46}$$

which follows also from Eqs. (1) and (2) (of M. Lax) if $\omega(\mathbf{r}', t' | \mathbf{r}, t)$ is likened to a Green function.

The practically common case is the one where one can use the approximation

$$K_g(\mathbf{r}_0, t_0, \mathbf{r}_0', t_0') = g_1(\mathbf{r}_0, t_0)\delta(\mathbf{r}_0' - \mathbf{r}_0)\delta(t'_0 - t_0). \tag{47}$$

It then follows from Eq. (44) that

$$K_p(\mathbf{r}, t, \mathbf{r}', t') = \omega(\mathbf{r}', t' | \mathbf{r}, t) \int_V \int_{-\infty}^{t} \bar{g}(\mathbf{r}_0, t_0) \omega(\mathbf{r}, t | \mathbf{r}_0, t_0)\, d\mathbf{r}_0\, dt_0$$
$$+ \int_V \int_{-\infty}^{t} [g_1(\mathbf{r}_0, t_0) - \bar{g}(\mathbf{r}_0, t_0)] \omega(\mathbf{r}, t | \mathbf{r}_0, t_0) \omega(\mathbf{r}', t' | \mathbf{r}_0, t_0)\, d\mathbf{r}_0\, dt_0. \tag{48}$$

One can show, at least for a number of particular instances, when the function ω is determined by the Fokker-Planck equation and $\bar{g}(\mathbf{r}_0, t_0) = \bar{g} = \text{const}$, $g_1(\mathbf{r}_0, t_0) = g_1 = \text{const}$, that the contributions of the first and second terms in Eq. (48) to the spectrum of the fluctuations in the quantity P differ only in multiplying constants. Finally, if the fluctuations in the rate of production are Poisson-like, i.e., $g_1(\mathbf{r}_0, t_0) = \bar{g}(\mathbf{r}_0, t_0)$ only the first term remains in general in Eq. (48). In this characteristic case Eq. (48) differs from Eq. (46) used in [5-7] and when absorbing screens are present only through the natural replacement of $\bar{p} = \text{const}$ by $\bar{p} = \bar{p}(\mathbf{r}, t)$ or $\bar{p} = \bar{p}(\mathbf{r})$ under conditions which are stationary as far as the particle production is concerned. We shall elucidate by an actual example how important the changes produced by this substitution are for the spectrum of the fluctuations in the quantity P.

Let the gas be enclosed in a rectangular parallelepipid and let one face of it (lying in the plane $x = l_1$ at a distance l_1 from the opposite face) be perfectly absorbing and the other faces be perfectly reflecting. We assume also that the random walk of each particle is described by a Fokker-Planck equation of the type

$$\frac{\partial}{\partial t}\omega(\mathbf{r}, t) = D\nabla^2\omega(\mathbf{r}, t), \tag{49}$$

which determines the conditional probability density (which is the same as the Green function of the problem) for the appropriate boundary conditions (the solution tends to zero at $x = l_1$ and its normal derivative vanishes at all reflecting faces).

We determine the density correlation from the equation

$$K_p(\mathbf{r}, t, \mathbf{r}', t') = \omega(\mathbf{r}', t' | \mathbf{r}, t) \int_V \int_{-\infty}^{t} \bar{g}\omega(\mathbf{r}, t | \mathbf{r}_0, t_0)\, d\mathbf{r}_0\, dt_0$$
$$= \bar{p}(\mathbf{r})\omega(\mathbf{r}', t' | \mathbf{r}, t), \tag{50}$$

which is a particular case of Eq. (48); the results corresponding to the theory developed in [5-7] are obtained by replacing in all formulae $\bar{p}(\mathbf{r})$ by

$$\bar{p} = V^{-1} \int_V p(\mathbf{r})\, d\mathbf{r}.$$

If the Green function is found as an expansion in terms of the functions of the problem, the standard formulae

$$K_P(t', t) = K_P(t' - t) = K_P(\tau) = \int_V \int_V K_p(\mathbf{r}, t, \mathbf{r}', t')\, d\mathbf{r}\, d\mathbf{r}', \tag{51}$$

$$S_P(\Omega) = \frac{2}{\pi} \int_0^{\infty} K_P(\tau) \cos\Omega\tau\, d\tau \tag{52}$$

($Sp(\Omega)$ is the spectrum of the fluctuations in the quantity P) lead to the expressions

$$S_P(\Omega) = \frac{16}{\pi^3} V\bar{g} \sum_{k=0}^{\infty} \frac{1}{(2k+1)^2} \frac{\tau_k^2}{1+\Omega^2\tau_k^2}, \tag{53}$$

$$S_{P_\Lambda}(\Omega) = \frac{16}{\pi^3} V\bar{g} \sum_{k=0}^{\infty} \frac{\pi^2}{12} \frac{\tau_k^2}{1+\Omega^2\tau_k^2},$$

$$\tau_k = \frac{4}{\pi^2} \frac{l_1}{D} \frac{1}{(2k+1)^2} \tag{54}$$

where $Sp_\Lambda(\Omega)$ is obtained when we assume $\bar{p}$ = const.

We can also derive the expressions for the spectra in closed form. In agreement with Eqs. (43) and (49) and using the independence (which is obvious in our case) of the average density of the coordinates y and z, we can find it from the equation

$$D\frac{d^2}{dx^2}\bar{p}(x) + \bar{g} = 0 \tag{55}$$

with the boundary conditions $\bar{p}(l_1) = 0$ and $d\bar{p}/dx|_{x=0} = 0$. This gives

$$\bar{p}(\mathbf{r}) = \bar{p}(x) = \frac{1}{2}\bar{g}\frac{l_1^2}{D}\left[1 - \frac{x^2}{l_1^2}\right]. \tag{56}$$

Moreover, performing in (51) the integration over y and z (perpendicular only to the reflecting walls) and using a Fourier transform to go directly to the fluctuations spectrum leads to the formula

$$S_P(\Omega) = \mathrm{Re}\frac{2}{\pi}\frac{V}{l_1}\int_0^{l_1}\int_0^{l_1} \bar{p}(x) g_0(x, x', \Omega)\, dx\, dx', \tag{57}$$

where the function $g_0(x, x', \Omega)$ is defined by the equation

$$i\Omega g_0(x, x', \Omega) - D\frac{d^2}{dx'^2} g_0(x, x', \Omega) = \delta(x' - x) \tag{58}$$

with the boundary conditions

$$\frac{d}{dx'} g_0(x, x', \Omega)|_{x'=0} = 0 \quad \text{and} \quad g_0(x, l_1, \Omega) = 0.$$

Altogether we get

$$S_P(\Omega) = \frac{2}{\pi} V\bar{g}\frac{1}{\Omega^2}\left[1 - \left(\mathrm{cth}\,\gamma + \frac{1}{2}\frac{\sin 2\gamma}{\mathrm{sh}^2\gamma}\right) \Big/ 2\gamma\left(1 + \frac{\cos^2\gamma}{\mathrm{sh}^2\gamma}\right)\right], \tag{59}$$

$$S_{P_\Lambda}(\Omega) = \frac{\sqrt{2}}{3\pi} V\bar{g}\left(\frac{l_1^2}{D}\right)^{1/2}\frac{1}{\Omega^{3/2}}\left(\mathrm{cth}\,\gamma - \frac{1}{2}\frac{\sin 2\gamma}{\mathrm{sh}^2\gamma}\right) \Big/ \left(1 + \frac{\cos^2\gamma}{\mathrm{sh}^2\gamma}\right), \tag{60}$$

where

$$\gamma = (\Omega l_1^2/2D)^{1/2}. \tag{61}$$

Equations (53) and (54) give the total, superposition pattern of the spectra[4)] and are convenient for studying their low-frequency characteristics since for $\Omega = 0$ one can easily sum the appropriate sums using the generalized Riemann zeta-function[9], and we find, in particular, $Sp(0)/Sp_\Lambda(0) = 1, 2, 3$.

More important differences in the spectra occur at high frequencies: for large values of γ we get from Eqs. (60) and (59), respectively,

$$S_{P_\Lambda}(\Omega) = \frac{\sqrt{2}}{3\pi} V\bar{g}\left(\frac{l_1^2}{D}\right)^{1/2}\frac{1}{\Omega^{3/2}}, \tag{62}$$

$$S_P(\Omega) = \frac{2}{\pi} V\bar{g}\frac{1}{\Omega^2}\left(1 - \frac{1}{2\gamma}\right) \tag{63}$$

where

$$S_P(\Omega) = \frac{2}{\pi} V\bar{g}\frac{1}{\Omega^2}. \tag{64}$$

For high frequencies we have thus

$$\frac{S_P(\Omega)}{S_{P_\Lambda}(\Omega)} = \frac{3\sqrt{2}}{(l_1^2\Omega/D)^{1/2}}. \tag{65}$$

and $Sp_\Lambda(\Omega)$ may appreciably exceed $Sp(\Omega)$. From the equations given here it follows that the high-frequency asymptotic expression $Sp(\Omega) \sim \Omega^{-3/2}$ for the fluctuations spectra has for the diffusion case a considerably less "universal" character than follows from[6,7].[5)]

Under conditions which are more complicated than in the example considered, the general Eq. (44) leads clearly to larger deviations from the results of[5-7]. This will occur if the fluctuations in the rate of production are non-Poisson-like or, all the more, when it is necessary to take into account the finite correlation radius of random production sources and, finally, when the trajectories of the gas particles have a non-Markovian character (in the latter case we must write Eq. (44), with $\omega(\mathbf{r}', t'|\mathbf{r}, t)$ on the right-hand side replaced by $\omega(\mathbf{r}', t'|\mathbf{r}, t, \mathbf{r}_0, t_0)$).

In this connection it is not without interest, from a methodical point of view, to elucidate within what limits Eq. (1) proposed by Lax[5] is valid when the single-time correlations are evaluated correctly; the definition of the latter by Eq. (44) and substitution into Eq. (1) shows that for Markovian trajectories of the gas particles Eq. (1) is in itself correct if we make the additional assumption of a δ-correlation in time of the random production sources; if, however, the motion of the particles is a non-Markovian process, Eq. (1) is not valid.

We note in conclusion that the calculation of the fluctuations correlations of a perfect gas using the minimally necessary and usually available information about the correlation characteristics of the motion of the particles and the production field strongly simplifies the problem. More detailed density distribution functions remain uncalculated but those are not necessary if the fluctuations are small and may be assumed to be Gaussian[8] since the correlation characteristics are a comprehensive description when they are known.[10]

4) Spectra of the form $\bar{g}\tau^2/(1 + \Omega^2\tau^2)$, corresponding to the simplest exponential correlation characteristic for Gaussian Markovian processes, are often found in applications; for an experimental study of spectra of the form (53) and (54) one usually manages to obtain at low frequencies a "resolution" of several of the first components.

5) The difference in the spectra $S_p(\Omega)$ for $\bar{p}$ = const and for $\bar{p}(\mathbf{r})$ can be treated also from the point of view of the difference in those two cases of the probability characteristics of the behavior of the particles: in Eq. (50) the integral

$$\int_V \omega(\mathbf{r}, t|\mathbf{r}_0, t_0)\, d\mathbf{r}_0 = \int_V \omega(\mathbf{r}_0, t|\mathbf{r}, t_0)\, d\mathbf{r}_0$$

is the probability for the "survival" up to time t in the volume V of a particle which at time t_0 is produced at the point $\mathbf{r}_0$, while the integral

$$\int_{-\infty}^{t}\int_V \omega(\mathbf{r}, t|\mathbf{r}_0, t_0)\, d\mathbf{r}_0\, dt_0$$

has the meaning of the average lifetime of such a particle in the volume V.

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[1] M. Smoluchowski, Wiener Ber. **123**, 2381 (1914).
[2] S. Chandrasekhar, Rev. Mod. Phys. **15**, 1 (1943).
[3] G. G. MacFarlane, Proc. Phys. Soc. **B63**, 807 (1950).
[4] K. M. van Vliet and A. van der Ziel, Physica **24**, 415 556 (1958).
[5] M. Lax, Rev. Mod. Phys. **32**, 25 (1960).
[6] M. Lax and P. Mengert, J. Phys. Chem. Solids **14**, 248 (1960).
[7] K. M. van Vliet and J. R. Fassett, Fluctuations in Solids (Ed. R. Burgess), Academic Press, New York, 1965, p. 268.
[8] L. D. Landau and E. M. Lifshitz, Statisticheskaya fizika (Statistical Physics), Nauka, 1964 [English translation published by Pergamon Press, 1968].
[9] E. T. Whittaker and G. N. Watson, Modern Analysis, Cambridge University Press, 1946.
[10] I. I. Gikhman and A. V. Skorokhod, Vvdenie v teriyu sluchaĭnykh protsessov (Introduction into the Theory of Random Processes), Nauka, 1965.

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