# *SOUND-LIKE WAVES IN THE ELECTRON PLASMA OF METALS IN A QUANTIZING MAGNETIC FIELD*

O. V. KONSTANTINOV and V. I. PEREL'

A. P. Ioffe Physico-technical Institute, Academy of Sciences, USSR

Submitted May 13, 1967

Zh. Eksp. Teor. Fiz. 53, 2034–2040 (December, 1967)

It is shown that in a metal, in the presence of a quantizing magnetic field, electron-gas longitudinal oscillations exist with an acoustic dispersion law. The number of acoustic branches is one less than the number of Landau levels below the Fermi surface. Their velocities lie in intervals between the allowed electron velocities along the magnetic field on the Fermi surface.

## 1. INTRODUCTION

CHARGE-density oscillations are possible in an electron gas—the Langmuir longitudinal oscillations. A change in the particle density in such a wave is accompanied by the appearance of a strong electric field. This leads to the result that the frequency of oscillations cannot be less than the limiting plasma frequency. Many authors[1,2] have shown that longitudinal waves of the acoustic type do not exist in the plasma of a metal with a single type of carrier. However, Pines and Schrieffer[3] have shown that in metals where there are electrons and holes, which differ widely in mass, there exist weakly damped longitudinal waves of the acoustic type. These can be called acoustic plasma oscillations. In our previous work,[4] such waves were considered in particular for the case of bismuth. It was noted there that the presence of electrons and holes is not obligatory for the existence of acoustic plasma waves. They also exist when there are two sorts (or more) of electrons. In the acoustic plasma waves, there are almost no oscillations of the charge density. The charge which arises because of a change in the concentration of carriers of a single type is almost totally compensated by the corresponding change in the concentration of carriers of the other type. For low collision frequency of ions and lattice (and of the ions with one another), the wave damping is basically Landau damping connected with the acceleration of particles whose velocity is equal to the phase velocity of the wave. In order that this damping be small, the masses of the different carrier types must differ widely.

A very special situation arises in a quantizing magnetic field. On the one hand, this field in quite natural fashion splits the carriers into groups related to the different Landau levels, which guarantees the possibility of quasineutral oscillations. On the other hand, as Gurevich, Skobov and Firsov have noted,[5] as a consequence of the discrete nature of the electron velocity along the magnetic field on the Fermi surface, Landau damping exists only when the velocity of the longitudinal wave propagating along the magnetic field is close to one of the allowed electron velocities (giant absorption oscillations). Therefore, if it were shown that the velocities of the acoustic plasma waves lie in the intervals between the allowed electron velocities, then such a wave would not experience Landau damping. It will be shown below that this is precisely the case in a quantizing magnetic field. It is shown that in this case, as many sound plasma branches exist as there are intervals between the Landau levels below the Fermi surface. The velocities of these waves are determined by Eq. (24). They are separated from the allowed electron velocities along the magnetic field by a quantity of the order of the ratio of the distance between the Landau levels to the Fermi momentum. Equation (26) gives the conditions under which this distance is sufficient to make the damping of the waves small.

The basic results referring to the acoustic plasma waves in a quantizing magnetic field were published by us earlier, together with S. L. Ginzburg,[6] who came independently to similar conclusions.

## 2. DISPERSION EQUATION AND ITS INVESTIGATION

Longitudinal electric oscillations of an electron gas, both of the optical and of the acoustic type, are described by the dispersion equation

$$\varepsilon(\omega, k) = 0, \tag{1}$$

where $\epsilon$ is the dielectric constant, computed under the assumption that the electric field in the wave is directed along the wave vector. Equation (1) is equivalent to the Poisson equation

$$ikE = 4\pi\rho / \varepsilon_0. \tag{2}$$

Here $\rho$ is the charge density determined by the field of the wave E. Thus, to find the spectrum of the plasma oscillations, it is necessary to compute $\rho$. The charge density is associated with the perturbation of the density matrix $f_{nn}(p)$ in the following way:

$$\rho = e\int \frac{dp}{2\pi\hbar}\frac{eH}{2\pi\hbar c}\sum_n f_{nn}. \tag{3}$$

Here the n refer to the Landau levels, and p is the quasimomentum of the electron along the magnetic field.

We shall assume that the wave vector **k**, meaning also the electric field **E**, is directed along the magnetic field **H**. Here only the diagonal part of the density matrix $f_{nn}(p)$ is perturbed, and we can write for it the equation

$$(-i\omega + \nu) f_{nn}(p) + ikv^{(n)} f_{nn}(p) + eEv^{(n)} \frac{f_{nn}(\varepsilon^n_{p+k/2}) - f_{nn}(\varepsilon^n_{p-k/2})}{\hbar k v^{(n)}} = 0. \tag{4}$$

Here $v^{(n)} = \partial\epsilon_p^n/\partial p$, $\epsilon_p^n$ is the electron energy at the n-th Landau level, and $f_0$ is the Fermi function. The quantity $\nu$ describes the relaxation; in what follows, this quantity will appear only in estimates.

Using Eqs. (3) and (4), we can obtain the following expression for $\rho$:

$$\rho = i(\rho' - i\rho''),$$
$$\rho' = -\frac{e^3 mEH}{4\pi^2\hbar^3 k^2 c}\sum_n \ln\left|\frac{u_n^{+2} - s^2}{u_n^{-2} - s^2}\right|, \tag{5}$$

where $s = \omega/k$ is the phase velocity of the wave,

$$u_n^+ = u_n + \frac{\hbar k}{2m}, \quad u_n^- = u_n - \frac{\hbar k}{2m}, \tag{6}$$

$$u_n = \left\{\frac{2}{m}\left[\varepsilon_F - \left(n + \frac{1}{2}\right)\hbar\Omega\right]\right\}^{1/2}. \tag{7}$$

The velocity $u_n$ is the longitudinal velocity of the electron on the Fermi surface in the n-th Landau level; $\epsilon_F$ is the Fermi energy, and $\Omega$ is the cyclotron frequency.

The quantity $\rho''$ which determines the damping differs from zero (and is equal to $\pi$) only in the intervals $u_n^{(-)} < s < u_n^{(+)}$. This behavior of the absorption was first noted by Gurevich, Skobov and Firsov, and was called by them giant oscillations.[5] This behavior of $\rho''$ takes place only in the absence of collisions when $\nu = 0$. In the presence of collisions, the dependence of $\rho''$ on s will be described not by a step wise but by a smooth, bell-shaped curve which has a maximum near $u_n$ and a width of the order of $\nu/k$ (if $\nu/k > \hbar k/m$). Apparently, the shape of the maximum cannot be obtained accurately with such a rough allowance for the relaxation processes. However, this form will not be of interest to us, since we shall always assume that s is distant from $u_n$ by an amount greater than $\nu/k$. For the same reason, $\nu$ is set equal to zero in Eq. (5) for $\rho'$.

In addition, we shall assume for simplicity, that $s - u_n \gg \hbar k/m$. Then Eq. (5) takes the form

$$\rho' = \frac{3n_0 e^2}{4\varepsilon_F k^2} kE\, g(s), \tag{8}$$

where

$$g(s) = -\frac{\hbar\Omega}{2mv_F}\sum_n\left(\frac{1}{u_n - s} + \frac{1}{u_n + s}\right); \tag{9}$$

here $n_0$ is the concentration of electrons, $v_F = (2E_F/m)^{1/2}$.

For oscillations of the acoustic type, the frequency of which is much less than the plasma frequency, we can assume that the dispersion equation (2) reduces to the equation

$$g(s) = 0. \tag{10}$$

Equation (9) shows that this equation has a solution in the intervals between each two neighboring velocities $u_n$. Thus, there generally appear as many different acoustic branches as there are Landau levels. However, in the derivation of Eq. (9), we have assumed several conditions to be satisfied. In order to establish what limitations are placed on the value of the magnetic field, temperature, frequency of oscillation, etc., it is necessary to know how close the roots of Eq. (10) are to the "dangerous" points $u_n$.

## 3. SOLUTION OF THE DISPERSION EQUATION

We shall assume that $\hbar\Omega \ll \epsilon_F$ and shall consider the two groups of roots separately.

1. The group of roots $s_n$ which fall in the intervals between several unchanged values of $u_n$. These values correspond to Landau numbers near the maximum. The velocities $u_n$ in this group are of the order $(\hbar\Omega/m)^{1/2}$ and the distances between them are of the same order. We shall show that the sum over n, which enters into (9), can be replaced by an integral under these conditions, isolating, however, one "dangerous" term $(u_{n_0} - s)^{-1}$, where $n_0$ is the number of that velocity $u_{n_0}$ which is near the value of s. We see that for this group of roots, the values of $s_n$ will be separated from the "dangerous" points by a distance than is much less than the spacing between the neighboring velocities $u_n$.

We divide the sum over n into two parts, isolating from them the "dangerous" term

$$\sum_n = (u_{n_0} - s)^{-1} + \Sigma_1 + \Sigma_2. \tag{11}$$

Here $u_{n_0}$ is the Landau velocity near the desired value of s; $\Sigma_1$ includes terms for which $N - n < n_1$; N is the maximum Landau number; $n_1$ is some large number. The sum $\Sigma_1$ thus includes terms with comparatively small velocities $u_n$. The sum $\Sigma_2$ includes all the remaining terms. Inasmuch as the summation over them begins with a number corresponding to a sufficiently large velocity, the adjacent terms in them will differ from one another. Thus, for the first terms of the sum $\Sigma_2$ we obtain the estimate

$$\frac{(u_{N-n_1} - s)^{-1} - (u_{N-n_1-1} - s)^{-1}}{(u_{N-n_1} - s)^{-1}} \sim \frac{1}{n_1}.$$

Thus this sum can be replaced by an integral. Integration leads to the following expression

$$\Sigma_2 = \frac{mv_F}{\hbar\Omega}\left[2 - \frac{s}{v_F}\ln\frac{v_F + s}{v_F - s} - 2\frac{u_{N-n_1}}{v_F} + \frac{s}{v_F}\ln\frac{u_{N-n_1} + s}{u_{N-n_1} - s}\right]. \tag{12}$$

The last three terms contain a small parameter $s/v_F$; therefore,

$$\Sigma_2 = 2mv_F/\hbar\Omega. \tag{13}$$

We now estimate the value of $\Sigma_1$ and show that it can be discarded. In fact,

$$\Sigma_1 < \frac{2n_1}{(2\hbar\Omega/m)^{1/2}},$$

so that

$$\Sigma_1/\Sigma_2 < (\hbar\Omega/4\varepsilon_F)^{1/2} n_1.$$

Although we have assumed $n_1$ to be a large number (in practice, several units), under the assumption $\hbar\Omega \ll \epsilon_F$ one can put $\Sigma_1 \ll \Sigma_2$. Thus,

$$g(s) = -1 - \frac{\hbar\Omega}{2mv_F}\frac{1}{u_{n_0} - s}. \tag{14}$$

The dispersion equation $g(s) = 0$ leads to the following expression for the velocity $s_{n_0}$ of the acoustic plasma waves associated with the $n_0$-th Landau level:

$$s_{n_0} = u_{n_0} + \hbar\Omega/2mv_F. \tag{15}$$

This expression is valid for a Landau level with quantum numbers $n_0$ close to the maximum. Here $u_{n_0} - u_{n_0+1} \sim (\hbar\Omega/mv_F)^{1/2}$, so that

$$\frac{s_{n_0} - u_{n_0}}{u_{n_0+1} - u_{n_0}} \sim \left(\frac{\hbar\Omega}{\varepsilon_F}\right)^{1/2} \ll 1,$$

i.e., the difference of the velocity of the $n_0$-th acoustic branch from $u_{n_0}$ is much less than the interval between the neighboring Landau velocities, as was assumed.

2. We now consider those roots of the dispersion equation which fall between the values of the Landau velocities corresponding to the quantum numbers are far from the maximum. In this case, the sum entering into Eq. (9) is again divided into two parts, separating two "dangerous" terms in it:

$$\sum_n = (u_{n_0-1} - s)^{-1} + (u_{n_0} - s)^{-1} + \Sigma_1 + \Sigma_2. \tag{16}$$

$u_{n_0+1}$ and $u_{n_0}$ are those Landau velocities between which lies the desired root s. The sum $\Sigma_1$ includes terms for which the Landau number n lies in the range $n_0 - n_2 < n < n_0 + n_1$ (except for the "dangerous" ones that have been separated). Here $n_1$ and $n_2$ are certain large numbers. The sum $\Sigma_2$ includes all the remaining terms. This sum, just as the previous, can be replaced by an integral, so that

$$\Sigma_2 = \frac{mv_F}{\hbar\Omega}\left[2 - \frac{s}{v_F}\ln\frac{v_F + s}{v_F - s} + \frac{s}{v_F}\ln\frac{s - u_{n_0+n_1}}{u_{n_0-n_2} - s} + \frac{u_{n_0-1-n_2} - u_{n_0-1+n_1}}{v_F}\right]. \tag{17}$$

For calculation of the sum $\Sigma_1$, we make use of the fact that the velocity $u_n$ can be written in the form

$$u_n = \left(u_{n_0}^2 + \frac{2\hbar\Omega}{m}(n_0 - n)\right)^{1/2} \approx u_{n_0} + \frac{\hbar\Omega}{mu_{n_0}}(n_0 - n). \tag{18}$$

Then

$$\Sigma_1 = \sum_{n=n_0+1}^{n_0+n_1-1}\frac{1}{u_n - s} + \sum_{n=n_0-n_2-1}^{n_0-2}\frac{1}{u_n - s} = \frac{mu_{n_0}}{\hbar\Omega}\left\{-\sum_{j=1}^{n_1-1}\frac{1}{z+j} + \sum_{j=2}^{n_2+1}\frac{1}{j-z}\right\}, \tag{19}$$

where

$$z = -mu_{n_0}(u_{n_0} - s)\,/\,\hbar\Omega. \tag{20}$$

The finite sums in Eq. (19) can be expressed in terms of the logarithmic derivative of the factorial, $\Psi$[7]:

$$\Sigma_1 = \frac{mu_{n_0}}{\hbar\Omega}\{\Psi(n_2 + 1 - z) + \Psi(z) - \Psi(n_1 - 1 + z) - \Psi(1 - z)\}. \tag{21}$$

The quantity z lies in the range $0 < z < 1$; the numbers $n_1$ and $n_2$ are large. Using the asymptotic expansion for $\Psi$, we get

$$\Psi(1 - z + n_2) - \Psi(n_1 - 1 + z) = \ln\frac{n_2 + 1 - z}{n_1 - 1 + z} \approx \ln\frac{u_{n_0-1-n_2} - s}{s - u_{n_0+n_1-1}}. \tag{22}$$

Thus, by using Eqs. (9), (16), (17), (21) and (22), we finally obtain

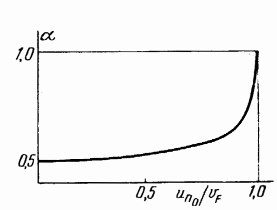
$$g(s) = -\frac{s}{2v_F}\ln\frac{v_F + s}{v_F - s} - 1 + \frac{u_{n_0}}{2v_F}\left[\frac{1}{z} + \frac{1}{z-1} + \Psi(1-z) - \Psi(z)\right], \tag{23}$$

where z is connected with s by Eq. (20). The argument of each function lies in the range 0 to 1 (inasmuch as we are seeking the root $s = s_{n_0}$, which lies between the Landau velocities $s_{n_0}$ and $s_{n_0+1}$). The function $\Psi(z)$ is tabulated in[7]. In the term with the logarithm in Eq. (23), we can replace s by $u_{n_0}$. Then the dispersion equation $g(s) = 0$ gives the connection between z and the parameter $u_{n_0}/v_F$. This dependence is shown in the drawing.

Using Eq. (20), we can write for the velocity of the $n_0$-th acoustic branch:

$$s_{n_0} = u_{n_0} + \frac{\hbar\Omega}{mv_F}\alpha, \tag{24}$$

where $\alpha = zv_F/u_{n_0}$ is the numerical coefficient which is shown in the drawing as a function of $u_{n_0}/v_F$. One can show that Eq. (24) in fact gives the velocity of all acoustic branches if $\hbar\Omega \ll \epsilon_F$), since it is identical




Dependence of the quantity $\alpha = zv_F/u_{n_0}$ on $u_{n_0}/v_F$.

with Eq. (15) when $u_{n_0}/v_F \ll 1$. It is seen from (24) that the departure of the velocity from the "dangerous" point $u_{n_0}$ is $\hbar\Omega/mv_F$ in order of magnitude.

We now discuss the criteria upon whose satisfaction the acoustic branches of plasma oscillations along the magnetic field will not have any significant damping. The region of the giant absorption peak has a finite width $\hbar k/m$. Moreover, it is smeared out by collisions and by the temperature. We require that the sound velocity be separated from the center of the peak by a quantity much larger than all these widths. This means that

$$s - u_{n_0} \gg \frac{\hbar k}{m}, \quad s - u_{n_0} \gg \frac{\nu}{k}, \quad s - u_{n_0} \gg \frac{T}{mu_{n_0}}, \tag{25}$$

where T is the temperature in energy units.

With account of Eq. (24), we can write down the following criterion for the existence of weakly damped oscillations:

$$\Omega \gg \omega\frac{v_F}{u_{n_0}}, \quad \frac{\hbar\Omega}{\varepsilon_F} \gg \frac{\nu}{\omega}\frac{v_F}{u_{n_0}}, \quad \frac{\hbar\Omega}{T} \gg \frac{v_F}{u_{n_0}}. \tag{26}$$

It is seen that all these criteria are most rigid for velocities $u_{n_0}$ appreciably smaller than the Fermi velocity; it is much weaker for $u_{n_0} \sim v_F$. However, for typical metals, the second of the conditions (26) is difficult to achieve.

We note that although the detailed analysis was given for the more realistic case $\hbar\Omega \ll \epsilon_F$, it is clear that acoustic plasma waves exist when $\hbar\Omega \sim \epsilon_F$, i.e., when there is a small number of Landau levels (two or more) below the Fermi surface. In these cases, the dispersion equation (10) must be solved numerically by using the general formula (9).

## 4. ACOUSTIC PLASMONS IN BISMUTH

The Fermi surface of bismuth is nonspherical. It consists of a single elongated (hole) ellipsoid of revolution, oriented along a threefold axis, and three electron surfaces. The latter can be approximated by triaxial ellipsoids with axis ratios 1:1.4:15, while the long axis is almost perpendicular to the threefold axis and the middle axis is almost parallel.[8] The dispersion equation for acoustic plasmons in bismuth can be written in the form

$$\frac{1}{\varepsilon_{eF}}g_e(s) + \frac{1}{\varepsilon_{hF}}g_h(s) = 0, \tag{27}$$

where $\epsilon_{eF}$ and $\epsilon_{hF}$ are the Fermi energies of the electrons and holes. We shall assume that the magnetic field and the direction of propagation of the sound wave are directed along the threefold axis. We shall not carry out the calculations for the ellipsoidal surfaces, but shall only write down the results. The formulas for $g(s)$ obtained in the previous section remain unchanged. By $\Omega$ must be understood the cyclotron frequency and by $v_F$ the Fermi surface in the direction of the threefold axis. The mass m means the mass along this axis, i.e., $m = 2\epsilon_F/v_F^2$.

The dispersion equation (27), together with the expression (23), shows that two systems of acoustic plasma branches are possible in bismuth. The electron branches will have velocities less than the electron velocity $v_{eF}$ in the direction of the threefold axis; the hole branch velocities are less than the analogous velocity $v_{hF}$ for holes. We note that $v_{eF} \gg v_{hF}$. Equation (27) is valid only for frequencies much less than the plasma frequency $(4\pi n_0 e^2/m\epsilon_0)^{1/2}$; it must be kept in mind that $\epsilon_0 \sim 100$ for bismuth.[9]

The methods for excitation of acoustic plasma waves require special consideration. In the isotropic case, for excitation of electromagnetic waves (as always in the excitation of longitudinal waves), it is required that there be a component of the electric field perpendicular to the surface of the sample.

[1] V. P. Silin, Zh. Eksp. Teor. Fiz. 35, 1243 (1958) [Sov. Phys.-JETP 8, 870 (1959)].

[2] L. P. Gor'kov, and I. E. Dzyaloshinskiĭ, Zh. Eksp. Teor. Fiz. 44, 1650 (1963) [Sov. Phys.-JETP 17, 1111 (1963)].

[3] D. Pines and J. Schrieffer, Phys. Rev. 124, 1387

[4] O. V. Konstantinov and V. I. Perel', Fiz. Tverd. Tela (in press).

[5] V. L. Gurevich, V. G. Skobov and Yu. A. Firsov, Zh. Eksp. Teor. Fiz. 40, 786 (1961) [Sov. Phys.-JETP 13, 552 (1961) [Sov. Phys.-JETP 13, 552 (1961)].

[6] S. L. Ginzburg, O. V. Konstantinov and V. I. Perel' (Fiz. Tverd. Tela.) (in press).

[7] E. Jancke and F. Emde, Tables of Functions (Russian translation, Gostekhizdat, 1948).

[8] V. S. Édel'man and M. S. Khaĭkin, Zh. Eksp. Teor. Fiz. 49, 107 (1965). [Sov. Phys.-JETP 22, 77 (1966)].

[9] A. A. Abrikosov, Zh. Eksp. Teor. Fiz. 44, 2039 (1963) [Sov. Phys.-JETP 17, 1372 (1963)].

Translated by R. T. Beyer
231