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NONLINEAR THEORY OF PENETRATION OF A HIGH FREQUENCY FIELD INTO A CONDUCTOR

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A nonlinear theory of penetration of a field into conductors with current carriers of both signs is developed assuming a normal skin effect and ordinary opacity, in which case the real part of the linear dielectric permittivity is negative. It is shown that with increase of field amplitude of the incident wave the boundary of the opacity region shifts towards lower frequencies. As the field frequency approaches the ordinary opacity limit, the critical field amplitude at which nonlinearity of the material field equations becomes important decreases. For amplitudes exceeding the critical amplitude, when nonlinear transparency becomes possible, the field penetrates the conductor at a depth greater than the usual skin layer. In this case the field in a conductor with low dissipation is first characterized by rapid anharmonic spatial oscillations with a period and amplitude that vary slowly with the penetration depth. Subsequently the period and amplitude in a small region of the order of thickness of the ordinary skin layer change into an aperiodic field where damping can be described at a greater depth by the linear theory.

1. As is well known, the nonlinear properties of a conductor become manifest under conditions when the oscillation energy of an electron in the high-frequency field becomes comparable with the electron kinetic energy. The study of the electromagnetic field in conductors under such conditions was undertaken in a number of papers[1-7]. We develop in this communication a nonlinear theory of the reflection of a plane monochromatic electromagnetic wave from the surface of a conductor. We confine ourselves to normal incidence of the wave. In addition, we confine ourselves to frequencies for which the films are not transparent in the linear theory. In other words, the frequency of the external field will be assumed to be smaller than the plasma Langmuir frequency of the carriers, when the real part of the ordinary linear dielectric constant is negative. Dissipative effects will be assumed small; this is possible when the alternating frequency of the field is large compared with the effective collision frequency.

Considering the penetration of the field into the conductor, we shall show that under such conditions, depending on the field amplitude, the transparency limit shifts towards lower frequencies. In addition to studying damped solutions of the field, corresponding to the opacity region, we develop also a theory of penetration into the conductor of a strong field attenuated by dissipative processes. In the latter case the field near the conductor surface is characterized by fast spatial oscillations. On penetrating deeper into the conductor, the minimum and maximum values of the field intensity in these oscillations decrease slowly, and then the dependence of the field on the coordinates becomes aperiodic and rapidly damped.

2. We assume that the conductor occupies the half-space $x > 0$. Then the electric field in the conductor can be written in the form

$$E(x,t) = E_1(x)\sin\omega t + E_2(x)\cos\omega t. \tag{2.1}$$

Assuming that the dependence of the field on the coordinate x is such that the square of the electric field averaged over a time longer than the oscillation period becomes dependent on the coordinate x, we can use the ordinary concepts of the potential of the high-frequency forces[8-11]. Then, for conductors with positive and negative carriers, we can write for the electric induction (see[7])

$$D(x,t) = \{1 - (c\varkappa_0/\omega)^2 f(E_1^2(x) + E_2^2(x))\}E(x,t). \tag{2.2}$$

To normalize the constant κ_0, we assume that $f(0) = 1$. In particular, if the mass of the electrons is much smaller than the mass of the positively charged particles, then $c^2\kappa_0^2 = \omega_{Le}^2 = 4\pi e^2 N_e/m$, where N_e is the number of electrons per cm^3 under conditions when there is no electric field.

In the case of a Maxwellian particle momentum distribution

$$f(\zeta) = \exp\{-\zeta/E_M^2\}, \quad E_M^2 = \frac{8\omega^2}{e^2}\frac{m_- m_+}{m_+ + m_-}\varkappa T. \tag{2.3}$$

On the other hand, if the charge carriers have a degenerate Fermi distribution, then

$$f(\zeta) = (1 - \zeta/E_F^2)^{3/2}, \quad E_F^2 = 2p_0^2\omega^2/e^2, \tag{2.4}$$

where p_0 is the end-point momentum of the distribution. For relatively weak fields, it is possible to retain in (2.3) and (2.4) only the terms linear in ζ. We consider specially the case

$$f(\zeta) = 1 - \zeta/E_0^2, \tag{2.5}$$

since, first, we can write in this case simple analytic expressions for the field, and, second, for weak fields (2.5) is equivalent to the formula obtainable from (2.3) and (2.4).

Using the material equation (2.2), we can easily obtain with the aid of Maxwell's equations

$$\begin{aligned} E_1'' + [k^2 - \varkappa_0^2 f(E_1^2 + E_2^2)]E_1 = 0, \\ E_2'' + [k^2 - \varkappa_0^2 f(E_1^2 + E_2^2)]E_2 = 0, \end{aligned} \tag{2.6}$$

where $k^2c^2 = \omega^2$.

The system (2.6) has as an integral

$$E_1E_2' - E_2E_1' = M_s \equiv \text{const}, \tag{2.7}$$

which is a reflection of the conservation of the electromagnetic-field energy flux. Another integral of the field equations (2.6) is

$$(E_1')^2 + (E_2')^2 + k^2(E_1^2 + E_2^2) - \varkappa_0^2 \int\limits_{E^2_{min}}^{E_1^2+E_2^2} d\zeta f(\zeta) = \mathscr{E} = \text{const}. \tag{2.8}$$

The integrals (2.7) and (2.8) enable us to find the sought-for solution.

Something must be said, however, concerning the boundary conditions. In the region $x < 0$ we have a wave incident on the conductor as well as a reflected wave

$$E(x, t) = E_{\text{inc}}\{\cos(kx - \omega t) + R\cos(kx + \omega t + \psi)\}. \tag{2.9}$$

Here E_{inc} is the specified amplitude of the incident wave, and R and ψ are the reflection coefficient and the phase shift of the reflected wave, which must be determined. According to the condition for the continuity of the tangential components of the electric and magnetic fields on the surface of the conductor, we have

$$E_1(0) = -E_{\text{inc}}R\sin\psi, \quad E_2(0) = E_{\text{inc}}(1 + R\cos\psi), \tag{2.10}$$
$$E_1'(0) = E_{\text{inc}}k(1 - R\cos\psi), \quad E_2'(0) = -E_{\text{inc}}kR\sin\psi.$$

The electric field (2.1) can be written in the form

$$E(x, t) = E(x)\sin[\omega t + \varphi(x)]. \tag{2.11}$$

Then the relations (2.7) and (2.8) take the form

$$-E^2(x)\varphi'(x) = M_s, \tag{2.12}$$
$$(E')^2 + \frac{M_s^2}{E^2} + k^2E^2 - \varkappa_0^2 \int\limits_{E^2_{min}}^{E^2} d\zeta f(\zeta) = \mathscr{E}. \tag{2.13}$$

From the boundary conditions (2.10) we get

$$R^2 = 1 - M_s(0)/kE_{\text{inc}}^2, \quad E(0) = E_{\text{inc}}\sqrt{1 + R^2 + 2R\cos\psi},$$
$$\operatorname{ctg}\varphi(0) = -\frac{R\sin\psi}{1 + R\cos\psi} \quad \operatorname{tg}\psi = \frac{E(0)E'(0)}{M_s(0) - kE^2(0) + 2kE_{\text{inc}}^2}. \tag{2.14}$$

We are interested in the case of a transparent conductor, when the field vanishes as $x \to +\infty$. According to (2.13), it is obvious that $E = 0$ is possible only if $M_s = 0$. This immediately leads to the equality $\varphi'(x) = 0$. The latter, according to (2.14), signifies that the reflection coefficient R is equal to unity. This, in particular, makes it possible to write down the relations

$$E(0) = 2E_{\text{inc}}\cos(\psi/2), \quad E'(0) = -2kE_{\text{inc}}\sin(\psi/2). \tag{2.15}$$

It follows further from (2.13) that for the case of an electromagnetic field that vanishes at infinity we have $\mathscr{E} = 0$. Then, taking into account the fact that $E_{min} = 0$ and considering the relation (2.13) at the point $x = 0$, we obtain the following equation for the phase shift of the reflected wave:

$$k^2 = \varkappa_0^2\cos^2\frac{\psi}{2}\int\limits_0^1 d\xi f\left(4\xi E_{\text{inc}}^2\cos^2\frac{\psi}{2}\right). \tag{2.16}$$

Finally, from (2.13) we get the relation

$$x = \pm\int\limits_{E(0)}^{E} d\xi\left(\varkappa_0^2\int\limits_0^{\xi^2} d\zeta f(\zeta) - k^2\xi^2\right)^{-1/2}, \tag{2.17}$$

which determines the dependence of the electric field on the coordinate.

3. Let us consider the applications of formulas (2.16) and (2.17) for the charged-particle density dependence described by formulas (2.3)—(2.5). We note that the function $f(\zeta)$ is by definition a positive quantity. This means that expressions (2.4) and (2.5) can be used only in the region $f \geq 0$. The vanishing of the function f reflects the possible containment of the charged particles by the high-frequency field.

Let us turn first to the simplest case when the particle distribution (as a function of the field) is described by formula (2.5). We then have from (2.17)

$$\cos^2\frac{\psi}{2} = \frac{E_0^2}{4E_{\text{inc}}^2}(1 - \alpha), \tag{3.1}$$

where

$$\alpha = \sqrt{1 - 8E_{\text{inc}}^2k^2/E_0^2\varkappa_0^2}. \tag{3.2}$$

We note that Eq. (2.17) also has a second solution, which, however, does not satisfy the condition that f be positive, and will therefore not be considered by us.

According to (3.2), the electric field intensity of the incident wave cannot exceed a critical value

$$E_{\text{cr}} = 2^{-3/2}(\varkappa_0/k)E_0. \tag{3.3}$$

At the same time, according to (3.1), when the amplitude of the incident wave approaches the critical value (3.3), the inequality $2E_{\text{inc}} > E_0$ should be satisfied. Therefore the electric field intensity of the incident wave can reach the critical value (3.3) only for values of k less than a critical value

$$k_{\text{cr}} = 2^{-1/2}\varkappa_0. \tag{3.4}$$

When $E_{\text{inc}} = E_{\text{cr}}$, formula (3.1) takes the form

$$\cos^2\frac{\psi}{2} = 2\left(\frac{k}{\varkappa_0}\right)^2. \tag{3.5}$$

The right side of this formula is twice the value obtained from the linear theory.

Let us proceed now to consider the field in the conductor. This will help, in particular, to understand the meaning of the results (3.3) and (3.4). According to (2.17) we have

$$E(x) = \frac{E_0\sqrt{2(1 - k^2/\varkappa_0^2)}}{\operatorname{ch}(x\sqrt{\varkappa_0^2 - k^2} + \Delta)}, \tag{3.6}$$

where

$$\Delta = \ln\left[\sqrt{2(1 - k^2/\varkappa_0^2)} - \sqrt{2(1 - k^2/\varkappa_0^2) - (1 - \alpha)}\right] - \ln\sqrt{1 - \alpha}.$$

This field, as was assumed by us in the derivation of (2.17), tends to zero with increase in the coordinate x. This property of the field, described by formula (3.6), is obtained upon integration of the right side of (2.17) only if the inequality

$$2E_0^2(1 - k^2/\varkappa_0^2) > E^2(0) = E_0^2(1 - \alpha). \tag{3.7}$$

is satisfied. Therefore the damping of the field on penetrating into the conductor takes place if

$$k^2 < \varkappa_0^2[1 - 1/2(1 - \alpha)]. \tag{3.8}$$

It follows hence that an increase of the incident-wave field amplitude leads to a shift of the point of the onset of opacity into the region of lower frequencies. The largest shift takes place when $E_{\text{inc}} = E_{\text{cr}}$, when the opacity point corresponds to the frequency $\omega = c\kappa_0/\sqrt{2} = ck_{cr}$, as against the value $c\kappa_0$ from the linear theory.

According to (3.6), we can readily see that when the incident-wave amplitude reaches the critical value (3.3), the field on the surface of the conductor becomes equal to E_0. In other words, the function f vanishes on the surface of the conductor in this case, and with it the carrier density, corresponding to complete containment of the carriers by the high-frequency-wave field.

It follows from (3.7) that at values of k larger than critical, the field on the surface of the conductor is smaller than E_0, and consequently, it is impossible to contain the carriers by means of the high-frequency field of the incident and reflected waves. Moreover, according to (3.7), which for $\kappa_0 > k > k_{cr}$ can be written in the form

$$E_{inc}^2 \leqslant {}^1/_2 E_0^2 [1 - k^2/\varkappa_0^2], \tag{3.9}$$

it follows that the maximum incident-wave field amplitude allowed by this condition decreases with increasing k. On approaching the usual transparency point ($k \to \kappa_0$), the field of an incident wave capable of producing in the conductor a field that decreases at infinity decreases like $(\kappa_0^2 - k^2)^{1/2}$. We note that when the maximum field amplitude allowed by (3.9) is reached, the function $\Delta(k)$ vanishes and consequently

$$E(x) = \frac{2E_{inc}}{\operatorname{ch}(x\sqrt{\varkappa_0^2 - k^2})}. \tag{3.10}$$

Finally, we write out here the field in the case of low frequencies ($\kappa_0 \gg k$), but at the same time $E_{inc} = E_{cr}$. Then

$$\frac{E(x)}{E_0} = \frac{\sqrt{2}}{\operatorname{ch}(x\varkappa_0 + \ln[\sqrt{2} - 1])}. \tag{3.11}$$

The field is substantially different from the result of the linear theory at relatively small values of $x\kappa_0$, or in other words, within the skin layer.

4. a) In the case of a Fermi particle distribution, when formula (2.4) can be used, we obtain for the phase shift of the reflected wave

$$\cos^2\frac{\psi}{2} = \frac{E_F^2}{4E_{inc}^2}\left\{1 - \left[1 - 10\frac{E_{\text{n}}^2}{E_F^2}\frac{k^2}{\varkappa_0^2}\right]^{2/5}\right\}. \tag{4.1}$$

Complete containment of the carriers by the high-frequency field takes place when $E_{inc}^2 = 0.1(\kappa_0/k)^2 E_F^2$, and the corresponding value of the phase shift is determined by the relation $\cos^2(\psi/2) = (5/2)(k/\kappa_0)^2$. Complete containment is possible here only for values of k smaller than the critical value $k_{cr} = \sqrt{0.4}\,\kappa_0$. The last quantity determines also the smallest possible frequency for which there are no solutions corresponding to opacity in a strong field.

The coordinate dependence of the field, according to (2.17), can be described by the relation

$$\varkappa_0 x \sqrt{\frac{2}{5}} = \int\limits_{E/E_F}^{E_{max}/E_F} dz\,[1 - az^2 - (1 - z^2)^{5/2}]^{-1/2}, \tag{4.2}$$

where $a = (5/2)(k/\kappa_0)^2$ and E_{max} represents the maximum possible amplitude of the field on the surface, corresponding to a decrease of the field at infinity. For values of k lower than critical we have $E_{max} = E_F$, and for $\kappa_0 > k > \sqrt{0.4}\,\kappa_0$, the maximum value of the field is determined by the equation

$$1 = a(E_{max}/E_F)^2 + [1 - (E_{max}/E_F)^2]^{5/2}. \tag{4.3}$$

The solution of this equation ($E_{max}(a)$) is shown in Table I. On approaching the point of ordinary transparency ($k \to \kappa_0$), we have

$$E_{max} = E_F\sqrt{4/3(1 - k^2/\varkappa_0^2)}. \tag{4.4}$$

Table II lists the values of the right side of (4.2), and Fig. 1 illustrates the solutions of this equation; the ordinates represent E/E_F and the abscissas $\sqrt{0.4}\,x\kappa_0$. It must be emphasized that the field on the boundary of the conductor, obviously, never reaches the value E_{max}. However, even in this case it is easy to obtain the values of the field in the conductor with the aid of Table II.

b) In the latter case, corresponding to a Maxwellian particle distribution, when formula (2.3) holds, we obtain for the phase shift of the reflected wave

$$\cos^2\frac{\psi}{2} = -\frac{E_M^2}{4E_{\text{n}}^2}\ln\left\{1 - \frac{4E_{inc}^2 k^2}{E_M^2\varkappa_0^2}\right\} = -\frac{k^2}{\varkappa_0^2}\beta\ln\left(1 - \frac{1}{\beta}\right), \tag{4.5}$$

where $\beta = E_M^2\kappa_0^2/4E_{inc}^2k^2$ is the ratio of the thermal motion of the particles to the field pressure. According to (4.5), obviously, $\beta \geq 1$. At the same time, the inequality

$$k^2 \leqslant k_{cr}^2 = \varkappa_0^2\frac{1}{\beta}\left\{-\ln\left(1 - \frac{1}{\beta}\right)\right\}^{-1}. \tag{4.6}$$

should be satisfied. We see therefore that when the amplitude of the incident-wave field increases, bringing the field pressure closer to the particle pressure, the boundary of the opacity region shifts towards lower frequencies. On approaching to the point of ordinary transparency ($k \to \kappa_0$) we get from (4.6)

Table I

a	$\leqslant 1$	1.25	1.5	1.75	2	2.25	2.5
E_{max}/E_F	1	0.885	0.775	0.66	0.53	0.37	0

Table II

E/E_F	a								
	0	0.10	0,25	0,50	0,75	1.00	1,25	1,50	2.25
1	0.00	0.00	0.00	0.00	0.00	0,00	—	—	—
0.9	0.10	0,105	0.114	0.136	0.18	0,39	—	—	—
0.8	0.20	0,21	0,23	0.25	0.33	0,60	0,45	—	—
0.7	0.31	0.32	0.34	0,40	0.48	0,78	0,70	0,47	—
0.6	0.43	0,44	0.47	0,53	0.64	0,97	0.91	0.78	—
0,5	0.56	0.58	0,61	0,68	0.8	1.14	1,13	1,05	—
0.4	0.71	0,73	0,77	0,86	0.99	1,35	1,36	1,33	—
0,3	0.90	0.93	0,97	1.07	1,22	1.61	1,65	1.65	1.29
0.2	1.16	1,19	1.25	1,37	1.55	1,95	2.03	2,08	2.41
0.1	1.60	1.65	1,72	1.8	2,07	2,53	2.66	2,79	3,92
0.05	2.04	2.09	2,18	2,36	2,60	3.07	3.25	3,50	5.28
0.01	3.06	3,14	3,26	3,49	3,82	4.40	4.69	5,12	8.51

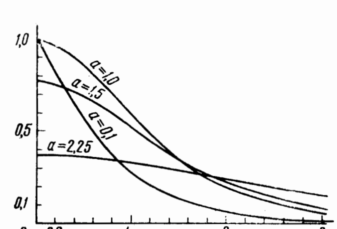

FIG. 1.

$$(E_{\text{inc}}/E_M)^2 < {}^1\!/_2(1 - k^2/\varkappa_0^2) \ll 1. \tag{4.7}$$

The field inside the conductor, according to (2.3) and (2.17), is determined by the relation

$$x\varkappa_0 = \int\limits_{E/E_M}^{E_{max}/E_M} dz\,(1 - (k/\varkappa_0)^2 z^2 - e^{-z^2})^{-1/2}, \tag{4.8}$$

where the upper limit of the integration corresponds to the vanishing of the radicand. Table III characterizes the dependence of (E_{max}/E_M) on $(k/\kappa_0)^2$.

Table III

$k^2/\varkappa_0^2$	0.04	0.1	0.2	0.3	0.4	0.5	0.6	0.7	0.8	0.9
E_{max}/E_M	5	3.16	2.24	1.79	1.49	1.26	1.06	0.87	0.68	0.46

Finally, Table IV lists the values of the right side of (4.8), and Fig. 2 shows for illustration the plot of E/E_M against $\kappa_0 x$ for several values of $(k/\kappa_0)^2$. Just as in the analysis of the solutions of (4.2), we can determine with the aid of Table IV the field in the conductor also in the case when the field on the surface is smaller than E_{max}.

In strong fields, when $E \gg E_M$, we obtain in accordance with (4.8)

$$E \cong (\varkappa_0/k) E_M \cos kx. \tag{4.9}$$

When the condition $k \ll \kappa_0$ is satisfied, formula (4.9) becomes inaccurate in the vicinity of $kx \cong \pi/2$. Noting that the formula (4.9), which was obtained with exponential accuracy, corresponds to a field in vacuum, it is easily seen that one can speak with the same degree of accuracy of compression of the carriers by the high-frequency field. An appreciable particle concentration is produced thereby precisely in the vicinity of $kx \cong \pi/2$.

5. Owing to the dissipative effects, as is well known, sufficiently thick conductors are opaque also when the real part of their dielectric constant is positive. We shall investigate below a similar effect under conditions when the nonlinearity of the material equations of the field is appreciable. We confine ourselves to the case of greatest interest, when the effective frequency ν of the collisions causing the dissipation is small compared with the frequency of the electromagnetic field, although a number of the general premises developed below are valid also in the case of appreciable dissipation. Then the material equation can be written in the form (see (2.2))[1)]

$$D(x,t) = \left\{1 - \left(\frac{c\varkappa_0}{\omega}\right)^2 f(E_1^2 + E_2^2)\left(1 + \frac{\nu}{\omega^2}\frac{\partial}{\partial t}\right)\right\} E(x,t). \tag{5.1}$$

Representing the electric field in the medium in the form (2.11), we can readily obtain from the electromagnetic-field equations the following two equations:

$$\frac{d^2E}{dx^2} = \frac{M_s^2}{E^3} - \{k^2 - \varkappa_0^2 f(E^2)\}E, \tag{5.2}$$

$$\frac{dM_s}{dx} = -\frac{\nu}{\omega}\varkappa_0^2 f(E^2) E^2. \tag{5.3}$$

Table IV

E/E_M	$k^2/\varkappa_0^2$: 0.04	0.1	0.2	0.3	0.4	0.5	0.6	0.7	0.8	0.9
0,05	10,10	7.55	6.09	5,78	5.76	6.05	6.50	6.68	7.95	9,46
0.1	9.39	6.63	5,26	4,90	4.95	4.96	5.21	5.56	6.09	6.95
0,2	8.68	5.89	4.49	4.06	4.04	3.97	4.10	4,27	4.49	4.63
0.3	8.26	5.46	4.03	3.57	3.50	3.38	3.43	3.49	3.51	3.09
0.4	7.96	5.14	3.69	3.21	3.12	2.94	2.94	2.91	2.74	1.64
0.5	7,72	4.90	3.43	2,92	2,80	2,59	2.53	2.41	2.05	
0.6	7,51	4.69	3.20	2.68	2.53	2.29	2.18	1,96	1,31	
0.7	7,34	4.51	3.01	2.46	2.30	2.02	1.84	1.51		
0.8	7,18	4.34	2.83	2.27	2.08	1.76	1.52	0.98		
0,9	7.04	4.19	2.66	2.09	1.87	1.50	1.16			
1.0	6,91	4.05	2.51	1,91		1.24	0.72			
1.1	6.78	3.92	2.36	1.74		0,94				
1.2	6.66	3.79	2.21	1.57		0.56				
1.3	6.54	3.66	2.07	1.39						
1,4	6.43	3.54	1.92	1.20						
1.5	6.32	3.42	1.77	1.00						
1.6	6.21	3.29	1,62	0.77						
1,7	6.10	3,17	1,46							
1.8	5.99	3.05	1,29							
1,9	5.88	2,92	1.10							
2.0	5.77	2.79	0,89							
2,1	5.66	2,66	0,62							
2.2	5.54	2,52								
2,3	5.43	2,38								
2.4	5.32	2.23								
2.5	5.20	2.07								
2.6	5.09	1,90								
2,7	4.97	1.71								
2.8	4.85	1,51								
2.9	4.73	1.28								
3.0	4,60	1.00								

M_s is defined in (2.12), but here this quantity is no longer conserved, since the energy of the field is absorbed by the medium if dissipation is taken into account.

Equations (5.2) and (5.3) enable us also to write the following:

$$\frac{d\mathscr{E}}{dx} = \frac{2M_s}{E^2}\frac{dM_s}{dx} = -2\frac{\nu}{\omega}\varkappa_0^2 f(E^2) M_s, \tag{5.4}$$

where $\mathscr{E}$ is defined by (2.13), and is likewise not conserved as a result of dissipation.

Since the right sides of (5.3) and (5.4) contain the small parameter ν/ω, we can speak of a relatively slow variation of M_s and $\mathscr{E}$ with increasing coordinate x. On the other hand, even at constant M_s, Eq. (5.2) describes a rapid dependence of the field E on the coordinate. The possibility of such a separation into fast and slow dependences afford a deeper insight into the behavior of the field in the nonlinear conductor considered by us.

Let us discuss first the fast alternating field dependence. We neglect here the possible slow dependence and assume that M_s and $\mathscr{E}$ are constant. Then the dependence of the field on the coordinate, according to (2.13), can be characterized by the relation

$$x = \pm \int\limits_{E(0)}^{E} \left(\mathscr{E} + \varkappa_0^2 \int\limits_{E_{min}^2}^{\xi^2} d\zeta f(\zeta) - k^2\xi^2 - M_s^2/\xi^2\right)^{-1/2} d\xi. \tag{5.5}$$

Formula (5.5) describes the nonlinear periodic solutions of the field equations in a medium without dissipation, which were first investigated by Volkov[1]. Indeed, when $M_s \neq 0$, the minimum of the field corresponds to the vanishing of the radicand in the integral of the right side of (5.5). It is therefore determined by the equation of the turning point:

$$M_s^2 + k^2 E_{min}^4 = \mathscr{E} E_{min}^2. \tag{5.6}$$

From this it follows, in particular, that $\mathscr{E} \geq 0$. Further, inasmuch as the radicand in the integral (5.5) vanishes like the first power of $\xi - E_{min}$ in the vicinity of the point in which $E = E_{min}$, the field assumes a minimum

1)We note that ν may contain a dependence on f. For example, in the case of a fully ionized plasma $\nu \sim f$. However, we shall disregard this dependence here. The necessary generalization is trivial, as can be readily verified by the reader.

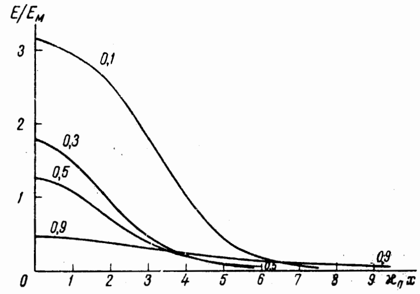

FIG. 2.

value at a finite value of the coordinate x. Then the field increases with increasing coordinate x. Such an increase of the field can be limited by two factors. First, the function f can vanish, corresponding to the absence of carriers, or, what is the same, to the presence of a second surface bounding the conductor. Second, a second turning point is possible, in which the field reaches the maximum value defined by the relation

$$\frac{M_s^2}{E_{max}^2} + k^2 E_{max}^2 - \varkappa_0^2 \int_{E_{min}^2}^{E_{max}^2} d\zeta f(\zeta) = \mathscr{E}. \tag{5.7}$$

We shall assume this last possibility, since the first is realized only for conductors of finite thickness[2)].

In the vicinity of the maximum field, the radicand of the integral (5.5) behaves like $E_{max} - E$. Therefore the maximum field occurs at a finite value of x. Thus, in the presence of two turning points the field in the conductor depends periodically on the coordinate x, and we have for the period of the spatial oscillations

$$L = 2 \int_{E_{min}}^{E_{max}} \left(\mathscr{E} - (M_s/\xi)^2 - k^2\xi^2 + \varkappa_0^2 \int_{E_{min}^2}^{\xi^2} d\zeta f(\zeta) \right)^{-1/2} d\xi. \tag{5.8}$$

According to (2.13) and (2.14), the reflection coefficient and the phase shift of the reflected wave are determined by the relations

$$M_s = k(1 - R^2) E_{\text{inc}}^2, \quad 2k^2 E_{\text{n}}^2 (1 + R^2) = \mathscr{E} + \varkappa_0^2 \int_{E_{min}^2}^{E_{\text{n}}^2(1+2R\cos\psi+R^2)} d\zeta f(\zeta). \tag{5.9}$$

Bearing in mind also relations (5.6) and (5.7), we can see that R and ψ are determined by the minimum and maximum values of the field in the conductor. For the approximation considered now, in which neglect dissipation completely, we can speak of the existence of an undamped high-frequency field in the conductor; this field determines the field of the reflected wave. It is easily seen that if damping is neglected, this field can be chosen such that wide ranges of arbitrary values can be obtained for R and ψ.

The foregoing demonstrates the importance of taking dissipative processes into account. Under conditions of small dissipation, one can speak of a slow variation of the parameters determining the fast oscillations of the field with increasing coordinate x. Namely, a slow change takes place in the period of the spatial oscillations and in the maximum and minimum values of the field, and therefore a change takes place also in M_S and $\mathscr{E}$. To describe such a slow dependence, we can use the method of averaging over the trajectories[12]. The equations for the quantities averaged over the period of the fast oscillations can be then written in the form

$$\frac{dM_s}{dx} = -\frac{\nu}{\omega}\varkappa_0^2 \langle E^2 f(E^2)\rangle, \quad \frac{d\mathscr{E}}{dx} = -2\frac{\nu}{\omega}\varkappa_0^2\langle f(E^2)\rangle M_s. \tag{5.10}$$

We shall use here the old notation for the averaged values of M_S and $\mathscr{E}$, and we define the averaging operation as:

$$\langle F(E^2)\rangle = \frac{2}{L}\int_{E_{min}}^{E_{max}} F(E^2)\left[\mathscr{E} - (M_s^2/E^2) - k^2E^2 + \varkappa_0^2 \int_{E_{min}^2}^{E^2} d\zeta f(\zeta)\right]^{-1/2} dE. \tag{5.11}$$

Equation (5.10) describes a slow variation of the parameters characterizing the fast oscillations. It should be noted that the field becomes weak upon sufficient penetration of the conductor, and therefore the nonlinear effects become negligibly small for sufficiently large x. Obviously, when $k^2 < \kappa_0^2$ it is possible to use here the results of linear electrodynamics, according to which the field decreases aperiodically and exponentially, like $\exp\{-x\sqrt{\kappa_0^2 - k^2}\}$, even without allowance for dissipation. The corresponding asymptotic solution of (5.2) and (5.3) can be obtained by putting f = 1. As a result we get[3)]

$$M_s(x) = E^2(x)\frac{\gamma\sqrt{\varkappa_0^2 - k^2}}{\sqrt{2}[1+\sqrt{1+\gamma^2}]^{1/2}}, \tag{5.12}$$

$$E(x) = \text{const}\cdot\exp\left\{-x\sqrt{\varkappa_0^2 - k^2}\frac{1}{\sqrt{2}}[1+\sqrt{1+\gamma^2}]^{1/2}\right\}, \tag{5.13}$$

where $\gamma = (\nu/\omega)\kappa_0^2/(\kappa_0^2 - k^2)$. In writing out these formulas it is assumed that $\kappa_0^2 > k^2$.

We can now describe the nonlinear picture of the penetration of the field into the conductor in the following manner. At large incident-field wave amplitude, the depth of the nonlinear skin layer can greatly exceed the depth of the linear skin layer $\sim (\kappa_0^2 - k^2)^{-1/2}$. Near the surface of the conductor, where the field is strong, slow nonlinear spatially-damped field oscillations are produced. These solutions go over into the aperiodic exponential field attenuation mode only after appreciable penetration into the conductor. It can be stated that a layer of order of magnitude equal to the linear skin layer separates the penetration region from the conductor region in which the field does not penetrate. In the penetration region the field is described by anharmonic spatial oscillations.

We have already seen that when dissipation is neglected there is an ambiguity in the determination of the field from the specified incident-wave intensity E_{inc}. The ambiguity remains also in the case of sufficiently weak dissipation. To demonstrate this, we must consider the question of satisfying the boundary conditions on the surface of the conductor.

Bearing in mind that the electric field E as a function of M_S is determined by the equation

$$\frac{\nu}{\omega}\varkappa_0^2 E^2 f(E^2)\frac{d}{dM_s}\left(\frac{\nu}{\omega}\varkappa_0^2 E^2 f(E^2)\frac{dE}{dM_s}\right) = \frac{M_s^2}{E^3} - [k^2 - \varkappa_0^2 f(E^2)]E, \tag{5.14}$$

---

2) A periodic structure may occur in the conductor when the action of the field gives rise to a conductor-insulator periodic structure.

3) We note that $M_S > 0$ leads, in accordance with (2.12), to the inequality $\varphi' < 0$. The latter denotes that the nonlinear wave travels from left to right.

and also taking into account the asymptotic connection between E and M_S when these two quantities are small, we easily see that the electric field and its spatial derivative are determined uniquely by M_S. Therefore, knowing the solution of (5.14) at specified values of k and E_{inc}, we can determine the boundary value $M_S(0)$ (and consequently also E(0) and E′(0)) with the aid of the equation

$$4E_{inc}^2 = [E'(0)/k]^2 + E^2(0)[1 + M_s(0)/kE^2(0)]^2, \qquad (5.15)$$

which follows from the boundary relations (2.14). Accordingly, the phase shift and the reflection coefficient can be calculated with the aid of relations (2.14) using the already determined boundary values of E, E′, and M_S. Finally, the spatial dependence of the field on the coordinate x is determined by integrating (5.3) after substituting into it the solution of (5.14), and $\varphi(x)$ is determined by integrating (2.12).

The ambiguity in the determination of the field[4)] is connected with the possible existence of several solutions of (5.15). In particular, when $\nu/\omega = 0$ and periodic anharmonic solutions are obtained, the same value of E corresponds to two values of E′ with opposite signs. When ν/ω differs from zero but is small, Eq. (5.15) can have, for specified E_{inc} and k, a large number of solutions, corresponding to a slow variation (and therefore a large number of close values) of E_{max} and E_{min}. With increasing dissipation, the possible number of the solutions of (5.15) decreases. We shall show in the next section, using an example of universal significance, that when the dissipation is sufficiently strong, the solution of the boundary-value problem for specified E_{inc} and k turns out to be unique.

6. Let us consider in detail the penetration of the field into a conductor near the point of the ordinary transparency ($\kappa_0^2 \gtrsim k^2$). As we have already seen, the nonlinear effects become manifest here already in a relatively weak field. This is precisely the case to which we shall confine ourselves below. In accordance with this, we can assume in (5.3) and (5.4) that f = 1, and we can substitute in (5.2) the value of f in the form (2.5). By making the change of variables

$$\tau = x\sqrt{\varkappa_0^2 - k^2}, \quad \nu/\omega = \gamma(1 - k^2/\varkappa_0^2), \quad \mathscr{E} = \varepsilon\varkappa_0^2 2E_0^2(1 - k^2/\varkappa_0^2)^2,$$

$$E^2 = r^2 2E_0^2[1 - k^2/\varkappa_0^2], \quad M_s^2 = \mu^2 4E_0^4(\varkappa_0^2 - k^2)^3/\varkappa_0^4, \qquad (6.1)$$

we can rewrite the system (5.2)–(5.4) in the form

$$\frac{d^2r}{d\tau^2} = r - 2r^3 + \frac{\mu^2}{r^3}, \quad \frac{d\mu}{d\tau} = -\gamma r^2, \quad \frac{d\varepsilon}{d\tau} = -2\gamma\mu. \qquad (6.2)$$

Equations (6.2) reveal the analogy between our problem and the problem of the motion of an anharmonic rotator whose moment (μ) dissipates, per unit time, in proportion to the square of the radius r.

When γ is small and we can speak of separation of the fast and slow variations of the quantities characterizing the field, it is meaningful to consider an approximate solution of (6.2) in which μ and ϵ are assumed constant. We then obtain

$$r^2 = r_{max}^2 - A\,\mathrm{sn}^2([\tau - \tau_0]\sqrt{A/k^2}, k), \qquad (6.3)$$

4)In the nonlinear theory of a longitudinal field in the plasma, the solution had three values near the plasma resonance [7].

where sn(z, k) is the Jacobi elliptic sine function. $A = r_{max}^2 - r_{min}^2$, $k = \sqrt{A/(2r_{max}^2 + r_{min}^2 - 1)}$, and the minimum and maximum "turning points" are determined by the relations

$$\varepsilon r_{min}^2 + r_{min}^4 - r_{min}^6 - \mu^2 = 0,$$

$$r_{max}^2 = {}^1/_2(1 - r_{min}^2) + {}^1/_2[(1 - r_{min}^2)^2 + 4\mu^2/r_{min}^2]^{1/2}.$$

We note that $r_{max}^2 + r_{min}^2 \geq 1$, and equality occurs only for $\epsilon = \mu = 0$, when k = 1 and the minimal turning point vanishés while the maximum becomes equal to unitv.

Formula (6.3) describes the spatial oscillations of the field, with a period

$$\frac{1}{\sqrt{\varkappa_0^2 - k^2}}\frac{2k}{A}\mathrm{K}(k), \qquad (6.4)$$

where

$$\mathrm{K}(k) = \int_0^{\pi/2} d\varphi/\sqrt{1 - k^2\sin^2\varphi}$$

is the complete elliptic integral. Owing to the dissipative effects, this period changes on penetrating deeper into the conductor, and the maximum and minimum fields also change. For the description of such a slow variation in the case of small γ we obtain, following the averaging method, the equations

$$\frac{d\varepsilon}{d\tau} = -2\gamma\mu, \quad \frac{d\mu}{d\tau} = -\gamma\left\{r_{max}^2 - A\frac{\mathrm{D}(k)}{\mathrm{K}(k)}\right\}, \qquad (6.5)$$

where

$$\mathrm{D}(k) = \int_0^{\pi/2} d\varphi\sin^2\varphi/\sqrt{1 - k^2\sin^2\varphi}$$

is the complete elliptic integral.

Equations (6.5) have a stationary point $\epsilon = 0$ and $\mu = 0$, in the vicinity of which

$$\varepsilon = \frac{1}{4}\mu^2\ln\frac{64}{\mu^2} + \text{const}, \quad \mu\ln\frac{64}{\mu} = -4\gamma\tau + \text{const}. \qquad (6.6)$$

It follows therefore that the representative point in the (ϵ, μ) plane arrives at a stationary point within a finite "time" τ. For the case of interest to us, when the field decreases at infinity, the passage through such a stationary point is necessary. Moreover, the solution of the averaged equations (6.5) is suitable only until this point is reached.

The system (6.5) can be solved in quadratures. Namely:

$$A = A(k^2) = A_0\exp\left\{\int_{k_0^2}^{k^2}\frac{dk^2}{k^2}\frac{\mathrm{C}(k) - \mathrm{D}(k)}{(1 - k^2)\mathrm{C}(k) - 2\mathrm{E}(k)}\right\}, \qquad (6.7)$$

$$\frac{2\gamma}{3\sqrt{3}}(\tau_0 - \tau) = \int_{k_0^2}^{k^2}\frac{dk^2}{k^3}\frac{A^2(k^2)(1 - k^2)\mathrm{K}(k)}{(1 - k^2)\mathrm{C}(k) - 2\mathrm{E}(k)} \qquad (6.8)$$

$$\times\{[k^2 + (1 + k^2)A(k^2)][k^2 + (1 - 2k^2)A(k^2)][-k^2 + (2 - k^2)A(k^2)]\}^{-1/2}.$$

Here k_0 and A_0 are the initial values of the functions k and A when $\tau = \tau_0$ and E(k) and C(k) are the complete elliptic integrals

$$\mathrm{E}(k) = \int_0^{\pi/2} d\varphi\sqrt{1 - k^2\sin^2\varphi}, \quad \mathrm{C}(k) = \int_0^{\pi/2}\frac{d\varphi\sin^2\varphi\cos^2\varphi}{\sqrt{1 - k^2\sin^2\varphi}}.$$

Formulas (6.7) and (6.8) describe the variation of the slow variables up to the stationary point, at which A = 1 and k = 1 (and accordingly $r_{max}^2 = 1$). Inasmuch as passage through the stationary point is necessary in order for the field to decrease at infinity, we get the condition

$$1 = A(1) \equiv A_0 \exp\left\{\int_{k_0^2}^{1} \frac{dk^2}{k^2} \frac{\mathrm{C}(k) - \mathrm{D}(k)}{(1-k^2)\mathrm{C}(k) - 2\mathrm{E}(k)}\right\}. \qquad (6.9)$$

Passage through the stationary point, and hence the decrease of the field at infinity, is possible only if the condition (6.9), which is imposed on the initial values A_0 and k_0, is satisfied.

After passing through the stationary point, the field oscillations stop and $r(\tau)$ tends monotonically to zero. The asymptotic solution of (6.2), which describes such a "rotator decay" and which is suitable for small values of r and μ, is of the form

$$r = \left[\frac{\sqrt{2}}{\gamma}(1+\sqrt{1+\gamma^2})^{1/2}\mu\right]^{1/2}\left\{1 - \left[\frac{\sqrt{2}}{\gamma}\frac{(1+\sqrt{1+\gamma^2})^{1/2}}{3+5\sqrt{1+\gamma^2}}\mu\right] - \frac{9}{2}\frac{1}{4+5\sqrt{1+\gamma^2}}\left[\frac{\sqrt{2}}{\gamma}\frac{(1+\sqrt{1+\gamma^2})^{1/2}}{3+5\sqrt{1+\gamma^2}}\mu\right]^2 \right.$$
$$\left. - \frac{1}{2}\frac{199+89\sqrt{1+\gamma^2}-140\gamma^2}{(4+5\sqrt{1+\gamma^2})(15+17\sqrt{1+\gamma^2})}\left[\frac{\sqrt{2}}{\gamma}\frac{(1+\sqrt{1+\gamma^2})^{1/2}}{3+5\sqrt{1+\gamma^2}}\mu\right]^3 + O(\mu^4)\right\}, \qquad (6.10)$$

$$\ln\mu + \frac{2\sqrt{2}}{\gamma}\frac{(1+\sqrt{1+\gamma^2})^{1/2}}{3+5\sqrt{1+\gamma^2}}\mu = -\gamma\tau\sqrt{2}(1+\sqrt{1+\gamma^2})^{1/2} + \text{const.} \qquad (6.11)$$

The fast decrease of the field goes over relatively rapidly into the decrease described by the linear theory.

Figures 3 and 4 show plots of r, $v = dr/d\tau$, and μ against $\tau\left(-\tau = \gamma^{-1}\int_{0.01}^{\mu} d\mu/r^2(\mu)\right)$, obtained by numerical integration of the system (6.2) and of the equation

$$\gamma^2 r^2 \frac{d}{d\mu}\left(r^2\frac{dr}{d\mu}\right) = r - 2r^3 + \frac{\mu^2}{r^3}. \qquad (6.12)$$

at values $\gamma = 0.2$ and 0.05.

We note that formula (6.10) for $\gamma \leq 0.5$ leads to results which coincide practically with the results of numerical calculation up to values of r corresponding to $r_{max} = 1$, i.e., up to the stationary point of (6.5). This indicates a direct transition of the asymptotic solution (6.10) into an oscillatory dependence, which can be conveniently described by formulas (6.3), (6.7), and (6.8).

Let us discuss, finally, the question of satisfying the boundary conditions on the surface of the conductor, and respectively on the uniqueness of the obtained solutions. Putting $E_{inc}^2 = \eta^{1/2}E_0^2(1-k^2/\kappa_0^2)$ and using (6.1), we can rewrite (5.15) in the form

$$\eta = \left[\frac{\varkappa_0^2}{k^2} - 1\right]v^2 + r^2\left[1 + \frac{\mu}{r^2}\sqrt{\frac{\varkappa_0^2}{k^2} - 1}\right]^2. \qquad (6.13)$$

Figure 5 shows plots of η against μ for $\kappa_0^2/k^2 - 1$ equal to 0.25, 0.0625, and 0.01 and $\gamma = 0.2$. We note that, as follows from (6.1), by bringing the values of k and κ_0 close together for a specified γ, we arrive at smaller values of ν/ω. We can therefore state on the basis of Fig. 5 that at small values of ν/ω each given value of η corresponds to several values of μ, and when the dissipation increases this ambiguity vanishes.

The discussed ambiguity of the solutions, which corresponds to the possible existence of several stationary solutions, is typical of a number of nonlinear-

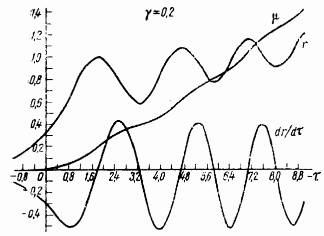

FIG. 3.

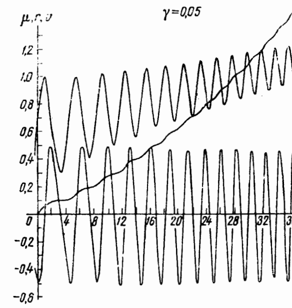

FIG. 4.

oscillation problems. It is possible to use here the concepts developed in the theory of nonlinear solutions[12] concerning the stability of such stationary amplitudes. In particular, one can speak of oscillatory hysteresis under adiabatic variation of the amplitude of the incident wave. The course of such an oscillatory hysteresis is indicated by the arrows in Fig. 5.

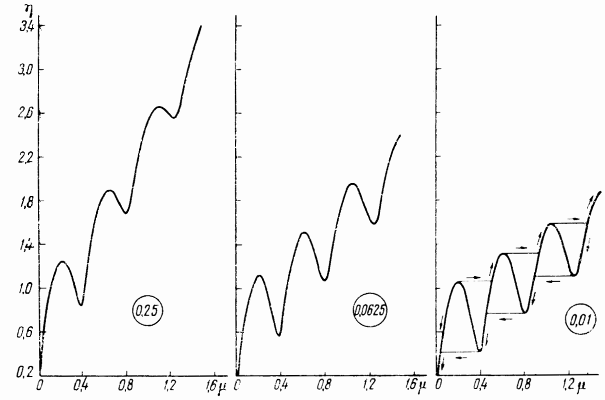

FIG. 5.

7. Conclusions. The foregoing analysis shows that the limiting frequency of the transparency region shifts with increasing incident-wave amplitude towards smaller values. We studied the penetration of the field into the plasma of a conductor under conditions of nonlinear transparency and weak damping. We have shown that several stationary solutions can exist under such conditions, a fact which can become manifest by oscillatory hysteresis under adiabatic variation of the amplitude of the incident wave. We note that the phenomena under consideration can take place in relatively weak fields, when the plasma instability known to occur in a high-frequency field do not develop. Finally, in order to prevent excessive heating, which can greatly hinder the understanding of the phenomenon, it would be necessary to use in the experiment high-frequency field pulses with duration not greatly exceeding the free path time $1/\nu$. For relatively weak fields, in which the aforementioned phenomena are possible, the stationary field distribution investigated here will apparently become established within such a time.

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