AN AXIOMATIC MODEL OF QUANTUM FIELD THEORY

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A variant of the axiomatic method of quantum field theory proposed in[1] is considered. Equations for the Heisenberg field operators are set up within the framework of this method. Solutions of these equations are derived. It is shown when this model only has a trivial solution.

1. INTRODUCTION

In earlier paper,[1] the author has considered a set of axioms which must be satisfied by a relativistic field theory. These axioms represent a certain modification of the axioms usually introduced in the axiomatic method of quantum field theory (cf., for example,[2-4]). It was asserted in[1] that the axioms formulated there permit the construction of a nontrivial example of the theory. This problem is at present essential for the entire axiomatic method, since there is a suspicion that the axioms proposed are only consistent with the trivial theory of free particles. In the present paper we attempt to construct such a nontrivial example.

It was remarked in[1] that the problem of constructing a theory which satisfies all axioms reduces to the construction of families of field quantities (operators of the field and of the canonical momentum) $A_\sigma(t, \mathbf{z}; T)$ which must satisfy the following requirements:[1)]

Axiom III. For each fixed t and T, the quantities $A_\sigma(t, \mathbf{z}; T)$ ($\sigma = \pm 1$) are operator-valued generalized functions on the space $S(R_3)$, i.e., in order to convert the quantities $A_\sigma(t, \mathbf{z}; T)$ into ordinary operators it suffices to average them with "good" [in $S(R_3)$] functions in the space variable $\mathbf{z}$ alone. This in turn allows one to introduce a time ordering of the operators A_σ, i.e., physically, to introduce the acts of creation and annihilation of particles. The example of the Lagrangian method shows that such an ordering is necessary for the construction of the S matrix.

Axiom IV. The region of definition and of the values of $A_\sigma(t, \mathbf{z}; T)$ is $D(T, n)$. This requirement is purely mathematical, and is connected with the unboundedness of the operators A_σ. The latter means that fields of arbitrary intensity can exist in nature.

Axiom V.

$$[A_\sigma(t, \mathbf{z}; T), A_\rho(t, \mathbf{u}; T)] = \frac{i}{2}(\sigma - \rho)\delta(\mathbf{z} - \mathbf{u}), \quad \sigma, \rho = \pm 1. \quad (1)$$

Axiom VI. The operators $A_\sigma(t, \mathbf{z}; T)$ are Hermitian.

Axiom VII. For each fixed t and T the ring of operators A_σ coincides with the ring $\mathfrak{A}$ of operators φ_σ satisfying the free field equations. Here

$$A_\sigma(t, \mathbf{z}; T) = \varphi_\sigma(t, \mathbf{z}) \text{ for } t = T. \quad (2)$$

It follows from this that A_σ can be written in the form of an expansion in terms of the φ_σ:

$$A_\sigma(t, \mathbf{z}; T) = \sum \int d\mathbf{u} J_i{}^\sigma(t; T; \mathbf{z}; \mathbf{u}_1, \ldots, \mathbf{u}_i)\varphi_{\sigma_1}(T, \mathbf{u}_1)\ldots\varphi_{\sigma_i}(T, \mathbf{u}_i), \quad (3)$$

and, vice versa, the φ_σ can be expanded in terms of the A_σ. This implies in turn that the fields $A_\sigma(t_1, \mathbf{z}; T)$ can be expanded in terms of the $A_\sigma(t_2, \mathbf{z}; T)$ for arbitrary t_1 and t_2, and vice versa. For us it will be more convenient to use an expansion in "normal products" of the fields A_σ.

We define the operators A_σ^+ and A_σ^- by the formula

$$A_\sigma^\pm(t, \mathbf{z}; T) = \frac{1}{2}A_\sigma(t, \mathbf{z}; T) \mp \frac{i\sigma}{2}\int d\mathbf{u}\Omega^{-\sigma}(\mathbf{z} - \mathbf{u})A_{-\sigma}(t, \mathbf{u}; T), \quad (4)$$

where

$$\Omega^\pm(\mathbf{z}) = (2\pi)^{-3}\int d\mathbf{k}(m^2 + \mathbf{k}^2)^{\pm 1/2}e^{i\mathbf{k}\mathbf{z}}. \quad (5)$$

When A_σ is replaced by φ_σ, formula (4) gives the usual definition of the positive- and negative-frequency parts of the operator φ_σ. By the normal product

$$:A_{\sigma_1}(t, \mathbf{z}_1; T)\ldots A_{\sigma_n}(t, \mathbf{z}_n; T): \quad (6)$$

we understand the following. In order to construct (6), one must take the usual product of the operators A_σ, write each of them as a sum of A_σ^+ and A_σ^- and then place all A_σ^+ to the left of all A_σ^-. It is easy to see that the usual product of operators A_σ can be expanded in terms of normal products and vice versa, where the expansion is accomplished by the usual Wick theorem with the same coefficient functions as in the case of free fields. The normal product of A_σ does not depend on the order of writing the operators. Thus the field $A_\sigma(t_1, \mathbf{z}; T)$ can be written in the form

$$A_\sigma(t_1, \mathbf{z}; T) = \sum_{i,j}\int d\mathbf{u}\, d\mathbf{v}\, J_{ij}{}^\sigma(t_1, t_2, T; \mathbf{z}; \mathbf{u}_1, \ldots, \mathbf{u}_i; \mathbf{v}_1, \ldots, \mathbf{v}_j)\cdot$$
$$\cdot :A_+(t_2, \mathbf{u}_1; T)\ldots A_+(t_2, \mathbf{u}_i; T)A_-(t_2, \mathbf{v}_1; T)\ldots A_-(t_2, \mathbf{v}_j; T):. \quad (7)$$

We can assume without loss of generality that the J_{ij}^σ are symmetric in all $\mathbf{u}$ and $\mathbf{v}$. We shall assume that the sum in (7) is finite and that the J_{ij}^σ are generalized functions with moderate growth such that

$$\int d\mathbf{z} f(\mathbf{z})J_{ij}{}^\sigma(\ldots; \mathbf{z}; \mathbf{u}_1, \ldots; \mathbf{v}_1 \ldots) \in S(R_{3(i+j)}),$$
$$\text{if } f(\mathbf{z}) \in S(R_3). \quad (8)$$

Axiom VIII. The functions $J_i^\sigma(t, T; \mathbf{z}; \mathbf{u}_1, \ldots)$ depend only on $t - T$, $|\mathbf{u}_1 - \mathbf{z}|, \ldots, |\mathbf{u}_i - \mathbf{z}|$. It follows from this that the functions J_{ij}^σ can depend only on $t_1, -t_2, t_2 - T$, $|\mathbf{u}_1 - \mathbf{z}|, \ldots, |\mathbf{u}_i - \mathbf{z}|, |\mathbf{v}_1 - \mathbf{z}|, \ldots, |\mathbf{v}_j - \mathbf{z}|$.

Axiom IX.

$$[A_\sigma(t_1, \mathbf{z}_1; T), A_\rho(t_2, \mathbf{z}_2; T)] = 0 \text{ for } |\mathbf{z}_1 - \mathbf{z}_2| > |t_1 - t_2|, \quad (9)$$

$$\text{or for } |\mathbf{z}_1 - \mathbf{z}_2| > |t_1 - t_2| + \eta, \quad (9a)$$

[1)]We only write down the requirements on A_σ, preserving the numbering of [1]. All notation which is not explained is the same as in [1].

where η is a small positive quantity.

The requirements VII and IX are two aspects of the causality principle; the requirement IX contains a negative assertion: events which are separated by a space-like distance do not depend on each other. Requirement VII reflects the positive aspect of the causality principle: actually, this requirement asserts that there exists a field equation [formula (7)]. This means physically that events at the times t_1 and t_2 are not independent, and there exists a causal connection between them.

2. THE TRANSITIVITY OF THE AXIOMS

It turns out that the axioms enumerated in Sec. 1 have the transitive property, which consists in the following. Let us assume that for some τ there exist operators $A_\sigma(T+\tau, \mathbf{z}; T)$ which satisfy all axioms; then the operators $A_\sigma(T+N\tau, \mathbf{z}; T)$ satisfying all axioms also exist. Here N is an arbitrary integer number. For economy of writing, we introduce the new notation

$$A_\sigma(T+t, \mathbf{z}; T) = B_\sigma(t, \mathbf{z}).$$

We assume that the operators $B_\sigma(\tau; \mathbf{z})$ exist. In particular formula (7) holds for these operators:

$$B_\sigma(\tau; \mathbf{z}) = \sum \int d\mathbf{u}\, d\mathbf{v}\, J_{ij}{}^\sigma(\tau; \mathbf{z}; \mathbf{u}_1, \ldots; \mathbf{v}_1, \ldots) : \varphi_+(T, \mathbf{u}_1)\ldots\varphi_-(T, \mathbf{v}_1)\ldots : . \tag{10}$$

Here we have replaced $J_{ij}^\sigma(T+\tau, T, T; \ldots)$ by the symbol $J_{ij}^\sigma(\tau; \ldots)$. Axiom VII requires that the operators $\varphi_\sigma(T, \mathbf{z})$ belong to the ring of operators $B_\sigma(\tau; \mathbf{z})$. Therefore there must exist such functions $J_{ij}^\sigma(-\tau; \ldots)$ that

$$\varphi_\sigma(T; \mathbf{z}) = \sum \int d\mathbf{u}\, d\mathbf{v}\, J_{ij}{}^\sigma(-\tau; \mathbf{z}; \mathbf{u}_1, \ldots; \mathbf{v}_1, \ldots) \times : B_+(\tau, \mathbf{u}_1)\ldots B_-(\tau; \mathbf{v}_1)\ldots : . \tag{11}$$

Let us show that the operators $B_\sigma(N\tau; \mathbf{z})$, defined by

$$B_\sigma((N\pm 1)\tau; \mathbf{z}) = \sum \int d\mathbf{u}\, d\mathbf{v}\, J_{ij}{}^\sigma(\pm\tau; \mathbf{z}; \mathbf{u}_1, \ldots; \mathbf{v}_1, \ldots) \times : B_+(N\tau; \mathbf{u}_1)\ldots B_-(N\tau; \mathbf{v}_1)\ldots : , \tag{12}$$

satisfy all axioms. We consider only the case where (12) is written with the plus sign; the other case is completely analogous. The proof is by induction. Thus we assume that the operators $B_\sigma(N\tau; \mathbf{z})$ satisfy all axioms.

Since the functions $J_{ij}^\sigma(\tau; \mathbf{z}; \ldots)$ satisfy (8) and the sum in (12) is considered finite, the requirements III and IV are evidently also satisfied for the operators $B_\sigma((N+1)\tau; \mathbf{z})$.

Let us now consider axiom V. From (1), written for the operators $B_\sigma(\tau; \mathbf{z})$, we easily obtain relations which must be satisfied by the functions $J_{ij}^\sigma(\tau)$. For this purpose it suffices to substitute $B_\sigma(\tau; \mathbf{z})$ in the form (10) in (1) and expand the result in normal products of $\varphi_\sigma(T, \mathbf{u})$. Equating the coefficient functions of identical normal products on the left and right-hand sides of (1), we obtain certain relations (*) for the $J_{ij}^\sigma(\tau; \ldots)$. These relations have a rather complicated form and will therefore not be quoted here. It is clear that the relations (*) are necessary and sufficient conditions for the fulfillment of (1) by $B_\sigma(\tau; \mathbf{z})$. Let us now consider the condition (1) for $B_\sigma((N+1)\tau; \mathbf{z})$. The commutators of $B_\sigma((N+1)\tau; \mathbf{z})$ appearing in (1) are now expanded in terms of the normal products of $B_\sigma(N\tau; \mathbf{u})$ with the help of (12). Since the commutation relations for $B_\sigma(N\tau; \mathbf{u})$ are the same as for $\varphi_\sigma(T, \mathbf{u})$, the coefficient functions in this expansion will be the same as in the expansion of $B_\sigma(\tau; \mathbf{z})$ in terms of $\varphi_\sigma(T, \mathbf{u})$. In virtue of (*), the equations (1) will also be satisfied for $B_\sigma((N+1)\tau; \mathbf{z})$.

The assertion VI is proved in an elementary fashion. Since $B_\sigma(\tau; \mathbf{z})$ is Hermitian, $J_{ij}^\sigma(\tau; \ldots)$ is real. Since the operators in a normal product can be commuted, the normal product of the Hermitian operators $B_\sigma(N\tau; \mathbf{u})$ is a Hermitian operator, and thus the operator $B_\sigma((N+1)\tau; \mathbf{z})$ is also Hermitian.

Let us now consider axiom VII. By construction, the $B_\sigma((N+1)\tau; \mathbf{z})$ lie in the ring of operators $B_\sigma(N\tau; \mathbf{u})$ and hence, by induction, also in the ring $\mathfrak{A}$. Let us show that reversely, the $B_\sigma(N\tau; \mathbf{z})$ lie in the ring of operators $B_\sigma((N+1)\tau; \mathbf{u})$. We note that the functions $J_{ij}^\sigma(\tau; \ldots)$ and $J_{ij}^\sigma(-\tau; \ldots)$ are not independent but are connected by certain relations (**). These relations can be obtained in the following way. We substitute $B_\sigma(\tau; \mathbf{z})$ in the form (10) in (11) and expand the result in the normal products of $\varphi_\sigma(T, \mathbf{u})$. Equating the coefficient functions of identical normal products of $\varphi_\sigma(T, \mathbf{u})$, we obtain (**). Again, we do not write down these relations because of their complexity.

It is easy to see that

$$B_\sigma(N\tau; \mathbf{z}) = \sum \int d\mathbf{u}\, d\mathbf{v}\, J_{ij}{}^\sigma(-\tau; \mathbf{z}; \mathbf{u}_1, \ldots; \mathbf{v}_1, \ldots) \times : B_+((N+1)\tau; \mathbf{u}_1)\ldots B_-((N+1)\tau; \mathbf{v}_1)\ldots : . \tag{13}$$

Indeed, let us substitute in (13) the quantities $B_\sigma((N+1)\tau; \mathbf{u})$ expressed through $B_\sigma(N\tau; \mathbf{u})$ with the help of (12), and expand the result in normal products of $B_\sigma(N\tau; \mathbf{u})$. Since the commutation relations for $B_\sigma(N\tau; \mathbf{u})$ are the same as for $\varphi_\sigma(T; \mathbf{u})$, the coefficient functions in this expansion will be the same as in the expansion of the right-hand side of (11) with respect to $\varphi_\sigma(T; \mathbf{u})$. Therefore the equation (13) becomes an identity in virtue of (**). Thus, $B_\sigma(N\tau; \mathbf{z})$ does indeed lie in the ring $B_\sigma((N+1)\tau; \mathbf{u})$, and hence this ring coincides with $\mathfrak{A}$.

Condition VIII is satisfied by construction.

Consider now condition IX. We have assumed that (9) is satisfied for $t_1 = T+\tau$, $t_2 = T$. It follows at once from this that the support of $J_{ij}^\sigma(\tau; \mathbf{z}; \ldots)$ lies within the region $|\mathbf{u}_l - \mathbf{z}| \le \tau$; $|\mathbf{v}_k - \mathbf{z}| \le \tau$; $l = 1, \ldots, i$; $k = 1, \ldots, j$. Let us now consider the commutator

$$[B_\sigma((N+1)\tau; \mathbf{z}), B_\rho(N'\tau; \mathbf{z}')] = \sum \int d\mathbf{u}\, d\mathbf{v}\, J_{ij}{}^\sigma(\tau; \mathbf{z}; \mathbf{u}_1, \ldots; \mathbf{v}_1, \ldots) \times [: B_+(N\tau; \mathbf{u}_1)\ldots B_-(N\tau; \mathbf{v}_1)\ldots : , \; B_\rho(N'\tau; \mathbf{z}')]. \tag{14}$$

Here $0 \le N' \le N$. By induction, the expression in the square brackets on the right-hand side of (14) vanishes when $|\mathbf{u}_l - \mathbf{z}| > (N - N')\tau$ or $|\mathbf{v}_k - \mathbf{z}| > (N - N')\tau$. From this and from the support properties of $J_{ij}^\sigma(\tau; \mathbf{z}, \ldots)$ we obtain at once that the right-hand part of (14) vanishes when $|\mathbf{z} - \mathbf{z}'| > (N + 1 - N')\tau$.

Thus, the transitivity of the axioms has to be proved for a discrete set of parameters t. The transitivity can also be shown for a continuum of t values, at least if axiom IX is taken of the form (9a). Let us define, for example, $B_\sigma(t, \mathbf{z})$ in the following manner:

$$B_+(N\tau+\xi;\ \mathbf{z}) = (2\pi)^{-3}\int d\mathbf{k}\, d\mathbf{u}\, e^{i\mathbf{k}(\mathbf{z}-\mathbf{u})}[\omega^{-1}\sin\omega\xi B_-(N\tau;\ \mathbf{u}) + \cos\omega\xi B_+(N\tau;\ \mathbf{u})], \tag{15}$$

$$B_-(N\tau+\xi;\ \mathbf{z}) = (2\pi)^{-3}\int d\mathbf{k}\, d\mathbf{u}\, e^{i\mathbf{k}(\mathbf{z}-\mathbf{u})}[\cos\omega\xi B_-(N\tau;\ \mathbf{u}) - \omega\sin\omega\xi B_+(N\tau;\ \mathbf{u})] \qquad (\omega = (\mathbf{k}^2+m^2)^{1/2}),$$

where $-\zeta \le \xi < \tau - \zeta$. Here $\tau > \zeta \ge 0$. Different ζ correspond to different ways of definition. The fulfillment of axioms III, IV, VI, and VIII for $B_\sigma(N\tau + \xi, \mathbf{z})$ is evident. The fulfillment of axiom V is proved by direct substitution of (15) in (1). Solving (15) for $B_\sigma(N\tau; \mathbf{u})$, we obtain the latter expressed through $B_\sigma(N\tau + \xi; \mathbf{z})$, i.e., the axiom VII is satisfied. Furthermore,

$$(2\pi)^{-3} \int d\mathbf{k}\, e^{i\mathbf{k}(\mathbf{z}-\mathbf{u})}\omega^{-1} \sin \omega\xi = 0 \quad \text{for} \quad |\mathbf{z}-\mathbf{u}| > \xi, \tag{16}$$

and analogously, when $\omega^{-1} \sin \omega\xi$ is replaced by $\cos \omega\xi$ and $\omega \sin \omega\xi$. Hence

$$[B_\sigma(N_1\tau + \xi_1; \mathbf{z}_1), B_\rho(N_2\tau + \xi_2; \mathbf{z}_2)] = 0 \tag{17}$$

for

$$|\mathbf{z}_1 - \mathbf{z}_2| > |N_1\tau - N_2\tau| + |\xi_1| + |\xi_2|,$$

or, setting $N_1\tau + \xi_1 = t_1$, $N_2\tau + \xi_2 = t_2$, we obtain

$$[B_\sigma(t_1; \mathbf{z}_1), B_\rho(t_2; \mathbf{z}_2)] = 0 \ \text{for} \ |\mathbf{z}_1 - \mathbf{z}_2| > |t_1 - t_2| + 2\tau, \tag{18}$$

i.e., axiom IX is satisfied for $\eta = 2\tau$.

We see that the theory is completely determined by the functions $J^\sigma_{ij}(\tau; \mathbf{z}, \ldots)$. The knowledge of these functions is equivalent to the knowledge of the Lagrangian in the sense that the functions $J^\sigma_{ij}(\tau; \ldots)$ determine completely the form of the field equations. The transitivity of the axioms has two important consequences. One is mathematical: instead of proving the existence of the required functions $J^\sigma_{ij}(t_1, t_2; \ldots)$ for all values of t_1 and t_2, it suffices to prove the existence of the functions $J^\sigma_{ij}(\tau; \ldots)$ for some fixed value τ of the difference of t_1 and t_2. The second consequence is physical. The fact that J^σ_{ij} depends only on the difference $t_1 - t_2$, reflects the uniformity of space-time in the time variable t, or more precisely, the periodicity with the small period τ.

The appearance of a nonvanishing τ implies actually the introduction of a small nonlocality in the theory. We see that within a time interval of order τ there may be a violation of the strict locality of the theory as well as a violation of the strict uniformity of space-time.

Let us now turn to the construction of specific functions J^σ_{ij} which satisfy all requirements enumerated in Sec. 1.

3. THE FREE FIELD

Let us assume that there exists an operator $U_0(\tau)$ such that

$$B_\sigma(\tau; \mathbf{z}) = U_0(\tau)\varphi_\sigma(T, \mathbf{z})U_0^{-1}(\tau), \quad U_0(\tau)|\Omega\rangle = c|\Omega\rangle. \tag{19}$$

Here $|\Omega\rangle$ is the vacuum vector, which was denoted by $|\Omega; T, n\rangle$ in[1]. It is then obvious that

$$B_+^-(\tau; \mathbf{z})|\Omega\rangle = 0. \tag{20}$$

Here B_+^- is given by (4). We show that in this case $B_\sigma(\tau; \mathbf{z})$ is expressed linearly through $\varphi_\sigma(T, \mathbf{z})$. Actually, it suffices to require (20); the formulas (19) will be obtained as a consequence. The assertion just formulated is a variant of the Haag theorem.[5,3]

Now let us prove it. Substituting in (20) the quantity $B_+^-(\tau; \mathbf{z})$ expressed through $B_\sigma(\tau; \mathbf{z})$ with the help of (4), with $B_\sigma(\tau, \mathbf{z})$ given by (10), we obtain

$$B_+^-(\tau; \mathbf{z})|\Omega\rangle = {}^1\!/_2 \sum (i)^k \int d\mathbf{u}\, d\mathbf{v}\, d\mathbf{v}'\, \Omega^+(\mathbf{v}_1 - \mathbf{v}_1')\ldots\Omega^+(\mathbf{v}_k - \mathbf{v}_k')$$
$$\times\Big\{J_{jk}^+(\tau; \mathbf{z}; \mathbf{u}_1, \ldots; \mathbf{v}_1, \ldots) + i\int d\mathbf{w}\, \Omega^-(\mathbf{z}-\mathbf{w}) J_{jk}^-(\tau; \mathbf{w}; \mathbf{u}_1, \ldots; \mathbf{v}_1, \ldots)\Big\}$$
$$\times\varphi_+^+(\mathbf{u}_1)\ldots\varphi_+^+(\mathbf{v}_1')\ldots|\Omega\rangle = 0. \tag{21}$$

We introduce the Fourier transform of J^σ_{ik}:

$$J^\sigma_{j-kk}(\tau; \mathbf{z}; \mathbf{u}_1, \ldots; \mathbf{v}_1, \ldots) = (2\pi)^{-3j}\int d\mathbf{p}\, d\mathbf{q}\, \mathcal{J}^\sigma_{j-kk}(\tau; \mathbf{q}_{k+1}, \ldots, \mathbf{q}_j; \mathbf{q}_1, \ldots, \mathbf{q}_k)$$
$$\times \exp\Big\{i\mathbf{p}\mathbf{z} + i\sum_{l=1}^{k} \mathbf{v}_l\mathbf{q}_l + \sum_{l=1}^{j-k} \mathbf{u}_l\mathbf{q}_{l+k}\Big\}\delta\Big(\mathbf{p} + \sum_{l=1}^{j}\mathbf{q}_l\Big). \tag{22}$$

Substituting (22) in (21), we obtain

$$\mathcal{J}_{j0}^-(\tau; \mathbf{q}_1, \ldots, \mathbf{q}_j) = i\omega\mathcal{J}_{j0}^+(\tau; \mathbf{q}_1, \ldots, \mathbf{q}_j)$$
$$+ \sum_{k=1}^{j} (i)^k P(\mathbf{q}_{k+1}, \ldots, \mathbf{q}_j | \mathbf{q}_1, \ldots, \mathbf{q}_k)$$
$$\times\omega_1, \ldots, \omega_k\{i\omega\mathcal{J}^+_{j-kk}(\mathbf{q}_{k+1}, \ldots, \mathbf{q}_{j-k}; \mathbf{q}_1, \ldots, \mathbf{q}_k) \tag{23}$$
$$- \mathcal{J}^-_{j-kk}(\mathbf{q}_{k+1}, \ldots, \mathbf{q}_{j-k}; \mathbf{q}_1, \ldots, \mathbf{q}_k)\},$$

where

$$\omega_l = (\mathbf{q}_l^2 + m^2)^{1/2}; \quad \omega = ((\mathbf{q}_1 + \ldots + \mathbf{q}_j)^2 + m^2)^{1/2};$$

$P(\mathbf{q}_{k+1}, \ldots)$ is the sum over all permutations of $\mathbf{q}_{k+1}, \ldots, \mathbf{q}_j$ with $\mathbf{q}_1, \ldots, \mathbf{q}_k$. Since J_{j-kk} is a generalized function of moderate growth with the support properties described in Sec. 2, the function $\tilde{J}_{j-kk}(\tau; \mathbf{q}_1, \ldots)$ will be an entire function.[6]

Let us analyze (23). We consider first the case $j \ne 1$. The different products of ω and ω_l on the right-hand side of (23) are not entire functions but have branch points. Since the coefficients of these products are entire functions and the singularities of the products themselves are at different points, a compensation of these singularities can occur only when all coefficient functions are equal to zero,

$$\mathcal{J}_{kl}{}^\sigma = 0 \ \text{for} \ k + l > 1; \tag{24}$$

i.e., the operators $B^\pm_\sigma(\tau; \mathbf{z})$ are expressed linearly through $\varphi^\pm_\sigma(T, \mathbf{z})$. Since, moreover, they satisfy the same commutation relations and there exists a vacuum for B^-_σ, there exists[7] a unitary operator $U_0(\tau)$ such that (19) is fulfilled.

Setting j = 1 in (23) and using (1), we obtain

$$\mathcal{J}_{01}{}^+ = \omega^{-1}(1 - (\mathcal{J}_{10}{}^+)^2)^{1/2},\ \mathcal{J}_{10}{}^- = -\omega(1 - (\mathcal{J}_{10}{}^+)^2)^{1/2},\ \mathcal{J}_{01}{}^- = \mathcal{J}_{10}{}^+. \tag{25}$$

The conditions (25) can be satisfied by setting

$$\mathcal{J}_{10}{}^+(\tau; q) = \cos\tau\omega, \tag{26}$$

so that all $\tilde{J}_{jk}$ are entire functions with the necessary properties. In this case the standard free field equations are obtained for B_σ:

$$B_{+0}(\tau; \mathbf{z}) = (2\pi)^{-3}\int d\mathbf{k}\, d\mathbf{u}\, e^{i\mathbf{k}(\mathbf{z}-\mathbf{u})}[\cos\tau\omega\varphi_+(T, \mathbf{u}) + \omega^{-1}\sin\omega\tau\varphi_-(T, \mathbf{u})].$$
$$B_{-0}(\tau; \mathbf{z}) = (2\pi)^{-3}\int d\mathbf{k}\, d\mathbf{u}\, e^{i\mathbf{k}(\mathbf{z}-\mathbf{u})}[-\omega\sin\omega\tau\varphi_+(T, \mathbf{u}) + \cos\omega\tau\varphi_-(T, \mathbf{u})]. \tag{27}$$

Here the operators B_σ are provided with the additional index 0 in order to indicate that they are obtained under the assumption that $|\Omega\rangle$ is the vacuum for B^-_σ. The operator $U_0(\tau)$ is constructed in an elementary fashion:

$$U_0(\tau) = :\exp\Big\{\int d\mathbf{u}\, d\mathbf{z}\, M(\tau; \mathbf{u}-\mathbf{z})\varphi_+^+(T, \mathbf{u})\varphi_+^-(T, \mathbf{z})\Big\}:, \tag{28}$$

where

$$M(\tau;\ \mathbf{u}) = (2\pi)^{-3}\int d\mathbf{k}\, 2\omega\, e^{i\mathbf{k}\mathbf{u}}(e^{i\omega\tau}-1). \tag{29}$$

It follows from this that if we want to introduce an interaction in a relativistic theory, one must drop the requirement that the vacuum for the interacting field coincides with the vacuum of the free field in terms of which a particular interpretation is made. We call attention to the fact that in the usual Lagrangian method, which despite all its shortcomings, does somehow describe interacting systems, these two vacua do not (because of the renormalization) in fact coincide.

4. INTERACTING FIELDS

The operators $B_{\sigma,0}(\tau;\ \mathbf{z})$ cannot describe interacting fields since they are linearly expressed through $\varphi_\sigma(T,\ \mathbf{z})$. Let us generalize (27), by introducing a nonlinearity:

$$\begin{gathered} B_+(\tau;\ \mathbf{z}) = (2\pi)^{-3}\int d\mathbf{k}\, d\mathbf{u}\, e^{i\mathbf{k}(\mathbf{z}-\mathbf{u})}[\cos\omega\tau_1\varphi_+(T,\ \mathbf{u}) + \omega^{-1}\sin\omega\tau_1\varphi_-(T,\ \mathbf{u})], \\ B_-(\tau,\ \mathbf{z}) = (2\pi)^{-3}\int d\mathbf{k}\, d\mathbf{u}\, e^{i\mathbf{k}(\mathbf{z}-\mathbf{u})}[-\omega\sin\omega\tau_1\varphi_+(T,\ \mathbf{u}) \\ + \cos\omega\tau_1\varphi_-(T,\mathbf{u})] + \sum_{j=0}^{s}\frac{1}{j!}\int d\mathbf{u}\, F_j(\tau_2;\mathbf{z},\mathbf{u}_1,\ldots,\mathbf{u}_j) \\ \times : B_+(\tau;\ \mathbf{u}_1),\ldots,B_+(\tau;\ \mathbf{u}_j):, \end{gathered} \tag{30}$$

Here $\tau = \tau_1 + \tau_2$, and $F_j(\tau_2;\ \mathbf{z},\ \mathbf{u}_1,\ \ldots,\ \mathbf{u}_j)$ satisfies the following conditions. As a function of $\mathbf{z},\ \mathbf{u}_1,\ \ldots,\ \mathbf{u}_j$, the function $F_j \in S'(R_{3(j+1)})$ and satisfies (8). Moreover, F_j is real, symmetric in $\mathbf{z},\ \mathbf{u}_1,\ \ldots,\ \mathbf{u}_j$, depends only on $|\mathbf{u}_1 - \mathbf{z}|,\ \ldots,\ |\mathbf{u}_j - \mathbf{z}|$, and vanishes when at least one of these differences is larger than τ_2.

By a direct proof we convince ourselves that $B_\sigma(\tau;\ \mathbf{z})$ satisfies all axioms of Sec. 1. According to Sec. 2, the operators $B_\sigma(N\tau;\ \mathbf{z})$, defined by the formulas

$$\begin{gathered} B_+((N+1)\tau;\ \mathbf{z}) = (2\pi)^{-3}\int d\mathbf{k}\, d\mathbf{u}\, e^{i\mathbf{k}(\mathbf{z}-\mathbf{u})}[\cos\omega\tau_1 B_+(N\tau,\ \mathbf{u}) \\ + \omega^{-1}\sin\omega\tau_1 B_-(N\tau,\ \mathbf{u})], \\ B_-((N+1)\tau;\ \mathbf{z}) = (2\pi)^{-3}\int d\mathbf{k}\, d\mathbf{u}\, e^{i\mathbf{k}(\mathbf{z}-\mathbf{u})}[-\omega\sin\omega\tau_1 B_+(N\tau,\ \mathbf{u}) \\ + \cos\omega\tau_1 B_-(N\tau;\mathbf{u})] + \sum_{j=0}^{s}\frac{1}{j!}\int d\mathbf{u}\, F_j(\tau_2;\mathbf{z};\mathbf{u}_1,\ldots): B_+((N+1)\tau,\mathbf{u}_1)\ldots:, \end{gathered} \tag{31}$$

will also satisfy all axioms. Formulas (31) define the operators $B_\sigma(t,\ \mathbf{z})$ for a discrete set of values of t; for continuous values of t the operators $B_\sigma(t,\ \mathbf{z})$ can be defined with the help of (15).

The recurrence relations (31) can be regarded as field equations. It is clear that they have a solution, and a unique one at that. Let us find it. We introduce the auxiliary operators $B_\sigma(N\tau,\ \mathbf{z};\ g)$ which satisfy formula (31a), which we do not write down because of its complexity. The equations (31a) are obtained from (31) by replacing the last term in the second equation by

$$\begin{gathered} \sum_{j=0}^{s}\frac{1}{j!}\int d\mathbf{x}\, d\mathbf{u}\, g(\mathbf{x})\,\Phi_j(\mathbf{z}-\mathbf{x},\mathbf{u}_1-\mathbf{x},\ldots,\mathbf{u}_j-\mathbf{x}) \\ \times : B_+((N+1)\tau,\ \mathbf{u}_1;\ g)\ldots B_+((N+1)\tau,\ \mathbf{u}_j;\ g):. \end{gathered} \tag{32}$$

Here $\Phi_j(\mathbf{u}_1,\ \ldots,\ \mathbf{u}_{j+1})$ satisfies (8), is symmetric in all arguments and depends only on their absolute values, is symmetric, and

$$\int d\mathbf{x}\,\Phi_j(\mathbf{u}_1-\mathbf{x},\ldots,\mathbf{u}_{j+1}-\mathbf{x}) = F_j(\tau_2;\ \mathbf{u}_1,\ldots,\mathbf{u}_{j+1}). \tag{33}$$

The function $g(\mathbf{x}) \in S(R_3)$ is also real.

It is clear that the equations (31a) with the initial condition (2) also have a unique solution. For $g(\mathbf{x}) \to 1$ in the sense of convergence in $S'(R_3)$ [the function $g(\mathbf{x})$ can be regarded as belonging to $S'(R_3)$] the quantities $B_\sigma(N\tau;\ \mathbf{z};\ g) \to B_\sigma(N\tau;\ \mathbf{z})$ in the sense of strong operator convergence. This is easily seen by induction, regarding (31a) as a recurrence formula.

We introduce the operator

$$\begin{gathered} V(g) = \exp\Big\{-i\sum_{j=0}^{s}\frac{1}{(j+1)!}\int d\mathbf{x}\, d\mathbf{u}\, g(\mathbf{x})\,\Phi(\mathbf{u}_1-\mathbf{x},\ldots,\mathbf{u}_{j+1}-\mathbf{x}) \\ \times : \varphi_+(T,\ \mathbf{u}_1)\ldots\varphi_+(T,\ \mathbf{u}_{j+1}):\Big\} \end{gathered} \tag{34}$$

and the operator

$$W(N, g) = [U_0(\tau_1)V(g)]^N. \tag{35}$$

We show that the operators

$$B_\sigma(N\tau;\mathbf{z};g) = W(N;g)\varphi_\sigma(T,\mathbf{z})W^+(N,g) \tag{36}$$

satisfy (31a). We do this by induction. Since V(g) commutes with $\varphi_+(T,\ \mathbf{z})$,

$$\begin{gathered} W(1;\ g)\varphi_+(T,\ \mathbf{z})W^+(1;\ g) = (2\pi)^{-3}\int d\mathbf{k}\, d\mathbf{u}\, e^{i\mathbf{k}(\mathbf{z}-\mathbf{u})} \\ \times[\cos\omega\tau_1\varphi_+(T,\ \mathbf{u}) + \omega^{-1}\sin\omega\tau_1\varphi_-(T,\ \mathbf{u})]. \end{gathered} \tag{37}$$

Multiplying (37) from the left by W(N; g) and from the right by $W^+(N;\ g)$, we obtain the first formula (31a). Furthermore,

$$\begin{gathered} [\varphi_-(T,\ \mathbf{z}),\ V^+(g)] = V^+(g)\sum\frac{1}{j!}\int d\mathbf{x}\, d\mathbf{u}\,\Phi_j(\mathbf{z}-\mathbf{x},\ \mathbf{u}_1-\mathbf{x},\ldots\ \mathbf{u}_j-\mathbf{x})\cdot \\ \cdot : \varphi_+(T,\ \mathbf{u}_1)\ldots\varphi_+(T,\ \mathbf{u}_j): g(\mathbf{x}). \end{gathered} \tag{38}$$

Thus

$$\begin{gathered} W(1;\ g)\varphi_-(T,\ \mathbf{u})W^+(1;\ g) = (2\pi)^{-3}\int d\mathbf{k}\, d\mathbf{u}\, e^{i\mathbf{k}(\mathbf{z}-\mathbf{u})}[\cos\omega\tau_1 \\ \times\varphi_-(T,\ \mathbf{u}) - \omega\sin\omega\tau_1\varphi_+(T,\ \mathbf{u})] + \sum\frac{1}{j!}\int d\mathbf{x}\, d\mathbf{u} \\ \times\Phi_j(\mathbf{z}-\mathbf{x},\ \mathbf{u}_1-\mathbf{x},\ldots,\mathbf{u}_j-\mathbf{x})g(\mathbf{x})U_0(\tau_1):\varphi_+(T,\ \mathbf{u}_1)\ldots\varphi_+(T,\mathbf{u}_j):U_0^+(\tau_1). \end{gathered} \tag{39}$$

Multiplying now (39) from the left by W(N; g) and from the right by $W^+(N,\ g)$, we obtain the second formula (31a). Thus the solution of the equations (31) is given by the operators

$$B_\sigma(N\tau;\ \mathbf{z}) = \lim_{g\to 1}[U_0(\tau_1)V(g)]^N\varphi_\sigma(T,\ \mathbf{z})[V^+(g)U_0^+(\tau_1)]^N. \tag{40}$$

Together with (15) this formula gives an expression for the Heisenberg field at an arbitrary instant t.

5. CONCLUSION

We have shown that one can construct a nontrivial, i.e., interacting model of relativistic field theory within the framework of the axiomatic method (in a variant of[1]). We emphasize that up to now only a "field theory" has been constructed, not a "theory for the scattering of particles." The point is that we have obtained an expression for the Heisenberg field for an arbitrary instant t. Knowing that, we can calculate the value of the field in a given state at any instant t. This is precisely the problem of "field theory." Thus we may conclude that the axioms proposed do not contradict the introduction of an interaction in the theory. However, for an actual comparison of the results of the theory with experiment we must still learn how to interpret the operators of the interacting field $B_\sigma(t, z)$ in terms of observable quantities. Actually, in order to turn a "field theory" into a "theory for the scattering of particles" one must give a particular interpretation of the state vectors. That is, one must construct vectors corresponding to the vacuum and one-, two-, etc.,-particle states of the ingoing (in) and outgoing (out) particles. It turns out that these states can indeed be constructed, so that one can establish not only a nontrivial "field theory," but also a nontrivial "theory for the scattering of particles;" one can in fact construct an S matrix different from unity.

We emphasize that in the construction of the operators we have dealt only with mathematically well-defined quantities, so that no divergencies could occur in our treatment.

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