BREMSSTRAHLUNG EFFECT ON RELATIVISTIC ELECTRONS IN A STRONG RADIATION FIELD

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Stimulated bremsstrahlung due to scattering of relativistic electrons by ions in the presence of a monochromatic electromagnetic wave is studied. The case of strong laser radiation is considered. Under certain assumptions, expressions are obtained for the integral cross sections for stimulated emission of several photons. The asymptotic case of a very strong field is considered. Expressions are obtained for the differential cross section of the process integrated over the directions of propagation of the scattered electrons.

1. A rather large number of recent papers (for example,[1-4]) are devoted to the study of the interaction between free electrons and a strong radiation field. The urgency of problems of this type is connected with the rapid progress in the development of methods of generating powerful coherent monochromatic radiation. Modern lasers produce fields corresponding to an energy density $I \gtrsim 10^7$ MW/cm^2 (after focusing the laser beam). In many cases the interaction between charged particles and a field of such high intensity cannot be described with the aid of the customarily employed perturbation theory, and requires a more exact analysis. Stimulated bremsstrahlung produced upon scattering of slow electrons by a Coulomb potential was considered by Bunkin and Fedorov[4]. In the present paper this problem is considered without being limited to the case of small electron energies and without any assumptions concerning the smallness of the field.

The possibility of considering a stimulated bremsstrahlung effect which is not confined to the framework of perturbation theory is determined by the circumstance that in the semiclassical approach (classical description of the field and quantum-mechanical description of the electron) there are known exact wave functions of the electron in the field of a plane electromagnetic wave. These functions are exact solutions of the Dirac equation. They were obtained by Volkov[5] and are of the form

$$\psi_{pr}(x)=\left[1+e\frac{\hat{k}\hat{A}}{2(kp)}\right]u_r(p)\exp\left[i\int ds\left(\frac{e(pA)}{(kp)}-\frac{e^2A^2}{2(kp)}\right)+i(px)\right], \quad (1)$$

where p_μ is a 4-vector whose components are the quantum numbers characterizing the state of the electron and coincide with the components of the electron 4-momentum in the absence of a field, $u_r(p)$ is an ordinary bispinor satisfying the Dirac equation, A_μ is the vector potential of the field, k_μ is its 4-momentum, and $s = (kx)$. In the case of an elliptically polarized wave, $A_4 = 0$, and $\mathbf{A} = \mathbf{a}_1 \cos s + \mathbf{a}_2 \sin s$, where the vectors $\mathbf{a}_1$ and $\mathbf{a}_2$ are orthogonal to each other and also to the vector $\mathbf{k}$. Unless the choice of the coordinate frame is specially stipulated, we shall henceforth assume that the z axis is directed along the vector $\mathbf{k}$, the x axis along the vector $\mathbf{a}_1$, and the y axis along the vector $\mathbf{a}_2$.

The probability of the transition of the electron between the states defined by the functions ψ_{pr}, in scattering by a Coulomb potential $Ze^2/|\mathbf{x}|$, can be obtained in the first Born approximation. The probability or the cross section of the electron scattering process can, as usual[2,3], be represented by a sum of the terms $d\sigma = \Sigma_n d\sigma^{(n)}$, which are characterized by changes of the zeroth component of the "quasimomentum" by an amount $n\omega$:

$$P_\mu = p_\mu - \frac{e^2(a_1^2+a_2^2)}{4(kp)}k_\mu,$$

where $\omega = |\mathbf{k}|$ is the frequency of the electromagnetic field, n are positive and negative integers, and

$$P_\mu^2 = -m_*^2 = -(m^2 + {}^1/_2 e^2(a_1^2+a_2^2)).$$

Using the usual computation procedures[6] we obtain, after summing and averaging over the polarizations of the electron in the final and in the initial state, the following expression for the cross section $d\sigma^{(n)}$:

$$d\sigma^{(n)} = \frac{Z^2e^4}{2v\mathscr{E}\mathscr{E}'}\frac{M}{q^4}d\mathbf{p}'\delta(\mathscr{E}'-\mathscr{E}+n\omega), \quad (2)$$

where

$$\mathscr{E} = P_0, \quad \mathbf{q} = \mathbf{P}'-\mathbf{P}+n\mathbf{k},$$

and the square of the matrix element M is given by

$$\begin{aligned}{}^1/_4 M = {} & |A|^2(m^2+\varepsilon\varepsilon'+\mathbf{pp}') + 2\,\mathrm{Re}\,\{A^*\mathbf{B}(\varepsilon\mathbf{p}'+\varepsilon'\mathbf{p})\} \\ & -|\mathbf{B}|^2(m^2-\varepsilon\varepsilon'+\mathbf{pp}') + 2\,\mathrm{Re}\,\{(\mathbf{Bp})(\mathbf{B}^*\mathbf{p}')\} - 2\,\mathrm{Re}\,\{A^*\mathbf{C}\cdot \\ & \times(p_z\mathbf{p}'-p'_z\mathbf{p})\} - 2\,\mathrm{Re}\,\{B_z^*\mathbf{C}(\varepsilon\mathbf{p}'-\varepsilon'\mathbf{p})\} \\ & + 2\,\mathrm{Re}\,\{(\mathbf{B}^*\mathbf{C})\}(\varepsilon p'_z-\varepsilon' p_z) + |\mathbf{C}|^2(\varepsilon\varepsilon'+m^2 \\ & + p_\perp p'_\perp - p_z p'_z) - 2\,\mathrm{Re}\,\{(\mathbf{Cp})(\mathbf{C}^*\mathbf{p}')\}. \end{aligned} \quad (3)$$

We have used here the notation

$$A = F_0\left(1+\frac{e^2(a_1^2+a_2^2)\omega^2}{4(kp)(kp')}\right) + F_3\frac{e^2(a_1^2-a_2^2)\omega^2}{2(kp)(kp')}, \quad (4)$$

$$B_{x,y} = -\omega F_{1,2}\frac{ea_{1,2}}{2}\left(\frac{1}{kp}+\frac{1}{kp'}\right),$$

$$B_z = -F_0\frac{e^2(a_1^2+a_2^2)\omega^2}{4(kp)(kp')} - F_3\frac{e^2(a_1^2-a_2^2)\omega^2}{2(kp)(kp')}, \quad (5)$$

$$C_{x,y} = -\omega F_{1,2}\frac{ea_{1,2}}{2}\left(\frac{1}{kp}-\frac{1}{kp'}\right), \quad C_z = 0. \quad (6)$$

Finally, the functions F_0, F_1, F_2, and F_3 are determined by the expressions

$$\{F_0, F_1, F_2, F_3\} = \frac{1}{2\pi}\int_0^{2\pi} ds\, e^{-ins+if(s)}\left\{1, \cos s, \sin s, \frac{1}{2}\cos 2s\right\} \quad (7)$$

where

$$f(s) = -e\left(\frac{\mathbf{a}_2\mathbf{p}}{kp} - \frac{\mathbf{a}_2\mathbf{p}'}{kp'}\right)\cos s + e\left(\frac{\mathbf{a}_1\mathbf{p}}{kp} - \frac{\mathbf{a}_1\mathbf{p}'}{kp'}\right)\sin s + \frac{e^2(a_1^2 - a_2^2)}{8}\left(\frac{1}{kp} - \frac{1}{kp'}\right)\sin 2s. \quad (8)$$

In the foregoing formulas, the vectors **P** and **P**′ determine the quasimomenta of the electron in the initial and final states, $p_\perp$ and $p'_\perp$ are their projections on the (x, y) plane, $\epsilon = p_0$, and $\epsilon' = p'_0$.

With the aid of a number of transformations, the expression (3) can be reduced to the same form as used by Yakovlev[3] to write down the matrix element of the photoproduction of a pair. (The corresponding matrix elements are obtained from one another with the aid of the substitution $p_\mu \to -p_\mu$.)

The formulas simplify somewhat in the particular case when the field has circular polarization ($a_1 = a_2$). With this, in accord with (7) and (8), the functions $F_{0,1,2}$ are expressed in terms of Bessel functions of order $|n|$ and their derivatives, and the function F_3 drops out from the expression for the matrix element (for details see[3]).

The differential cross sections $d\sigma^{(n)}$ (Eq. (2)) can be interpreted as determining the probability of electron scattering in a definite direction, with a change of energy by a definite amount. The quantum-mechanical mean value of the electron energy in the state (1) is determined by the expression

$$E = \left\langle \psi_{pr}{}^+ i\frac{\partial}{\partial t}\psi_{pr}\right\rangle = \mathscr{E} + \frac{\omega^2 e^2}{2\mathscr{E}(kp)^2}\left[(p a_1)^2 + (p a_2)^2 + \frac{e^2(a_1^2 - a_2^2)^2}{32}\right] - \frac{\omega^3 \varepsilon e^2 a_1 a_2}{2\mathscr{E}(kp)^2}\langle u_{pr}{}^+(1-\alpha_z)\Sigma_z | u_{pr}\rangle, \quad (9)$$

where α_Z and Σ_Z are ordinary spin matrices. The average energy E coincides neither with the quasienergy $\mathscr{E}$ nor with the free-electron energy ϵ, and is connected with them by a very complicated relation. We can use as a simpler energy characteristic of the electron the quantity ϵ, assuming that the field decreases very slowly as $|\mathbf{x}| \to \infty$. The quantities ϵ and **p** determine in this case the energy and the momentum of the electron at infinity. At a fixed value of the quasienergy and a fixed direction of electron motion at infinity, the electron energy ϵ is given by

$$4(\varepsilon - \sqrt{\varepsilon^2 + m^2}\cos\theta_0)(\mathscr{E} - \varepsilon) = e^2(a_1^2 + a_2^2), \quad (10)$$

where $\cos\theta_0 = p_Z/|\mathbf{p}|$.

As shown by Yakovlev[3], the number of real solutions of this equation can differ and depends on the values of the parameters contained in it. Both the average energy E and the "energy at infinity" ϵ differ from the quasienergy $\mathscr{E}$ and depend, for a specified value of the quasienergy, on the direction of propagation of the electrons. Therefore, when an electron is scattered by a Coulomb potential and the quasienergy is changed by an amount $n\omega$, its energy can be different and depends on the direction of motion of the scattering electron.

Thus, the energy spectrum of the electrons scattered in each given direction is always discrete. But if we are interested in the integral effect summed over the propagation directions of the scattered electrons, then at each value of the quasienergy (i.e., for each n) the electron spectrum comprises a certain band. With increasing field, the bands corresponding to different n can overlap, and consequently the energy spectrum of the scattered electrons becomes continuous. The condition for the width of the energy bands to be smaller than the distance between them is

$$\left(\frac{ea}{m}\right)^2\frac{p'}{2\omega} < 1. \quad (11)$$

If this condition is satisfied, then the energy of the scattered electrons is almost isotropic, their energy spectrum differs little from a discrete spectrum, and the integral cross sections $\sigma^{(n)}$ determine the scattered-electron energy distribution function. Since the condition (11) is in practice always more stringent in the optical frequency region then the condition $ea \ll m$, under which the difference between the energy ϵ and the quasienergy $\mathscr{E}$ is small, the cross sections $\sigma^{(n)}$ are in this case the cross sections for emission (when $n > 0$) or absorption (when $n < 0$) of $|n|$ quanta.

In the case when $(ea/m)^2 p'/2\omega \geq 1$, the interpretation of the differential cross sections (2), as well as of the cross sections integrated over the azimuthal angle φ of the momentum **p**′, remains the same. Together with (10) these cross sections determine the distribution function with respect to the energies of the electrons scattered in a given direction or at a specified angle to the z axis. But the meaning of the integral quantities changes in this case. For each ϵ' and n the conservation law in (2) determines the electron motion direction, i.e., $\cos\theta = p'_Z/|\mathbf{p}|$,

$$\cos\theta_n = \frac{1}{p'}\left(\varepsilon' - \frac{e^2(a_1^2 + a_2^2)}{4(\mathscr{E} - n\omega - \varepsilon')}\right). \quad (12)$$

The distribution with respect to the energies of the electrons scattered in all the directions is determined by the differential cross section

$$d\sigma = \frac{Z^2 e^4 (a_1^2 + a_2^2)}{8v\mathscr{E}}\sum_n \frac{1}{(\mathscr{E} - n\omega)(\mathscr{E} - n\omega - \varepsilon')^2}\int_0^{2\pi} d\varphi \frac{M}{q^4}\bigg|_{\cos\theta = \cos\theta_n} \varepsilon' d\varepsilon'. \quad (13)$$

The sum extends over those values of n, which are determined by the inequalities

$$\mathscr{E} - \varepsilon' - \frac{e^2(a_1^2 + a_2^2)}{4(\varepsilon' + p')} \geqslant n\omega \geqslant \mathscr{E} - \varepsilon' - \frac{e^2(a_1^2 + a_2^2)}{4(\varepsilon' - p')}. \quad (14)$$

2. The calculation of the integral cross sections $\sigma^{(n)}$ is meaningful in the case when condition (11) is satisfied. If at the same time the frequency of the field ω is much lower than the kinetic energy of the electron, $\omega \ll p^2/\epsilon$, which corresponds to the cases realized in practice, then the electron can both absorb and emit energy. The total effect is determined by the total cross section for the emission of n photons, $\sigma_T^{(n)} = \sigma_e^{(n)} - \sigma_a^{(n)}$ where $\sigma_e^{(n)} = \sigma^{(n)}$ and $\sigma_a^{(n)} = \sigma^{(-n)}$ are the cross sections for the emission and absorption of n quanta; $n > 0$.

Integration of the cross sections (2) over the direction of motions of scattered electron can be performed analytically only in certain particular cases under definite assumptions concerning the magnitude of the field and the initial state of the electron. From the point of view of the role of the nonlinear effects we can distinguish between three regions of field values: $ea \ll \omega$, $\omega \ll ea \ll m$, and $ea \gg m$.

The region $ea \ll \omega$ is the weak-field region. In this case the functions F_i (Eq. (7)) can be approximated by the first terms of the expansions in powers of the field. The corresponding results are those obtained by ordinary perturbation theory. The small parameter which determines the field and with respect to which the expansion is carried out is $(ea/\omega)p/m$. On going over to relativistic energies, the critical field at which perturbation theory ceases to be valid decreases compared with the case of slow electrons[4]. The stimulated bremsstrahlung of one photon in a weak field was investigated earlier[7]. The multiquantum-process cross sections $\sigma_T^{(n)}$, as well as the corrections to the cross section $\sigma_T^{(1)}$, are in this case very small

$$\sigma_T^{(n)} \sim \sigma_T^{(1)}\left(\frac{ea}{\omega}\frac{p}{m}\right)^{2n}.$$

The case $\omega \ll ea \ll m$ is of greatest interest, since these are precisely the conditions corresponding to fields that can be obtained with the aid of modern lasers. The inequalities presented for the amplitude of the vector potential correspond to the following limitations on the energy flux density I:

$$5\cdot 10^7\ \mathrm{W/cm^2} \ll I \ll 10^{18}\ \mathrm{W/cm^2}$$

Condition (11) can be satisfied simultaneously with the requirement $ea \gg \omega$, provided the electron energy is such that $p\omega \ll m^2$. Owing to the smallness of the frequency ($\omega \ll m$) in the optical region, this inequality is satisfied for practically all electron energies obtainable in laboratory conditions.

Let us consider in greater detail the case of a field of linear polarization, $a_1 = a$ and $a_2 = 0$, assuming the conditions $\omega \ll ea \ll m$ and $(ea/m)^2 p/\omega < 1$ are satisfied. Under these assumptions, the quasimomenta of the electron P_μ and P'_μ differ little from its momenta p_μ and p'_μ. The matrix element M is determined by the functions F_i, which reduce in our case to an expression of the type

$$F_0 = \frac{1}{2\pi}\int_0^{2\pi} ds \exp\{-ins + i(\alpha\sin s + \beta\sin 2s)\},$$

$$\alpha = ea\left(\frac{p_x}{kp} - \frac{p_x'}{kp'}\right), \quad \beta = \frac{e^2a^2}{8}\left(\frac{1}{kp} - \frac{1}{kp'}\right). \quad (15)$$

By virtue of the assumptions made concerning the magnitude of the field, the parameters α and β satisfy the conditions $|\alpha| \gg 1$ and $|\alpha| \gg |\beta|$ everywhere except in a narrow region of directions of the vector p′. This makes it possible to use the saddle-point method to calculate the functions F_0. It also shows that the asymptotic form of the functions (15) differs little in this case from the asymptotic form of the Bessel functions $J_n(\alpha)$. The additional saddle points are located far from the real axis and make an exponentially small contribution $\sim \exp(-\alpha^2/|\beta|)$. The principal saddle points experience a small shift. This leads to small corrections to the asymptotic expression for the functions F_0 compared with the Bessel functions, and also to a change in their oscillatory dependence. The inequality $|\alpha| \gg |\beta|$ can be violated only in a very narrow region of angles, when $|\alpha| \ll 1$.

It is easy to see that in the nonrelativistic case ($p \ll m$, $p' \ll m$) the expressions (2) and (3) lead to previously obtained results[4] the stimulated bremsstrahlung effect on slow electrons.

By virtue of the smallness of the parameter ea/m in the region $|\alpha| \gg 1$, expressed (3) for the matrix element M can be reduced to the following simpler form

$$M_n = \frac{8}{\pi}(m^2 + \varepsilon\varepsilon' + p_zp_z')|\alpha|^{-1}\cos^2\left(\alpha - n\frac{\pi}{2} - \frac{\pi}{4} - \frac{4\beta^2}{\alpha}\right). \quad (16)$$

In the region $1 \gg |\alpha|$ when $|n| = 1$ we can obtain for the matrix element M the expression

$$M_{\pm 1} = \alpha^2(m^2 + \varepsilon\varepsilon' + \mathbf{pp}') \pm \alpha\frac{ae}{2}\left[p_x\left(1 + \frac{\varepsilon' + p_z'}{\varepsilon - p_z}\right) + p_x'\left(1 + \frac{\varepsilon + p_z}{\varepsilon' - p_z'}\right)\right] \quad (17)$$
$$+ \frac{e^2a^2}{8}\left[\frac{\varepsilon' + p_z'}{\varepsilon - p_z} + \frac{\varepsilon + p_z}{\varepsilon' - p_z'} - \frac{2}{(\varepsilon - p_z)(\varepsilon' - p_z')}(m^2 + p_yp_y' - p_xp_x')\right].$$

In the case $|n| > 1$ in the region $|\alpha| \ll 1$, obviously $M_n \sim (ea)^{2|n|}$, but the corresponding formulas become more complicated.

The contribution to the integral cross sections $\sigma^{(n)}$ from the region with large values of the argument $|\alpha| > 1$ can be estimated by using the representation (16) for the matrix element, averaging the integrand over the rapid oscillations, and going over in the integration with respect to the angles to a coordinate system (1, 2, 3) in which the axis 2 coincides with the y axis, and the axis 3 lies in the (xz) plane and makes an angle $\cos[1 + (\epsilon - p_z)^2/p_x^2]^{-1/2}$ with the z axis. In the general case this leads to very complicated formulas. The corresponding contribution to the cross sections $\sigma^{(n)}$ is as a rule $\sim(ea/\omega)^2$. An exception is the case when the value $p_x/(\epsilon - p_z)$ is close to its maximal value p/m

$$\frac{p^2}{m^2} - \frac{p_x^2}{(\varepsilon - p_z)^2} \ll \frac{\omega}{ea}\frac{p}{m}. \quad (18)$$

In this case the contribution to the cross sections $\sigma^{(n)}$ from the regions of large values of the argument $|\alpha| \gg 1$ will be $\sim ea/\omega$.

The contribution to the cross sections $\sigma^{(n)}$ from the region $|\alpha| \ll 1$ will be $\sim(ea/m)^{2|n|}$. In the general case it is therefore necessary to take into account in the calculation of the cross sections $\sigma^{(\pm 1)}$ all the regions, and the cross sections $\sigma^{(n)}$ with $|n| > 1$ are determined essentially by the region $|\alpha| \gg 1$. The total cross section $\sigma_T = \Sigma n\sigma_T^{(n)}$ is determined by a large number of terms, and processes with absorption (emission) of several quanta are important. On the other hand, if condition (18) is satisfied, the cross sections of lower order are determined by the region $|\alpha| \ll 1$. Calculation of the cross sections $\sigma_{e,a}^{(1)}$ with the aid of expression (17) yields in this case

$$\sigma_{e,a}^{(1)} \cong -\frac{\pi Z^2e^4}{4}\frac{(ea)^2\varepsilon^2}{\omega^2p^2m^6}\{(\varepsilon^2 + m^2)^2 \mp 2\varepsilon\omega(\varepsilon^2 + m^2)\}, \quad (19)$$

$$\sigma_T^{(1)} \cong -\frac{\pi Z^2e^4(ea)^2\varepsilon^2(\varepsilon^2 + m^2)}{\omega p^2m^6}. \quad (20)$$

The higher-order cross sections are small in this case, $\sigma_T^{(n)} \sim (ea/m)^{(2n-1)}\sigma_T^{(1)}$. At sufficiently large n, when $(ea/m)^{2(n-1)} \lesssim (\omega/ea)(m/\omega)^3$, the cross sections are determined essentially by the region of large values of the argument and their order of magnitude is

$$\sigma_T^{(n)} \cong Z^2e^4(ea)/m^3. \quad (21)$$

Thus, in the considered case when $\omega \ll ea \ll m$ and $ea \ll m(2\omega/p)^{1/2}$, deviations from perturbation theory occur essentially in the cross sections for the emission and absorption of several photons. In the case when condition (18) is satisfied, the higher-order cross sections remain small as before, but the smallness parameter is changed. The character of the field dependence of the total cross section σ_T remains unchanged, $\sigma_T \sim (ea)^2$.

3. If inequality (11) is not satisfied, then it is meaningful to consider the stimulated bremsstrahlung effect from the following two points of view. First, it is of interest to determine the cross sections $d\sigma_T^{(n)}$ and $d\sigma_e^{(n)} - d\sigma_a^{(n)}$, integrated over the azimuthal angle φ, which defines the vector $\mathbf{p}'$ at a specified value of the angle θ. Second, the question arises of calculating the differential cross section (13), which determines the energy distribution of the scattered electrons regardless of the propagation direction. We shall henceforth bear in mind only the second problem and confine ourselves to the case of a very strong field $ea \gg \epsilon$, $ea \gg \epsilon'$.

In this case, for practically all scattered-electron energies, the number of terms in the sum (13) is quite large. This makes it possible to go over from summation to integration, and leads, with allowance for (12), to the following expression for the differential cross section:

$$d\sigma = \frac{Z^2e^4}{2v\mathscr{E}}\frac{p'\varepsilon' d\varepsilon'}{\omega}\int do\frac{M}{q^4\mathscr{E}'}, \tag{22}$$

where do is the element of solid angle in the direction of the vector $\mathbf{p}'$. The possibility of going over to integration in the sum over n presupposes also that the dependence of all the functions on n becomes sufficiently slow. This makes it necessary to exclude from consideration in such an approach a certain region of scattered-electron energies near the value $\epsilon' = \epsilon$, since the quantity q^{-4} can become infinite when $\epsilon' = \epsilon$. The corresponding limitation can be written in the form

$$(\varepsilon' - \varepsilon)^2 \gg 8\omega p^3/(ea)^2. \tag{23}$$

With increasing field, the energy region excluded in this manner becomes very narrow.

The matrix elements M in the strong-field approximation in the form

$$M = \frac{(ea)^4}{8(\varepsilon - p_z)(\varepsilon' - p_z')}|F_0 + 2F_3|^2. \tag{24}$$

Calculation of the functions F_i (15) becomes much simpler if the parameter β does not vanish. Such a situation occurs at sufficiently low energies of the scattered electron

$$\varepsilon' < \frac{(\varepsilon - p_z)^2 + m^2}{2(\varepsilon - p_z)}. \tag{25}$$

In this case the parameters α and β are such that $|\beta| \gg |\alpha|$ and $|\beta| \gg 1$. This makes it possible to use the saddle-point method. All the saddle points lie on the real axis. For the functions F_0, the following asymptotic representation hold:

$$F_0 \cong \frac{1}{(2\pi|\beta|)^{1/2}}\left\{\cos\left[\beta + \frac{\alpha}{\sqrt{2}} - n\frac{\pi}{4} - \frac{\pi}{4}\operatorname{sgn}\beta\right] + \cos\left[\beta - \frac{\alpha}{\sqrt{2}} - n\frac{3\pi}{4} - \frac{\pi}{4}\operatorname{sgn}\beta\right]\right\}. \tag{26}$$

The integration in (22) can be performed if the rapidly oscillating terms in the integrand are neglected. This makes it possible to write for the cross section

$$d\sigma = \frac{3\cdot 2^5 Z^2 e^4}{ve^2a^2}p'\varepsilon' d\varepsilon'(\varepsilon - p_z) \times \int_{-1}^{1} dx \frac{(\varepsilon' - p_z')[(\varepsilon' - p_z' - \varepsilon + p_z)^2 + p_\perp^2 + p'^2 - p_z'^2]}{|\varepsilon' - p_z' - \varepsilon + p_z|} \times \frac{1}{\{[(\varepsilon' - p_z' - \varepsilon + p_z)^2 + p_\perp^2 + p'^2 - p_z'^2]^2 - 4p_\perp^2(p'^2 - p_z'^2)\}^{3/2}}, \tag{27}$$

where $p_z' = (p'x)$.

This integral can obviously be calculated exactly, but in the general case the formulas are quite cumbersome. We confine ourselves therefore to the case when the direction of initial electron momentum is sufficiently close to the z axis:

$$p_\perp^2 \ll (\varepsilon' - \varepsilon + p_z)^2 + p'^2.$$

As a result of integration of (27) we get

$$d\sigma = \frac{3\cdot 2^5 Z^2 e^4}{v(ea)^2}\frac{p'\varepsilon' d\varepsilon'(\varepsilon - p_z)}{[(\varepsilon' - \varepsilon + p_z)^2 - p'^2]^2}\left[2\frac{|m^2 - (\varepsilon - p_z)^2|}{(\varepsilon' - \varepsilon + p_z)^2 - p'^2} + \frac{\varepsilon - p_z}{p'}\ln\frac{(|\varepsilon' - \varepsilon + p_z| - p')^2}{(\varepsilon' - \varepsilon + p_z)^2 - p'^2}\right]. \tag{28}$$

If the energy of the scattered electrons does not satisfy the inequality (25), then the parameter β can vanish. In this case it is necessary to estimate the contribution made to the integral (22) by different regions. The contribution of the region $|\beta| > |\alpha|$ can be estimated by using the asymptotic representation (26). In estimating the integral (27) it is necessary to take into account the fact that on approaching the point $\beta = 0$ the integrand increases rapidly, making it possible to retain in the results only the logarithmically large terms. The region $|\beta| < |\alpha|$ breaks up under the condition

$$\frac{\omega}{ea}(\varepsilon - p_z) \ll \sqrt{p'^2 - (\varepsilon - p_z - \varepsilon')^2}$$

into two subregions defined by the inequalities $|\alpha| > 1$ and $|\alpha| < 1$. The contribution of the first subregion may not be small if the condition

$$\frac{\omega}{ea}(\varepsilon - p_z) \ll \sqrt{p'^2 - p_x^2 - (\varepsilon' - \varepsilon + p_z)^2}$$

is satisfied. If in addition we assume satisfaction of the inequality

$$\frac{\omega}{ea}(\varepsilon - p_z) \ll |\sqrt{p'^2 - (\varepsilon' - \varepsilon + p_z)^2 - p_x^2} - p_y|,$$

which makes it possible to regard the function q^{-4} as sufficiently smooth, then the contribution of the region $|\alpha| > 1$, $|\alpha| > |\beta|$ can be estimated by using for the functions F_i the asymptotic form of the Bessel functions $J_n(\alpha)$ for large values of the argument and retaining the logarithmically large quantities. Taking all the foregoing into account, the cross section $d\sigma$ (22) can be represented in the form

$$d\sigma = \frac{3\cdot 2^4 Z^2 e^4 (\varepsilon - p_z)^2 \varepsilon' d\varepsilon'}{v(ea)^2}\left\{\frac{6[p_\perp^2 - (\varepsilon' - \varepsilon + p_z)^2 + p'^2]}{[p_\perp^2 + (\varepsilon' - \varepsilon + p_z)^2 - p'^2]^3} \times \ln\frac{eap'}{(\varepsilon - p_z)\sqrt{p'^2 - (\varepsilon' - \varepsilon + p_z)^2}} + \left(\frac{p'^2 - (\varepsilon' - \varepsilon + p_z)^2}{p'^2 - (\varepsilon' - \varepsilon + p_z)^2 - p_x^2}\right)^{1/2} \times (\sqrt{p'^2 - (\varepsilon' - \varepsilon + p_z)^2 - p_x^2} - p_y)^{-4}\ln\frac{ea\sqrt{p'^2 - (\varepsilon' - \varepsilon + p_z)^2}}{\omega(\varepsilon - p_z)}\right. \tag{29}$$

The conditions presented above, under which expression (29) is valid, are satisfied in the strong-field approximation everywhere with the exception of small regions of the initial momentum and the final energy of the electron. If one of these conditions is not satisfied, i.e., the region $|\alpha| < 1$ is missing or the contribution of the region $|\alpha| > 1$, $|\alpha| > |\beta|$ is small, then the second term in expression (29) should be discarded, for then the cross section $d\sigma$ is determined exclusively by the region $|\beta| > |\alpha| \gg 1$.

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154