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# SPIN WAVES AND CORRELATION FUNCTIONS IN A FERROMAGNETIC

V. G. VAKS, A. I. LARKIN, and S. A. PIKIN

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We consider the spin waves and correlation functions in a Heisenberg ferromagnet in the complete temperature range below the transition temperature $T_C$. We find the damping of the spin waves and show that the damping is small for all $T < T_C$ for long wavelengths. The damping increases with increasing temperature and the range of wave vectors for which the damping is less than the frequency tends to zero as $T \to T_C$. We give expressions for the correlation functions of the longitudinal and transverse spin components as functions of the wave vector, frequency, temperature and magnetic field.

## 1. INTRODUCTION

ONE usually assumes that spin waves exist in a ferromagnetic only at low temperatures $T \ll T_C$ and that "the concept of spin waves completely loses its meaning"[1] at temperatures of the order of $T_C$. This point of view is connected with the fact that as the transition temperature is approached the number of spin waves and the fluctuations in the moment cease to be small so that the damping of the spin waves must increase.

We show in the present paper that, nevertheless, the damping of long-wavelength spin waves remains small for all temperatures below the critical one in a Heisenberg ferromagnet. We obtain for the ratio of the damping of a spin wave with wavevector $k$ to its frequency an expression of the form $k^3 f_1(T) + k^2 f_2(T)$. The coefficients $f_1$ and $f_2$ increase when $T_C$ is approached so that the range of $k$ for which spin waves exist shrinks. However, if $k^2 < (1 - T/T_C)^{\alpha}$ where $\alpha$ is a number of order unity the relative damping is small and spin waves exist right up to $T = T_C$.

This result is connected with the fact that long-wavelength spin waves are simply small oscillations in the direction of the total moment of the system. However complicated the structure of the system, the rotation of the average moment as a whole does not change the energy of the system, when there is no magnetic anisotropy, and, especially, does not lead to dissipative processes. Therefore, both the frequency and the damping of the spin waves tend for all $T$ to zero with $k$ and it seems natural that then the damping tends to zero faster.

We do not take into account in this paper relativistic effects of the magnetic interaction or anisotropy. In most ferromagnetics they are small compared with the exchange interaction and can influence the damping of only the very long wavelength spin waves, for instance, the width of ferromagnetic resonance. For spin waves with microscopic wave lengths these effects are usually small; quantitative criteria are given in the Conclusion.

We use the previously-developed temperature-dependent diagram technique.[2] We obtain a general expression for the correlation function of the transverse spin components $K_{+-}$ as function of $\mathbf{k}$, the frequency $\omega$, the temperature $T$, and the magnetic field $H$. For a large interaction range $r_0$ this expression is valid everywhere below $T_C$, except in the immediate vicinity of the transition point $T_C - T \lesssim T_C r_0^{-6}$, $\mu H \lesssim T_C r_0^{-9}$. At low temperatures the results are valid for all $r_0$. We show, in particular, that the dependence of the spin wave spectrum on $T$ and $H$ leads by no means to a similar dependence of the average moment. In the low temperature range the results are the same as those given earlier[1,3,4] and generalize them.

We study also the dependence on $\mathbf{k}$, $\omega$, $T$, and $H$ of the correlation function $K_{zz}$ of the longitudinal spin components. We find that in the range of small $k$ the most important contribution to $K_{zz}$ is connected with the presence of spin waves. We obtain an expression for this contribution, valid for all temperatures below $T_C$. Near the critical point this contribution is appreciably larger than the contribution from static fluctuations described by formulae of the Van Hove-de Gennes-Villain type[5,6] and has a different dependence on $k$, $\omega$, and $T$. We discuss the possibility of an experimental verification of the results, in particular in experiments on the critical scattering of neutrons.

## 2. THE CORRELATION FUNCTION OF THE TRANSVERSE SPIN COMPONENTS

We consider an ideal Heisenberg ferromagnetic with Hamiltonian

$$\hat{\mathcal{H}} = -\mu H \sum_{\mathbf{r}} S_{\mathbf{r}}^z - \frac{1}{2} \sum_{\mathbf{r} \neq \mathbf{r}'} V(\mathbf{r} - \mathbf{r}') \mathbf{S}_{\mathbf{r}} \mathbf{S}_{\mathbf{r}'}. \quad (1)$$

Here $\mathbf{S_r}$ is the spin operator of the atom which is assumed to be fixed in a site of the crystal lattice, and $\mathbf{r}$ is the coordinate of the site. $V(\mathbf{r} - \mathbf{r}')$ is the effective potential of the interaction between the spins, $H$ the external magnetic field, directed along the $z$ axis, and $\mu\mathbf{S}$ the magnetic moment of the atom.

In a previous paper[2], referred to in the following as I, we suggested a temperature-dependent diagram technique to describe systems with spin-spin interactions. We showed in I that the correlation function of the spins can be written in the form (see I (13)):

$$K_{\alpha\gamma}(\mathbf{k}, i\omega_n) = \sum_{\mathbf{r}_2} e^{i\mathbf{k}(\mathbf{r}_1 - \mathbf{r}_2)} \frac{1}{2\beta} \int_{-\beta}^{\beta} e^{i\omega_n t} dt \, \langle \hat{T}(S_{\mathbf{r}_2}^{\alpha}(t) - \langle S^{\alpha} \rangle)$$

$$\times (S_{\mathbf{r}_1}^{\gamma}(0) - \langle S^{\gamma} \rangle)\rangle = \frac{\Sigma_{\alpha\gamma}(\mathbf{k}, i\omega_n)}{1 - \beta V_{\mathbf{k}} \Sigma_{\alpha\gamma}(\mathbf{k}, i\omega_n)}, \quad (2)$$

where the index pair $\alpha\gamma$ takes on the values $+-$, $-+$, or zz;

$$K_{-+}(\mathbf{k}, i\omega_n) = K_{+-}(\mathbf{k}, -i\omega_n), \quad \beta = \frac{1}{T}, \quad V_{\mathbf{k}} = \sum_{\mathbf{r}} V(\mathbf{r}) e^{i\mathbf{kr}}.$$

$\Sigma_{\alpha\gamma}$ is the irreducible self-energy part. We showed that in the first approximation in the self-consistent field

$$\Sigma_{+-}^{(0)}(\mathbf{k}, i\omega_n) = b(y) G_n(y) = \frac{b(y)}{y - i\beta\omega_n},$$

$$K_{+-}^{(0)}(\mathbf{k}, i\omega_n) = \frac{b(y)}{y - \beta V_{\mathbf{k}} b(y) - i\beta\omega_n}. \tag{3}$$

The quantities y and b(y) are defined by Eqs. I(3) and I(5):

$$y = \beta(V_0 \langle S^z \rangle + \mu H),$$

$$b(y) = \frac{\operatorname{Sp} S^z \exp(yS^z)}{\operatorname{Sp} \exp(yS^z)} = \left(S + \frac{1}{2}\right) \operatorname{cth}\left(S + \frac{1}{2}\right) y - \frac{1}{2} \operatorname{cth} \frac{y}{2}, \tag{4}$$

and in the first approximation taken in (3), the dependence of y on H and T is determined by Eq. I(4)

$$\langle S^z \rangle = (y - \beta\mu H) / \beta V_0 = b(y). \tag{5}$$

The excitation spectrum is determined by the poles of the analytical continuation $K(\mathbf{k}, \omega)$ of the correlation function $K(\mathbf{k}, i\omega_n)$.[7,8] Replacing in the second Eq. (3) $i\omega_n \to \omega$ we get, using (5), for the spin wave spectrum in first approximation

$$\omega = \varepsilon_{\mathbf{k}} = b(y)(V_0 - V_{\mathbf{k}}) + \mu H. \tag{6}$$

The spin wave energy (6) depends quadratically on k for small k and to simplify the notation in the following it is convenient to introduce the mass of the spin waves m defining it by the relation

$$\varepsilon_{\mathbf{k}} = \frac{k^2}{2m} + \mu H, \quad \mathbf{k} \to 0,$$

$$k^2 = \sum_{\alpha,\beta=1}^{3} k_\alpha k_\beta \sum_{\mathbf{r}} x_\alpha x_\beta \frac{V(\mathbf{r})}{V_0} = \sum_{i=1}^{3} k_i^2 x_{0i}^2, \quad \frac{1}{m} = V_0 b(y). \tag{7}$$

As in I(24) we have denoted here by $x_{0i}^2$ the principal values of the tensor $V_0^{-1} \sum_{\mathbf{r}} x_\alpha x_\beta V(\mathbf{r})$, and by $k_i$ the components of k along its principal axis. For cubic lattices each of the quantities $x_{0i}^2$ is equal to one third of the mean square interaction range $R_0^2$ and $k^2 = |\mathbf{k}|^2 R_0^2/3$. In the general case $k^2$ is not proportional to $|\mathbf{k}|^2$ and by k we shall in the following always understand the dimensionless quantity $(k^2)^{1/2}$ from (7), and not $|\mathbf{k}|$.

At low temperatures, $T \ll T_c S^{-1}$, b = S and we find from (6) the Bloch formula[9] for the spin wave spectrum

$$\omega = S(V_0 - V_{\mathbf{k}}) + \mu H. \tag{8}$$

At finite T the quantity b(y) is determined by the condition that (5) be self-consistent. In particular, for H = 0 and $T \to T_c = V_0 S(S+1)/3$, Eq. (6) gives according to I(6) and I(21)

$$\varepsilon_{\mathbf{k}} = \left(\frac{3a^3}{c} \frac{T_c - T}{T_c}\right)^{1/2} (V_0 - V_{\mathbf{k}}), \quad a = \frac{S(S+1)}{3}, \quad c = \frac{a(6a+1)}{10}. \tag{9}$$

However, the region where the temperature dependence of the spectrum is determined by Eqs. (6) and (9) (as also is the case for similar results from Sec. 5 of I) occurs only for large $r_0$. At low temperatures (6) gives an exponential temperature dependence of the

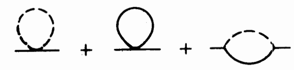

FIG. 1

spectrum while the following terms which take into account the interaction of spin waves lead to a power law. We find therefore the correlation function $K_{+-}$ in the next approximation.

To do this we must take into account in the polarization operator $\Sigma_{+-}$, which occurs in (2), the next terms which are depicted in the diagrams of Fig. 1. As in I(16a), the dotted lines correspond in this figure to the effective interaction of $S^Z$ with $S^Z$ and the full-drawn line which we shall use here instead of the wavy line in I, corresponds to the effective interaction of $S^+$ with $S^-$ (I.16b):

$$V^{zz}(\mathbf{k}, i\omega_n) = \frac{V_{\mathbf{k}}}{1 - \beta V_{\mathbf{k}} b' \delta_{n0}}, \quad V^{+-}(\mathbf{k}, i\omega_n) = \frac{V_{\mathbf{k}}}{1 - \beta V_{\mathbf{k}} b G_n(y)}. \tag{10}$$

The vertex points in Fig. 1 correspond to the vertex blocks given by Eq. I(11).

After summing over the frequencies of the internal lines we find for the first correction $\Sigma_{+-}^{(1)}$ to $\Sigma_{+-}$

$$\Sigma_{+-}^{(1)}(\mathbf{k}, i\omega_n) = G_n(y) \left\{ \frac{b''}{2} \sum_{\mathbf{q}} \frac{\beta V_{\mathbf{q}}}{1 - \beta V_{\mathbf{q}} b'} + \sum_{\mathbf{q}} [n_y - n_{\mathbf{q}}(1 - \beta V_{\mathbf{q}} b')] \right\}$$
$$+ b G_n^2(y) \beta \left[ \sum_{\mathbf{q}} n_{\mathbf{q}} (V_{\mathbf{q}-\mathbf{k}} - V_{\mathbf{q}}) + \frac{b'}{b} \sum_{\mathbf{q}} \frac{V_{\mathbf{q}-\mathbf{k}} - V_{\mathbf{q}}}{1 - \beta V_{\mathbf{q}} b'} + \right.$$
$$\left. + b' \sum_{\mathbf{q}} \frac{(V_{\mathbf{q}-\mathbf{k}} - V_{\mathbf{q}})^2}{(\varepsilon_{\mathbf{q}} - i\omega_n)(1 - \beta V_{\mathbf{k}-\mathbf{q}} b')} \right]. \tag{11}$$

where, as in I(18)

$$n_{\mathbf{k}} = [\exp(\beta\varepsilon_{\mathbf{k}}) - 1]^{-1}, \quad n_y = (e^y - 1)^{-1}.$$

Substituting (11) into (2) and performing the analytical continuation through the substitution $i\omega_n \to \omega$ we obtain after elementary transformations

$$K_{+-}^{(1)}(\mathbf{k}, \omega) = T\langle S^z \rangle \left[ -\omega + \mu H + (V_0 - V_{\mathbf{k}}) \langle S^z \rangle - \right.$$
$$- \sum_{\mathbf{q}} n_{\mathbf{q}} (V_{\mathbf{q}-\mathbf{k}} - V_{\mathbf{q}}) - \frac{b'}{b} \sum_{\mathbf{q}} \frac{V_{\mathbf{q}-\mathbf{k}} - V_{\mathbf{q}}}{1 - \beta V_{\mathbf{q}} b'} -$$
$$\left. - b' \sum_{\mathbf{q}} \frac{(V_{\mathbf{q}-\mathbf{k}} - V_{\mathbf{q}})^2}{(\varepsilon_{\mathbf{q}} - \omega - i\delta)(1 - \beta V_{\mathbf{q}-\mathbf{k}} b')} \right]^{-1} \tag{12}$$

Here $\langle S^Z \rangle$ is determined from the first approximation formula I(18):

$$\langle S^z \rangle = \frac{y - \beta\mu H}{\beta V_0} = b + \frac{b''}{2} \sum_{\mathbf{q}} \frac{\beta V_{\mathbf{q}}}{1 - \beta V_{\mathbf{q}} b'} + \sum_{\mathbf{q}} [n_y - n_{\mathbf{q}}(1 - \beta V_{\mathbf{q}} b')], \tag{13}$$

The spin wave spectrum is determined by the poles of (12). It is clear that when there is no magnetic field the energy in the small k region is, as before, proportional to $k^2$. It is also clear that the temperature dependence of the spin wave spectrum reduces to the T-dependence of $\langle S^Z \rangle$[10] only in the zeroth approximation (6), as the last three terms in the denominator in (12) are of the same order of magnitude as the terms in Eq. (13) for the moment.

## 3. THE SPIN WAVE SPECTRUM AT LOW TEMPERATURES

In the low temperature region the deviation of the moment from the saturation value is proportional to $T^{3/2}$ (see, e.g.,[3] or I(31) and (32)):

$$\langle S^z\rangle = S - \left(\frac{3T}{2\pi V_0 S}\right)^{3/2}\frac{1}{r_0^3}Z_{3/2}(\beta\mu H). \quad (14)$$

Here $Z_\alpha(x) = \sum_n n^{-\alpha}e^{-nx}$, the average relative interaction range $r_0$ is determined as in I(28):

$$r_0^3 = v_c^{-1}\,\mathrm{Det}^{1/2}\|3V_0^{-1}\sum_r x_\alpha x_\beta V(\mathbf{r})\| = 3\sqrt{3}x_{01}x_{02}x_{03}v_c^{-1}; \quad (15)$$

$v_c$ is the volume of the elementary cell and $x_{0i}$ the same as in (7).

However, in the spectrum the terms proportional to $T^{3/2}$ are cancelled by the latter terms in the denominator in (12). Dropping exponentially small terms we get in that region

$$\omega = \mu H + S(V_0 - V_{\mathbf{k}}) - \sum_{\mathbf{q}} n_{\mathbf{q}}(V_0 - V_{\mathbf{k}} - V_{\mathbf{q}} + V_{\mathbf{q-k}}). \quad (16)$$

In the last term in (16) the expression in brackets is for small k and q proportional to $k^2q^2$ so that the correction to the spin wave spectrum due to interactions turns out to be proportional to $T^{5/2}$ for small T.

One can also find the low-temperature corrections to the spin wave spectrum without making any assumptions about the range of the interactions being large. As in thermodynamics (Sec. 6 in I) the result reduces to replacing the Born amplitude $V_0 - V_k - V_q + V_{q-k}$ in the last term in (16) by the complete scattering amplitude $A(\mathbf{k}, \mathbf{q}, \mathbf{k}, \mathbf{q})$ for which we can use the ladder approximation I(35).

We consider, for example, the small k region and find the mass renormalization for the case of cubic lattices when $k^2$, determined by Eq. (7), is equal to $|\mathbf{k}|^2R_0^2/3$. In that case the correction term is after summing over q in (16) also proportional to $|\mathbf{k}|^2$. Replacing in (16) the Born amplitude by $A(\mathbf{k}, \mathbf{q}, \mathbf{k}, \mathbf{q})$ leads to the fact that the last term in (16) is multiplied by some factor $Q(S, r_0)$. This factor leads also to an analogous correction to the thermodynamic formulae and for the case of nearest neighbor interactions it can be taken from the paper by Dyson (Eqs. (117) and (138) of[3]). For instance, for a simple cubic lattice with nearest neighbor interactions this factor is equal to

$$Q(S) \approx \frac{1 + 0.03S^{-1}}{1 - 0.1S^{-1}} + \frac{0.17}{S}, \quad Q\left(\frac{1}{2}\right) = 1.68. \quad (17)$$

At low temperatures the spin wave spectrum for cubic lattices has thus for small k the form

$$\omega = \mu H + V_0 S\frac{|\mathbf{k}|^2R_0^2}{6}\left[1 - \frac{\nu Q(S, r_0)}{Sr_0^3}\pi\left(\frac{3T}{2\pi V_0 S}\right)^{5/2}Z_{5/2}(\beta\mu H)\right]. \quad (18)$$

Here $Z_\alpha$ and $r_0$ are the same as in (14) and $\nu r_0^4$ denotes the mean quartic interaction range:

$$\nu r_0^4 = \sum_r r^4 V(r)/V_0 v_c^{4/3}.$$

Thus for nearest neighbor interactions $\nu = 1$, and $r_0$ is equal, respectively, to 1, $\sqrt{2}$, and $\sqrt{3}$ for simple, face-centered, and body-centered cubic lattices.

It is clear from Eqs. (17) and (18), and also from sections 6 and 7 of I, that as in thermodynamics the Born approximation (16) is applicable not only for a large interaction range $r_0$, but also for large S. In that case Eq. (16) is applicable not only for $T \ll V_0 S \sim T_c/S$, but also in the much wider range $T \ll T_c$. As we discussed in Sec. 7 of I in the interval $V_0 S \ll T \ll V_0 S^2 \sim T_c$ the temperature becomes then larger than the spin wave energy for all k. The spin wave density can then be expanded in the neighborhood of its classical expression $T/\epsilon_k$ and the correction to the spectrum is in first approximation proportional to $T/T_c$. The spectrum in a simple cubic lattice has thus, for instance, in that region for nearest neighbor interactions the form

$$\omega = \mu H + S(V_0 - V_{\mathbf{k}})\left[1 - \frac{T}{V_0 S^2}\left(1 + \frac{1.51\mu H}{V_0 S} - \frac{V_0 S}{2T}\right)\right]. \quad (19)$$

## 4. THE SPECTRUM NEAR THE TRANSITION

As was discussed above, at temperatures of the order of $T_c$ we can use Eq. (12) only for large interaction ranges $r_0$ and not too close to the transition, i.e., for $T_c - T \gg T_c r_0^{-6}$, $\mu HS \gg T_c r_0^{-9}$. For $T_c - T \ll T_c$, $\mu HS \ll T_c$ we can then expand the denominator in Eq. (12) in powers of y and $T - T_c$. As is discussed in Sec. 5, weakly-damped excitations will then occur only for small k. The expression for the spin wave mass M in that region has the form

$$\frac{1}{M} = \frac{1}{m}\left[1 + \frac{\gamma}{\sqrt{u}}\left(1 + \frac{2}{3}\sqrt{\frac{v}{u}}\right) + \frac{2}{9}\frac{\gamma}{\sqrt{u}}\frac{1 - \sqrt{v/u}}{(1 + \sqrt{v/u})^2}\right], \quad (20)$$

where m is determined by Eq. (7) and, in accordance with the notation in I(25) to I(27)

$$u = \frac{c}{a}y^2 + \tau, \quad v = \frac{c}{3a}y^2 + \tau, \quad \tau = \frac{T - T_c}{T_c}, \quad T_c = aV_0,$$

$$\gamma = \frac{c}{a^2}\frac{3\sqrt{6}}{2\pi r_0^3} = \frac{3}{5}\left(1 + \frac{1}{2S(S+1)}\right)\frac{3\sqrt{6}}{2\pi r_0^3},$$

$$cy^3 + 3ay\tau - 3ah = 0, \quad h = \mu H / T_c. \quad (21)$$

We give the explicit form of the corrections to the mass (20) for the cases of weak and strong magnetic fields, considered in I(29). We have

a) $\tau > 0$, $ch^2/a\tau^3 \ll 1$:

$$\frac{m}{M} = 1 + \frac{5}{3}\frac{\gamma}{\sqrt{\tau}} - \frac{28}{27}\frac{\gamma}{\sqrt{\tau}}\frac{ch^2}{a\tau^3}; \quad (22a)$$

b) $ch^2/a|\tau|^3 \gg 1$:

$$\frac{m}{M} = 1 + \left(\frac{\gamma^6 a}{9ch^2}\right)^{1/6}\left[2 - \frac{1}{\sqrt{3}} + \tau\left(\frac{a}{9ch^2}\right)^{1/3}\left(4 - \frac{31}{6\sqrt{3}}\right)\right]; \quad (22b)$$

c) $\tau < 0$, $ch^2/a|\tau|^3 \ll 1$:

$$\frac{m}{M} = 1 + \frac{\gamma}{\sqrt{2|\tau|}}\left[\frac{11}{9} - \frac{13}{36}\left(\frac{ch^2}{3a|\tau|^3}\right)^{1/2}\right]. \quad (22c)$$

The second term in the square brackets in (20) corresponds to the correction to the average moment $\langle S^z\rangle$, and the third one to the last three terms in the square brackets in (12). It is clear that also at low temperatures both terms are of the same order of magnitude and that the temperature dependence of the spectrum and its dependence on the magnetic field do not reduce to the analogous dependence for the average moment. For instance, in the weak magnetic field region below the transition the correction (22c) to the mass does not contain a term proportional to $\sqrt{H}$, which according to (20) and I(29c) occurs in the moment.

## 5. SPIN WAVE DAMPING

The damping of spin waves and also the absorption of a variable external field is determined by the imaginary part $-\Gamma(k, \omega)$ of the denominator of the correlation function $K_{+-}$ in (12):

$$\Gamma(\mathbf{k}, \omega) = \pi b' \sum_{\mathbf{q}}\frac{(V_{\mathbf{q-k}} - V_{\mathbf{q}})^2}{1 - \beta V_{\mathbf{k-q}}b'}\delta(\varepsilon_{\mathbf{q}} - \omega). \quad (23)$$

In the small k region Eq. (23) becomes

$$\Gamma(\mathbf{k},\omega)=\frac{3\sqrt{6}}{2\pi r_0^3}\frac{b'V_0}{b(1-\beta V_0 b')}\left(\frac{\omega-\mu H}{bV_0}\right)^{1/2}\frac{k^4}{4}\times f\left(\frac{\beta V_0 b'}{1-\beta V_0 b'}\frac{k^2}{2},\ \frac{2(\omega-\mu H)}{k^2 bV_0}\right), \tag{24}$$

where

$$f(x,t)=\frac{(1+xt)^2}{4x^3\sqrt{t}}\ln\frac{1+x(1+\sqrt{t})^2}{1+x(1-\sqrt{t})^2}-\frac{1+x(t-1)}{x^2},\qquad f(0,t)=1+\frac{4}{3}t, \tag{25}$$

and the quantities $k^2$ and $r_0^3$ are the same as in (7) and (15). To find the damping of the spin waves or the width of the ferromagnetic resonance we must put in Eq. (24) $2(\omega-\mu H)=bV_0k^2$ after which the second argument of the function f is equal to unity.

The damping determined by Eqs. (23) and (24) is connected with the scattering of spin waves by fluctuations in the moment $S^z$. For small k it decreases proportional to $k^5$. When the transition point is approached the scattering by fluctuations increases. Thus, in the region near the transition, considered in section 4, the complete expression for the spectrum for small k can according to (20) and (24) be written in the form

$$\omega=\mu H+yT_c\frac{k^2}{2}\left[1+\delta-i\frac{\gamma}{\sqrt{u}}\left(\frac{k^2}{2u}\right)^{1/2}\frac{a}{c}\times\left\{\begin{array}{ll}\frac{7}{3}\frac{k^2}{2y^2} & \text{when } k^2\ll u\\ \frac{u}{4y^2}\ln\frac{2k^2}{u} & \text{when } k^2\gg u\end{array}\right.\right]. \tag{26}$$

We have denoted by $\delta$ the relative correction to the mass determined by (20).

We give the explicit form of the relative damping for small k given by the last term in the brackets in (26) for the regions near the transition considered in (22):

a) $\tau>0,\quad ch^2/a\tau^3\ll 1$:

$$\frac{\Gamma(\mathbf{k})}{\varepsilon_{\mathbf{k}}-\mu H}=\frac{7}{3}\left(\frac{k^2}{2\tau}\right)^{3/2}\frac{a\tau^3}{ch^2}; \tag{27a}$$

b) $ch^2/a|\tau|^3\gg 1$:

$$\frac{\Gamma(\mathbf{k})}{\varepsilon_{\mathbf{k}}-\mu H}=\frac{7}{3}\left(\frac{\gamma^6 a}{9ch^2}\right)^{1/6}\left[\frac{k^2}{2}\left(\frac{a}{9ch^2}\right)^{1/3}\right]^{3/2}; \tag{27b}$$

c) $\tau<0,\quad ch^2/a|\tau|^3\ll 1$:

$$\frac{\Gamma(\mathbf{k})}{\varepsilon_{\mathbf{k}}-\mu H}=\frac{7}{18}\frac{\gamma}{\sqrt{|\tau|}}\left(\frac{k^2}{2|\tau|}\right)^{3/2}. \tag{27c}$$

It is clear from (26) and (27) that notwithstanding the increase in absorption near $T_c$ the damping is small in the very small k region and spin waves exist although for $k^2>\max(|\tau|, h^{2/3})$ damping increases fast. It is also clear that in a weak magnetic field the damping has an appreciably different form above and below the transition. When $T<T_c$ the relative damping is according to (27c) of the order $\gamma k^3|\tau|^{-2}$ while above $T_c$ this factor is multiplied by the large quantity $\tau^3 h^{-2}$. The following terms of the expansion in $\gamma$ for $T>T_c$ will contain higher powers in $H^{-2}$. The concept of spin waves or paramagnetic resonance has therefore above the transition a meaning only for sufficiently strong fields and small k. We give a more detailed discussion of this in the following.

At low temperatures the damping described by Eq. (23) is exponentially small. Therefore the damping described by the next terms in the expansion in $r_0^{-3}$ and corresponding to spin wave-spin wave scattering become more important. Moreover, the next approximation contains corrections to the first approximation (23) which describe the damping due to scattering by fluctuations. We must check that these corrections do not change the dependence of the damping on the wave vector **k**, i.e., that also in the following approximations in $r_0^{-3}$ the fluctuation damping is for small k proportional to $k^5$. We find therefore the next approximation for the scattering at arbitrary temperatures.

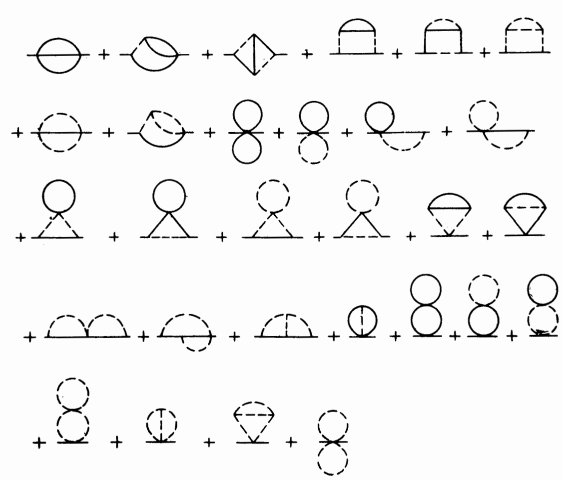

FIG. 2.

In Fig. 2 we give the second approximation diagrams $\Sigma_{+-}^{(2)}$ for $\Sigma_{+-}$. We have dropped in that figure terms containing isolated dotted lines with zero momentum; these terms correspond to substituting the corrections to y from (13) into the first approximation Eq. (11) and they give for the damping the same dependence on **k** and $\omega$ as (23). In the expression for the spectrum the quantity $\Sigma_{+-}^{(2)}/bG_n^2$ occurs, as can be verified using (2) and (3). Retaining in the answer only those terms which have a nonvanishing imaginary part (which means, in particular, that we drop the last four diagrams of Fig. 2) we obtain after very complicated calculations

$$\frac{\Sigma_{+-}^{(2)}(\mathbf{k},\omega)}{bG^2(\omega)}=-\Sigma_s-\Sigma_f, \tag{28a}$$

$$\Sigma_s=\frac{1}{2}\sum_{\mathbf{p},\mathbf{q}}\frac{(V_{\mathbf{p}}+V_{\mathbf{q}}-V_{\mathbf{p}-\mathbf{k}}-V_{\mathbf{q}-\mathbf{k}})^2}{\varepsilon_{\mathbf{p}}+\varepsilon_{\mathbf{q}}-\varepsilon_{\mathbf{p}+\mathbf{q}-\mathbf{k}}-\omega}[n_{\mathbf{p}+\mathbf{q}-\mathbf{k}}(1+n_{\mathbf{p}}+n_{\mathbf{q}})-n_{\mathbf{p}}n_{\mathbf{q}}], \tag{28b}$$

$$\begin{aligned}\Sigma_f=&+\frac{\partial n_y}{\partial y}\sum_{\mathbf{p}}\frac{(V_{\mathbf{p}}^{\mathbf{k}})^2}{\varepsilon_{\mathbf{p}}-\omega}+b(b')^2\sum_{\mathbf{p},\mathbf{q}}\frac{\beta V_{\mathbf{q}}(V_{\mathbf{p}}^{\mathbf{k}})^2(V_{\mathbf{p}-\mathbf{q}}^{\mathbf{k}})^2L_{\mathbf{k}-\mathbf{p}}L_{\mathbf{q}-\mathbf{p}}G(\omega)}{(\varepsilon_{\mathbf{p}}-\omega)(\varepsilon_{\mathbf{q}}-\omega)(\varepsilon_{\mathbf{p}-\mathbf{q}-\mathbf{k}}-\omega)}\\&-2bb'\sum_{\mathbf{p},\mathbf{q}}\sum_{m}\frac{V_{\mathbf{q}}V_{\mathbf{p}}^{\mathbf{k}}V_{\mathbf{p}-\mathbf{q}}^{\mathbf{k}}V_{\mathbf{p}-\mathbf{k}}^{\mathbf{q}}L_{\mathbf{k}-\mathbf{p}}G_m}{(\varepsilon_{\mathbf{p}}-\omega)(\varepsilon_{\mathbf{p}-\mathbf{q}-\mathbf{k}}-i\omega_m)(\varepsilon_{\mathbf{q}}-i\omega_m)}+\frac{b''}{b}\sum_{\mathbf{p},\mathbf{q}}\frac{V_{\mathbf{p}}^{\mathbf{k}}V_{\mathbf{q}}^{\mathbf{k}}L_{\mathbf{k}-\mathbf{p}}L_{\mathbf{k}-\mathbf{q}}L_{\mathbf{q}-\mathbf{p}}}{\varepsilon_{\mathbf{p}}-\omega}\\&+\frac{(b')^2}{b}\sum_{\mathbf{p},\mathbf{q}}\frac{\beta V_{\mathbf{q}}V_{\mathbf{p}}^{\mathbf{k}}(V_{\mathbf{p}}^{\mathbf{k}}-V_{\mathbf{p}-\mathbf{q}}^{\mathbf{k}})L_{\mathbf{k}-\mathbf{p}}L_{\mathbf{q}-\mathbf{p}}G(\omega)}{\varepsilon_{\mathbf{q}}-\omega}+b''\sum_{\mathbf{p},\mathbf{q}}\frac{V_{\mathbf{p}}^{\mathbf{k}}V_{\mathbf{q}}^{\mathbf{k}}V_{\mathbf{p}}^{\mathbf{q}}L_{\mathbf{k}-\mathbf{p}}L_{\mathbf{k}-\mathbf{q}}L_{\mathbf{q}-\mathbf{p}}}{(\varepsilon_{\mathbf{p}}-\omega)(\varepsilon_{\mathbf{q}}-\omega)}\\&-2(b')^2\sum_{\mathbf{p},\mathbf{q}}\frac{\beta V_{\mathbf{q}}(V_{\mathbf{p}}^{\mathbf{k}})^2V_{\mathbf{p}-\mathbf{q}}^{\mathbf{k}}L_{\mathbf{k}-\mathbf{p}}L_{\mathbf{q}-\mathbf{p}}G(\omega)}{(\varepsilon_{\mathbf{p}}-\omega)(\varepsilon_{\mathbf{q}}-\omega)}+2bb'\sum_{\mathbf{p},\mathbf{q}}\frac{\beta V_{\mathbf{q}}V_{\mathbf{k}-\mathbf{p}}(V_{\mathbf{p}}^{\mathbf{k}})^2L_{\mathbf{k}-\mathbf{p}}}{\varepsilon_{\mathbf{p}}-\omega}\sum_{m}\frac{G_m^2}{\varepsilon_{\mathbf{q}}-i\omega_m}\\&-2b'\sum_{\mathbf{p},\mathbf{q}}\frac{\beta V_{\mathbf{q}}^{\mathbf{k}}(V_{\mathbf{p}}^{\mathbf{k}})^2L_{\mathbf{k}-\mathbf{p}}G(\omega)n_{\mathbf{q}}}{\varepsilon_{\mathbf{p}}-\omega}\end{aligned}$$

$$
\begin{aligned}
&+2b'\sum_{\mathbf{p},\mathbf{q}}\frac{\beta V_{\mathbf{k-p}}V_{\mathbf{p}}^{\mathbf{k}}V_{\mathbf{q}}^{\mathbf{k}}L_{\mathbf{k-p}}(n_{\mathbf{q}}-n_y)}{(\varepsilon_{\mathbf{p}}-\omega)(y-\beta\varepsilon_{\mathbf{q}})}\\
&+2b'\sum_{\mathbf{p},\mathbf{q}}\frac{V_{\mathbf{p}}^{\mathbf{k}}V_{\mathbf{p-q}}L_{\mathbf{k-p}}(n_{\mathbf{q}}-n_y)}{\varepsilon_{\mathbf{p}}-\omega}-b'\sum_{\mathbf{p},\mathbf{q}}\frac{(V_{\mathbf{p}}^{\mathbf{k}})^2V_{\mathbf{q}}^{\mathbf{p}}L_{\mathbf{k-p}}n}{(\varepsilon_{\mathbf{p}}-\omega)^2}\\
&+b'\left(G(\omega)-\frac{b'}{b}\right)\sum_{\mathbf{p},\mathbf{q}}\frac{V_{\mathbf{q}}(V_{\mathbf{p}}^{\mathbf{k}})^2L_{\mathbf{q}}L_{\mathbf{k-p}}}{(\varepsilon_{\mathbf{p}}-\omega)^2}\\
&+b(b')^2\sum_{\mathbf{p},\mathbf{q}}\frac{\beta V_{\mathbf{q}}(V_{\mathbf{p}}^{\mathbf{k}})^2(V_{\mathbf{p}}^{\mathbf{q}})^2L_{\mathbf{k-p}}L_{\mathbf{q-p}}G(\omega)}{(\varepsilon_{\mathbf{p}}-\omega)^2(\varepsilon_{\mathbf{q}}-\omega)}\\
&-2(b')^2\sum_{\mathbf{p},\mathbf{q}}\frac{\beta V_{\mathbf{q-k}}V_{\mathbf{q}}^{\mathbf{k}}(V_{\mathbf{p}}^{\mathbf{k}})^2L_{\mathbf{k-p}}L_{\mathbf{k-q}}G(\omega)}{(\varepsilon_{\mathbf{p}}-\omega)(\varepsilon_{\mathbf{q}}-\omega)}\\
&-2\frac{(b')^2}{b}\sum_{\mathbf{p},\mathbf{q}}\frac{\beta V_{\mathbf{k-p}}V_{\mathbf{p}}^{\mathbf{k}}V_{\mathbf{q}}^{\mathbf{k}}L_{\mathbf{k-p}}L_{\mathbf{k-q}}G(\omega)}{\varepsilon_{\mathbf{p}}-\omega}\\
&+b^2b'\sum_{\mathbf{p},\mathbf{q}}\sum_m\frac{\beta V_{\mathbf{k-p}}V_{\mathbf{q}}V_{\mathbf{p-q-k}}(V_{\mathbf{p}}^{\mathbf{k}})^2L_{\mathbf{k-p}}G_m^2}{(\varepsilon_{\mathbf{p}}-\omega)(\varepsilon_{\mathbf{q}}-i\omega_m)(\varepsilon_{\mathbf{p-q-k}}-i\omega_m)}+\sum_{\mathbf{p},\mathbf{q}}\frac{(V_{\mathbf{p}}^{\mathbf{k}})^2L_{\mathbf{k-p}}^2}{\varepsilon_{\mathbf{p}}-\omega}\\
&\times\Big[\frac{b'''}{2}\beta V_{\mathbf{q}}L_{\mathbf{q}}+\frac{(b'')^2}{2}\beta^2V_{\mathbf{q}}V_{\mathbf{k-q}}L_{\mathbf{q}}L_{\mathbf{k-q}}+V_{\mathbf{q}}\sum_m\frac{b''-2b'G_m}{\varepsilon_{\mathbf{q}}-i\omega_m}\\
&+V_{\mathbf{q}}V_{\mathbf{p-q-k}}\sum_m\frac{(b')^2-2bb'G_m}{(\varepsilon_{\mathbf{q}}-i\omega_m)(\varepsilon_{\mathbf{p-q-k}}-i\omega_m)}+b'V_{\mathbf{k-p}}\frac{n_{\mathbf{k-q}}-n_{\mathbf{q}}}{\varepsilon_{\mathbf{q}}-\varepsilon_{\mathbf{k-q}}}\Big].
\end{aligned}
\tag{28c}
$$

Here $V_{\mathbf{p}}^{\mathbf{k}} = V_{\mathbf{p}} - V_{\mathbf{p-k}}$, $L_{\mathbf{p}} = (1 - \beta V_{\mathbf{p}} b')^{-1}$, $G(\omega) = (y - \beta\omega)^{-1}$, $G_m = (y - i\beta\omega_m)^{-1}$, and in the sums over m the quantities $i\omega_m = 2m\pi iT$ are the imaginary frequencies of the temperature-dependent diagram technique.[7] To find the imaginary parts in Eqs. (28) we must replace one or three energy denominators by a δ-function: $(\epsilon_i - \omega)^{-1} \to i\pi\delta(\epsilon_i - \omega)$.

Let us first of all consider the term $\Sigma_s$. Replacing the denominator by a δ-function we find for the damping $\Sigma_s$ connected with the spin-wave-spinwave scattering

$$
\Gamma_s(\mathbf{k},\omega)=\frac{\pi}{2}(e^{\beta\omega}-1)\times\sum_{\mathbf{p},\mathbf{q}}\frac{(V_{\mathbf{p}}+V_{\mathbf{q}}-V_{\mathbf{p-k}}-V_{\mathbf{q-k}})^2n_{\mathbf{p}}n_{\mathbf{q}}\delta(\varepsilon_{\mathbf{p}}+\varepsilon_{\mathbf{q}}-\varepsilon_{\mathbf{p+q-k}}-\omega)}{1-\exp(-\beta\varepsilon_{\mathbf{p+q-k}})}. \tag{29}
$$

It is clear from (29) that in the region of small $\omega$ and k the quantity $\Gamma_s$ is proportional to $\omega k^2$. The spin wave damping (or the ferromagnetic resonance width) is obtained from (29) through the substitution $\omega = \epsilon_k = \mu H + b(V_0 - V_k)$. We give the explicit form of the relative spin wave damping given by (29) for small k, $k^2 \ll 1$, in different temperature ranges:

a) $\varepsilon_{\mathbf{k}}=\dfrac{bV_0k^2}{2}+\mu H\gg T$:

$$
\frac{\Gamma_s(\mathbf{k})}{\varepsilon_{\mathbf{k}}}=\frac{27}{8r_0^6}k^3\frac{V_0}{b\varepsilon_{\mathbf{k}}}\left(\frac{T}{2\pi bV_0}\right)^{5/2}Z_{5/2}(\beta\mu H); \tag{30a}
$$

b) $\varepsilon_{\mathbf{k}}\ll T$:

$$
\frac{\Gamma_s(\mathbf{k})}{\varepsilon_{\mathbf{k}}}=\frac{9}{4\pi^3r_0^6}\frac{k^2}{2}\frac{T^2}{V_0^2b^4}\begin{cases}\ln^2(T/\varepsilon_{\mathbf{k}}) & \text{when } T\ll bV_0\\ \ln^2(bV_0/\varepsilon_{\mathbf{k}}) & \text{when } \varepsilon_{\mathbf{k}}\ll bV_0\ll T.\end{cases} \tag{30b}
$$

Krivoglaz and Kashcheev[4] obtained earlier formulae similar to (30) for low temperatures.

We consider now the low temperature region. Since all terms $\Sigma_f$ of (29c) contain as factors derivatives of $n_y$ or $b(y)$, for $T \ll T_c/S$, the fluctuation term $\Sigma_f$, as also the first approximation term of (23) is exponentially small. The damping is here therefore determined by the spin wave term $\Sigma_s$ and given by Eqs. (29) and (30). As in thermodynamics and in the spectrum (Sec. 3) the damping at low T can also be found without assuming $r_0$ (or S) to be large. As before, to obtain an exact answer we must then in Eq. (29) replace the Born approximation $V_{p-k}V_p - V_q + V_{q-k}$ by the exact amplitude $A(\mathbf{k}, \mathbf{p} + \mathbf{q} - \mathbf{k}, \mathbf{p}, \mathbf{q})$ determined by Eq. I(34). As in the case of nearest neighbor interactions in cubic lattices the exact expression for A in the small momenta range, important at low T, can be taken from Dyson's paper (Eq. (79) in[1]).

We note that for large spins S Eq. (29) for the damping is valid not only for $T \ll T_c/S$ but also in the wider interval $T \ll T_c$. In the range $T_c/S \lesssim T \ll T_c$ the quantity $y \sim \beta V_0 S \lesssim 1$ and the functions $n_y$ and $b'(y) \approx -n_y'$ can no longer be assumed to be small. However, in that range the first order term of (23) (as also the similar corrections in the thermodynamics case discussed in section 7 of I) contains higher powers of $S^{-1}$ i.e., $T/T_c$. Therefore also in that range is the damping determined by the term $\Gamma_s$ of (29).

When approaching the transition point the spin wave damping $\Gamma_s$, as also the fluctuation term in (23), increases. Thus, in the regions considered above in (22) and (27), which are near the transition, the relative damping has according to the second of Eqs. (30b) the form

a) $\tau > 0$, $ch^2/a\tau^3 \ll 1$:

$$
\frac{\Gamma_s}{\varepsilon_{\mathbf{k}}}=\frac{\gamma^2}{6\pi\tau}\frac{k^2}{2\tau}\left(\frac{a\tau^3}{ch^2}\right)^2\ln^2\frac{1}{k^2/2+\tau}; \tag{31a}
$$

b) $ch^2/a|\tau|^3 \gg 1$:

$$
\frac{\Gamma_s}{\varepsilon_{\mathbf{k}}}=\frac{1}{6\pi}\left(\frac{\gamma^6a}{9ch^2}\right)^{1/3}\frac{k^2}{2}\left(\frac{a}{9ch^2}\right)^{1/3}\ln^2\frac{1}{k^2/2+(ch^2/3a)^{1/3}}; \tag{31b}
$$

c) $\tau < 0$, $ch^2/a|\tau|^3 \ll 1$:

$$
\frac{\Gamma_s}{\varepsilon_{\mathbf{k}}}=\frac{1}{54\pi}\frac{\gamma^2}{|\tau|}\frac{k^2}{2|\tau|}\ln^2\frac{1}{k^2/2+(ch^2/3a|\tau|)^{1/2}}. \tag{31c}
$$

It is clear from (31) that near the transition the T- and H-dependence of the spin wave damping has the same character as that of the fluctuation term of (27). As in (27), notwithstanding the increase of the damping near $T_c$, for very long wavelength spin waves with $k^2 < |\tau|$ the relative damping is small for $T < T_c$. Above $T_c$ the damping is small only for sufficiently strong magnetic fields.

We now compare the contributions to the damping of the spin wave term $\Gamma_s$ of (29) and of the first order fluctuation term in (23) in different temperature and frequency ranges. These expressions have different $\omega$- and k-dependences: for small $\omega$, k the quantity $\Gamma_s$ is proportional to $\omega k^2$ while the term (23) is according to (24) and (25) proportional to $(\omega - \mu H)^{1/2}k^4$ or $(\omega - \mu H)^{3/2}k^2$. Therefore, (23) is non-vanishing only when $\omega > \mu H$ while (29) is non-vanishing for all $\omega$. For spin waves $\omega - \mu H = k^2/2m$ so that the ratio of (24) to (29) is proportional to k. At low temperatures the term (23) as was discussed above is exponentially small while (29) decreases as a power. Therefore, both for low temperatures and for small k at all T the damping is determined by the spin wave term (29). However, at temperatures of the order $T_c$ and not too small k ($k^2 \gtrsim |\tau| r_0^{-6}$) the term (23) becomes more important, in particular if we bear in mind the small numerical factor in (31c).

We discuss now the second order fluctuation term $\Gamma_f$ of (28c). We note firstly that the first term of $\Gamma_f$ is, in contrast to all others, proportional to $r_0^{-3}$ and not to $r_0^{-6}$. In reality, however, the term $\sim r_0^{-3}$ is completely cancelled by the corresponding contribution from $\Sigma_S$ in (28b) which arises from the regions when one of the momenta p or q is not small; $p > r_0^{-1}$ or $q > r_0^{-1}$. In the discussion of the term $\Gamma_S$ given above the contribution from these regions was not taken into account for the sake of simplicity; moreover, these terms are proportional to the expression (23) and as was discussed above are small for low T and small k. However, in a formal expansion in $r_0^{-3}$ the first term of $\Sigma_f$ should be included in $\Sigma_S$ after which both (28c) and (28b) contain only terms $\sim r_0^{-6}$. This was not done above for the sake of clarity and simplicity of the presentation.

One verifies easily that for each of the terms of $\Sigma_f$ the frequency and wave vector dependence of the imaginary part has the same character for small $\omega$ and k as the first order expression (24). Each of them is proportional to $(\omega - \mu H)^{1/2}k^4$, $(\omega - \mu H)^{3/2}k^2$ or higher powers of $\omega - \mu H$ and k. As in (24), the imaginary part occurs only when $\omega > \mu H$. At low temperatures all terms of $\Sigma_f$ are exponentially small. When approaching the transition the damping increases but as in the case (27) the relative damping $\Gamma(k)(\epsilon_k - \mu H)^{-1}$ for $T < T_C$ turns out to be, after separating off the factor $\gamma^2 |\tau|^{-1}$, proportional to half-odd-integer powers of $k^2 |\tau|^{-1}$, starting with $^3\!/_2$, and to be small if $k^2 < |\tau|$. Therefore, the evaluation of the fluctuation damping in second order does not change the $\omega$- and k-dependence of the damping from the first approximation (24).

This means, in particular, that near and below $T_C$ when there is no field the contribution from both orders to the fluctuation damping can be written in the form $f_1(\gamma |\tau|^{-1/2})(k^2 |\tau|^{-1})^{3/2}$ where the first two terms in the expansion of $f_1$ are given by Eqs. (27) and the corresponding terms from (28c). Similarly, the relative damping connected with spin wave-spin wave scattering for the same region can be written in the form $f_2(\gamma |\tau|^{-1/2})k^2 |\tau|^{-1} \ln^2 k$ and (31) gives the first term in the expansion of $f_2$.

Above we considered only two approximations. However, it is natural that also in higher orders in $\gamma |\tau|^{-1/2}$ the k-dependence of the fluctuation and the spin wave damping will have the same character. In that case, the general expression for the relative damping in the region considered can be written in the form

$$\frac{\operatorname{Im}\omega}{\omega} = f_1(\gamma|\tau|^{-1/2})\left(\frac{k^2}{|\tau|}\right)^{3/2} + f_2(\gamma|\tau|^{-1/2})\frac{k^2}{|\tau|}\ln^2\frac{1}{k}. \tag{32}$$

It is clear from Eq. (32) that in the region considered where the self-consistent field approximation is applicable the spin wave damping is small when $k \lesssim |\tau|^{-1/2}$. However, according to I(24) or inequality (35) below the quantity $|\tau|^{-1/2}$ is the Ornstein-Zernike expression for the spin correlation radius $r_C(\tau)$ near $T_C$ which characterizes the self-consistent field. It is therefore natural to assume that also outside the region of applicability of the self-consistent field method the condition for the existence of spin waves is given by the inequality $\lambda = 1/k \lesssim r_C(\tau)$. The condition for the existence of spin waves $\operatorname{Im}\omega/\omega \lesssim 1$ can then be written as

$$k \lesssim \frac{1}{r_c(\tau)} \sim \left(\frac{T_C - T}{T_C}\right)^{\alpha}. \tag{33}$$

The spin waves are therefore not damped if the wavelength exceeds the correlation radius. This result is natural since, as we noted in the introduction, when the wavelength further increases the spin wave goes over into an adiabatic rotation of the average moment. The value of the exponent $\alpha$ in (33) is determined by the temperature dependence of the correlation radius $r_C$. If, for instance, we assume for this quantity the dependence $r_C(\tau) \sim |\tau|^{-2/3}$ which has recently been assumed by a number of authors on the basis of numerical calculations and phenomenological considerations, the quantity $\alpha$ in Eq. (33) equals $^2\!/_3$.

Finally, we consider the region above the transition temperature. As we have already emphasized, in that region the damping is large for small H and the concept of an excitation loses its meaning. From Eqs. (29) and (30) we can also see that in that region for low frequencies $\omega$ the expansion parameter of the self-consistent field is proportional to $k(T - T_C)^2(\mu H)^{-2}$. Therefore, even though in the thermodynamic case (I) the self-consistent field method was applicable in the whole of the T-H-plane apart from a limiting vicinity of the transition point, and the rigor of it improved with increasing $T - T_C$, the consideration of low-frequency kinetics by this method in the region $T - T_C > \mu H\sqrt{k}$ turns out to be impossible. Physically this is connected with the fact that for small $\omega$ in the given region relaxation processes become more important and to describe these one needs a derivation of the hydrodynamic type which is different from the self-consistent field method used.

## 6. LONGITUDINAL SPIN COMPONENT CORRELATION FUNCTION

The correlation function $K_{ZZ}$ is defined by Eq. (2) with $\alpha = \gamma = z$. In first order in the self-consistent field we must substitute in this formula the zeroth approximation for $\Sigma_{ZZ}$ which is equal to $b'\delta_{n0}$(I(11a)) after which $K_{ZZ}$ is equal to (I(14a))

$$K_{zz}^{(0)}(\mathbf{k}, i\omega_n) = \delta_{n0}\frac{b'}{1 - \beta V_{\mathbf{k}} b'}. \tag{34}$$

Near the transition Eq. (34) becomes (I(24)):

$$K_{zz}^{(0)}(\mathbf{k}, i\omega_n) = \delta_{n0}\frac{a}{k^2/2 + \tau + cy^2/a}, \tag{35}$$

where the H- and T-dependence of y is determined by Eq. (21). It follows from (35) that the static susceptibility

$$\chi_T = \sum_n K_{zz}(0, i\omega_n)$$

satisfies in this approximation the Curie-Weiss law while the two-fold law is satisfied for the ratio of the susceptibility above and below $T_C$.

Formulae such as (34) and (35) were earlier obtained by Van Hove[5] and de Gennes and Villain.[6] When $T \sim T_C$ these formulae, as are all self-consistent field approximations, are applicable only for large interaction ranges $r_0$, and not too close to the transition: $|\tau| \gtrsim r_0^{-6}$ or $h \gtrsim r_0^{-9}$.

Equation (34) describes only the static correlation of $S^z$ and to study its time dependence we must find the next approximation, describing, in particular, the interaction with spin waves. At low T the correlations

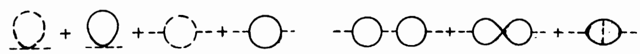

FIG. 3.

given by (34) tend exponentially to zero while the next spinwave terms in $r_0^{-3}$ decrease as a power. Finally, in the range of very small k, as we shall see, the spin wave term is more important for all T than (34). We find therefore the next approximation for $K_{zz}$.

The first order diagrams in $\Sigma_{zz}$ are given in Fig. 3a. After some calculations we find

$$\Sigma_{zz}(\mathbf{k}, i\omega_n) = \delta_{n0}\frac{\partial n_y}{\partial y} + \delta_{n0}\sum_{\mathbf{q}}\beta V_{\mathbf{q}}\Big[\frac{b'''}{2}L_{\mathbf{q}} + \frac{(b'')^2}{2}\beta V_{\mathbf{k+q}}L_{\mathbf{q}}L_{\mathbf{k+q}} + b''n_{\mathbf{q}} + \frac{(b')^2\beta V_{\mathbf{k+q}} - 2b'}{\beta(\varepsilon_{\mathbf{k+q}}-\varepsilon_{\mathbf{q}})}(n_{\mathbf{q}} - n_{\mathbf{k+q}})\Big] + T\sum_{\mathbf{q}}\frac{n_{\mathbf{q}}-n_{\mathbf{k+q}}}{\varepsilon_{\mathbf{k+q}}-\varepsilon_{\mathbf{q}} - i\omega_n} \qquad (36)$$

where $n_y$, $n_q$, and $L_q$ are the same as in (11) and (28).

Together with the usual corrections $\sim r_0^{-3}$ to the zeroth approximation (34) a term of a new kind (the last term) occurs in (36) which does not contain $\delta_{n0}$ and therefore does not vanish when we analytically continue in the frequency $i\omega_n \to \omega$. This term describes the scattering of a spin wave from the state with momentum $\mathbf{q}$ into a state with momentum $\mathbf{k} + \mathbf{q}$. In the following we shall be interested in the case of small frequencies $\omega$ and $k \ll 1$. For small k also $q \sim k$ is important in this term. The interaction of such long wavelength spin waves is small, as was discussed above. Therefore we need not take into account any complications corresponding to the interaction of spin waves in the limit of small k. In particular, the complication from higher powers of $\Sigma^{(1)}$ arising from the expansion of the denominator in (2) will be compensated by complications of the quantity $\Sigma$ itself. The three diagrams given in Fig. 3b corresponds thus, for example, simply to the interaction of virtual spin waves and for small k one can check that they cancel one another. Therefore it is in the given case sufficient simply to take for the quantity $K_{zz}(\mathbf{k}, \omega)$ the first order expression $\Sigma^{(1)}(\mathbf{k}, \omega)$:

$$K_{zz}(\mathbf{k}, \omega) = \Sigma_{zz}^{(1)}(\mathbf{k}, \omega) = T\sum_{\mathbf{q}}\frac{n_{\mathbf{q}}-n_{\mathbf{k+q}}}{\varepsilon_{\mathbf{k+q}}-\varepsilon_{\mathbf{q}}-\omega-i\delta}. \qquad (37)$$

As we said already the behavior of $K_{zz}$ in the small $k \ll 1$ region is of most interest. In that case Eq. (37) depends on the ratio of $\omega$ and k. We consider firstly the region of large frequencies $\omega \gg bV_0k = k/m$. Here $K_{zz}$ is real and is given by the expression

$$K_{zz}(\mathbf{k}, \omega) = -\frac{1}{\omega^2}\sum_{\alpha,\beta}k_\alpha k_\beta\sum_{\mathbf{q}}n_{\mathbf{q}}\frac{\partial^2\varepsilon_{\mathbf{q}}}{\partial q_\alpha \partial q_\beta}. \qquad (38)$$

In the region of low and high temperatures (38) takes the following form

a) $T \ll bV_0$.

$$K_{zz} = -\frac{2}{r_0^3}\frac{k^2}{m}\frac{T}{\omega^2}\Big(\frac{3mT}{2\pi}\Big)^{3/2}. \qquad (39a)$$

b) $\mu H \ll bV_0 \ll T$:

$$K_{zz} = \frac{T^2}{\omega^2}\sum_{\alpha,\beta}k_\alpha k_\beta\sum_{\mathbf{q}}\frac{\partial^2 V_{\mathbf{q}}}{\partial q_\alpha \partial q_\beta}\frac{1}{V_0 - V_{\mathbf{q}}}. \qquad (39b)$$

In Eq. (39b) the integral over $\mathbf{q}$ is a numerical tensor. For instance, for a simple cubic lattice with nearest neighbor interactions this expression equals $-0.18\, R_0^2\,\delta_{\alpha\beta}$.

For small frequencies $\omega \ll k/m$ and not too low temperatures $T \gg \epsilon_k$ the quantity $K_{zz}$ is given by the expression

$$K_{zz}(\mathbf{k}, \omega) = \frac{3\sqrt{3}}{2\pi r_0^3}\frac{m^2T^2}{k}\, i \ln\frac{2m\omega + k^2 + ik\sqrt{8m\mu H}}{2m\omega - k^2 + ik\sqrt{8m\mu H}}$$
$$= \frac{3\sqrt{3}}{2\pi r_0^3}\frac{m^2T^2}{k}\Big[\operatorname{arctg}\frac{k^2 - 2m\omega}{k\sqrt{8m\mu H}} + \operatorname{arctg}\frac{k^2 + 2m\omega}{k\sqrt{8m\mu H}} + \frac{i}{2}\ln\frac{(2m\omega + k^2)^2 + 8k^2m\mu H}{(2m\omega - k^2)^2 + 8k^2m\mu H}\Big]. \qquad (40)$$

For the imaginary part of $K_{zz}$ we can in this case obtain a closed expression even without assuming $\epsilon_k \ll T$:

$$\operatorname{Im} K_{zz}(\mathbf{k}, \omega) = \frac{3\sqrt{3}}{4\pi r_0^3}\frac{m^2T^2}{k}\ln\frac{1 - \exp[-\beta(2m\omega + k^2)^2/8mk^2 - \beta\mu H]}{1 - \exp[-\beta(2m\omega - k^2)^2/8mk^2 - \beta\mu H]}. \qquad (41)$$

It is clear from (40) and (41) that in a weak magnetic field there is in the correlation function $K_{zz}$ a peculiar "weak resonance" at the spin wave frequency $\omega = k^2/2m$. The imaginary part then increases logarithmically and the real part remains finite.

We can find from (40) the law for the decrease of the correlations in time for small k and large times t:

$$K_{zz}(\mathbf{k}, t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} d\omega\, e^{-i\omega t}K_{zz}(\mathbf{k}, \omega)$$
$$= \frac{3\sqrt{3}}{2\pi r_0^3}\frac{m^2T^2}{k}\frac{1}{t}\sin\frac{k^2t}{2m}t\exp\Big\{-tk\Big(\frac{8\mu H}{m}\Big)^{1/2}\Big\}. \qquad (42)$$

It is clear that thanks to the non-localized character of the interaction the spin diffusion through spin waves has an unusual character. When approaching the transition point the diffusion is retarded proportional to $(\mu H/m)^{1/2}$. To describe the diffusion in the case of very weak fields H and large times we must also take into account the spin wave damping discussed in Sec. 5, owing to which the law for the decrease of the correlations at large t will be exponential even at H = 0.

We note that for small k, $\omega$, H Eq. (37) for $K_{zz}$ increases according to (40) as $\min\{k^{-1}, k(m\omega)^{-1}, (\mu Hm)^{-1/2}\}$ and may become much larger than the first approximation term (34). This increase is connected with the large density of low momentum spin waves. We noted above that the interaction of low momentum spin waves is small so that the next approximation to $K_{zz}$ which takes spin wave interactions into account does not change the results. The $\mathbf{k}$-, $\omega$-, and H-dependence of the correlation function, given by Eqs. (38) to (41) is therefore in the region where these quantities are small the same also when the interaction range is not large or near the transition temperature when the parameter $\gamma|\tau|^{-1/2}$ is not small. In these cases only the temperature dependence changes. This can be seen from the sample of the term caused by spin waves which is proportional to $\sqrt{H}$ in the moment $\langle S^z \rangle$ in Eq. I(29c). This term, as we noted in I, leads in the small H region to a value of the susceptibility which increases as $H^{-1/2}$ and this can also be obtained from (40) putting $\omega = 0$, $k \to 0$. Taking into account the next approximation in $\gamma|\tau|^{-1/2}$ in this term in Eq. I(29c) corresponds as can be seen using (22c) simply to a correction to the spin wave mass.

When k increases the spin wave damping increases and near the transition point they exist only for small k determined by condition (33). In the derivation of relations (40) and (41) the spin wave damping was not taken into account. Therefore (40) and (41) will be valid near the transition only in the case if the $q^2$ which are important in the integral in (37) are $\sim \max(k^2, m\mu H, m^2\omega^2 k^{-1}) < |\tau|^\alpha$. For larger k, $\omega$, or H Eq. (40) is of the same order as the other terms in the expansion of $K_{zz}$ so that there is no longer any sense in splitting it off.

We considered above only the first approximation in $\Sigma_{zz}$ given by (36) and (37) and the diagrams of Fig. 3. More complicated diagrams describe not only corrections to the temperature dependence of the quantities but also relaxation processes and most importantly they influence the first order term in (34) and (35). Taking these processes into account must, in particular, lead to the fact that the "δ-function" term in (34) acquires a finite width $\Gamma_z$:

$$K_{zz}^{(0)}(\mathbf{k}, \omega) \to \frac{b'}{1 - \beta V_{\mathbf{k}} b'} \frac{\Gamma_z}{-i\omega + \Gamma_z}. \tag{43}$$

We did not succeed in finding the relaxation time $1/\Gamma_z$ since the self-consistent field method, as we reminded ourselves already, is not suitable for describing relaxation processes. However, as we noted above, the terms (37) to (41) are for small k, $\omega$, H and all temperatures below the critical one larger than (43) provided the interaction range $r_0$ is not very large.

## 7. NEUTRON SCATTERING

Nowadays there are two basic experimental methods to determine the correlation functions $K_{+-}$ and $K_{zz}$. The first method is the measurement of the magnetic susceptibility in a constant or high-frequency magnetic field. The transverse susceptibility then has a pole at the frequency corresponding to ferromagnetic resonance. However, the wavelength of ferromagnetic resonance is usual macroscopic, larger than or of the order of the dimensions of the sample. Therefore, its width is not determined by Eqs. (23)(29) but by relativistic effects of the magnetic interaction, the anisotropy field, and so on. For spinwaves with microscopic wavelengths these effects are small and they were not considered in the present paper.

A more effective method for the study of the **k**- and $\omega$-dependence of the correlation function is neutron scattering. The magnetic scattering cross-section for neutrons involving a momentum change by **k** and an energy change of $\omega$ is proportional to the spin correlation function: [5,11,7,8]

$$d\sigma \propto \sum_{\mathbf{r}} e^{-i\mathbf{kr}} \int_{-\infty}^{\infty} e^{i\omega t} dt \, \langle S_0^\alpha(0) S_r^\beta(t) \rangle = \frac{\operatorname{Im} K_{\alpha\beta}(\mathbf{k}, \omega)}{1 - e^{-\beta\omega}}. \tag{44}$$

We consider the $\omega$-dependence of the cross-section for different **k** and T. For all $T < T_c$ and small k satisfying condition (33) the quantities Im $K_{+-}$ and Im $K_{-+}$ which appear in (44) have steep maxima, respectively, at $\omega = \epsilon_k$ and $\omega = -\epsilon_k$; the width of these maxima is determined by Eqs. (23), (24), (29), and (30). Im $K_{zz}$ is in that region determined by Eq. (41) and is a smooth function with a width $\sim\epsilon_k$ which for H = 0 has only a logarithmic singularity in the point $\omega = \epsilon_k$. In the limiting case when $\omega \ll T$ and if one of the inequalities $k^2 \ll 2m\mu H$, $k^2 \ll 2m\omega$, $k^2 \gg 2m\omega$ is satisfied, we have

$$\operatorname{Im} K_{zz} = \frac{3\sqrt{3}}{2\pi r_0^3} \frac{T^2 m\omega k}{\omega^2 + k^2/4m^2 + 2k^2\mu H/m}. \tag{45}$$

If we take Van Hove's expression $\Gamma_z \sim |\tau| k^2$ for the width $\Gamma_z$ in (43) then there will be for small $\omega$ apart from the term (45) in Im $K_{zz}$ also a small peak with width $\sim|\tau| k^2$ described by Eq. (43).

When k increases or when we approach the transition point for fixed k the relative width of the spin wave peak increases. When $k^2 \gg |\tau|$ the correlation functions depend weakly upon $\tau$ and for a large interaction range are described by Van Hove's formula (43) both above and below the transition. The width of the peak in that region is not found in our paper.

In present-day experiments the cross-section integrated over frequencies is measured. When evaluating the integral of the right-hand side of (44) we must bear in mind that in the case considered of small k the important frequencies $\omega$ are small compared with the temperature and the denominator in (14) equals $\beta\omega$. Using the analyticity of $K(\mathbf{k}, \omega)$ in the upper half-plane of $\omega$ and the Kramers-Kronig relation we find that the cross-section integrated over the frequencies can be expressed in terms of the zero-frequency correlation function:

$$\int \sigma \, d\omega \sim K_{\alpha\beta}(\mathbf{k}, 0). \tag{46}$$

We consider the behavior of the **k**- and T-dependence of the total cross sections. Above the transition, all three static correlation functions have the Ornstein-Zernike form for large $r_0$. Everywhere below the transition $K_{+-}(\mathbf{k}, 0)$ is proportional to $k^{-2}$ in agreement with the results of de Gennes and Villain.[6] As we showed in section 6 the spin waves determine not only the behavior of $K_{+-}$ but also that of $K_{zz}$ which is proportional to 1/k.

The above described behavior of the correlation functions can also be used to study the polarization of neutrons.[11,12] The existing experimental data[13] agree qualitatively with the results obtained here, but they are insufficient for a quantitative comparison.

## 8. CONCLUSION

The basic result of this paper on the existence of spin waves even near the transition is valid only for sufficiently small momenta which satisfy condition (33). On the other hand, the momenta of the excitations must be sufficiently large so that the effects of anisotropy and magnetic interactions can be neglected. These effects lead to an additional damping which does not vanish for small k and $\omega$.[14] Taking this into account, condition (33) for the existence of spin waves has the form

$$\frac{(1+\beta)\mu^2}{V_0 v_c} \ll k^2 \ll \left(\frac{T_c - T}{T_c}\right)^{2\alpha}, \tag{47}$$

where $\beta$ is the anisotropy constant and $\mu$ the Bohr magneton. The expression on the left-hand side of (47) is of order of magnitude of $10^{-4}$ to $10^{-5}$. Therefore, for temperatures not extremely close to the transition there is a wide region of applicability of the results obtained.

We did not consider in this paper metallic ferromagnetics. The qualitative results are apparently also applicable to metals although there must occur additional limitations at the small k side which are connected with conduction electron damping. This damping can be considered, using Fermi liquid theory.[15]

In the foregoing we considered only ferromagnetics with one magnetic atom in the cell, such as EuO. To describe long-wavelength excitations the details of the structure are unimportant. The results obtained are therefore applicable also for a description of the low-frequency branch of spin waves in ferrites, i.e., ferromagnetics with a cell containing several magnetic atoms with an uncompensated spin.

The remarks made about the applicability of the results also apply to the correlation functions since the main contribution even to $K_{zz}$ is for small k given by the spin waves.


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