THEORY OF ELECTRON TUNNELING IN A SOUND FIELD IN SUPERCONDUCTORS

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Electron tunneling through a structure consisting of a normal metal, dielectric, and superconductor is considered for the case when a sound wave passes through this structure. The frequency dependence of the excess current produced by the sound field is found.

RECENTLY, a significant quantity of research has been devoted to the study of the tunnel current of superconductors subjected to the action of a sound field.[1-3] The problem in question took on special interest in connection with the study of the effects of the action of phonons on the volt-ampere characteristics of superconducting tunnel diodes.[3]

The present research is devoted to the study of the tunnel current of a structure consisting of a normal metal, dielectric, and superconductor in the case in which a sound wave passes through this structure. The sound wave modulates the tunnel current at its frequency, and as a result an effective tunnel current, which depends on the sound frequency, is added to the constant tunnel current. We shall study the volt-ampere characteristic of this current as a function of the sound frequency close to absolute zero temperature.

1. The Hamiltonian of the considered system we can write as follows:

$$H = H_0 + H_t, \quad H_0 = H_n + H_s + H_{int}, \tag{1}$$

where H_n and H_s are the Hamiltonians of the metals to the left and to the right of the barrier, and H_t and H_{int} describe the contact terms (see[4]) and the interaction of the sound with the electrons, respectively. The latter have the values

$$H_t = \iint d\mathbf{r}_1 d\mathbf{r}_2 [T(\mathbf{r}_1, \mathbf{r}_2)\psi^+(\mathbf{r}_1, t)\varphi(\mathbf{r}_2, t) + T^*(\mathbf{r}_1, \mathbf{r}_2)\varphi^+(\mathbf{r}_2, t)\psi(\mathbf{r}_1, t)],$$
$$H_{int} = \int d\mathbf{r}\, \hat{\lambda}_{\alpha\beta} u_{\alpha\beta}(x)\psi^+(\mathbf{r}, t)\psi(\mathbf{r}, t). \tag{2}$$

Here T is the amplitude of the transmission of the electron through the barrier, $u_{\alpha\beta}(x)$ is the deformation tensor of the body, $\hat{\lambda}_{\alpha\beta}$ is a tensor whose components are equal in order of magnitude to the Fermi energy, and $\psi(\mathbf{r}, t)$ and $\varphi(\mathbf{r}, t)$ are the electron field operators to the right and to the left of the barrier.

We introduce the following functions:[5]

$$G_>(\mathbf{r}, t, \mathbf{r}', t') = \frac{i\,\mathrm{Sp}\,\{\exp(iH_0\tau - iN\lambda)\psi^+(\mathbf{r}, t)\psi(\mathbf{r}, t)\}}{\mathrm{Sp}\,\{\exp(iH_0\tau - iN\lambda)\}}, \quad t > t',$$
$$G_<(\mathbf{r}, t, \mathbf{r}', t') = \frac{-i\,\mathrm{Sp}\,\{\exp(iH_0\tau - iN\lambda)\psi(\mathbf{r}', t')\psi^+(\mathbf{r}, t)\}}{\mathrm{Sp}\,\{\exp(iH_0\tau - iN\lambda)\}}, \quad t' > t, \tag{3}$$

where $i\lambda = \beta\mu$, $i\tau = \beta$ (μ is the chemical potential, β a quantity inverse to the temperature) and $N(\mathbf{r}, t) = \psi^+(\mathbf{r}, t)\psi(\mathbf{r}, t)$ is the operator corresponding to the number of particles. Let us find the time change in the particle number operator to the right of the barrier. Using the equation of motion of the operator and the first formula of (2), we obtain the following expression for the tunnel current between two metallic specimens (after averaging over the grand canonical ensemble) separated by a dielectric partition:

$$\langle \dot{N}(\mathbf{r}, t)\rangle = -\frac{2}{\hbar}\mathrm{Re}\int d\mathbf{r}'\, d\mathbf{r}_1\, d\mathbf{r}_2 T(\mathbf{r}_1, \mathbf{r}_2)T^*(\mathbf{r}, \mathbf{r}')$$
$$\times \int_{-\infty}^{t} dt' [G_<^{(1)}(\mathbf{r}_2, t', \mathbf{r}', t)G_>^{(2)}(\mathbf{r}, t, \mathbf{r}_1, t')$$
$$- \exp\{i(\lambda_1 - \lambda_2)\}\, G_<^{(1)}(\mathbf{r}_2, t', \mathbf{r}', t+\tau)G_>^{(2)}(\mathbf{r}, t+\tau, \mathbf{r}_1, t')]. \tag{4}$$

The functions $G_{\gtrless}(\mathbf{r}, t, \mathbf{r}', t')$ can be found by means of the solution of the Gor'kov equation, obtained with account of the interaction of the sound wave with the electrons.[6] By means of this solution and Eq. (3), one can obtain the following expression for the correction $\delta G_>(\mathbf{r}, t, \mathbf{r}', t')$ to the function $G_>(\mathbf{r}, t, \mathbf{r}', t')$ that is linear in the deformation tensor:

$$\delta G_>(\mathbf{r}, t, \mathbf{r}', t') = e^{-i\omega_0 t}\int_{-\infty}^{+\infty}\frac{d\omega}{2\pi}e^{i\omega(t-t')}$$
$$\times \int d\mathbf{r}''\hat{\lambda}_{\alpha\beta}u_{\alpha\beta}(\mathbf{r}'')[g_>(\mathbf{r}'', \mathbf{r}', \omega)g_>(\mathbf{r}, \mathbf{r}', \omega+\omega_0)$$
$$+ f_>(\mathbf{r}'', \mathbf{r}', \omega)f_>^+(\mathbf{r}, \mathbf{r}'', \omega+\omega_0)], \tag{5}$$

where $g_{\gtrless}(\mathbf{r}, \mathbf{r}', \omega)$, $f_{\gtrless}(\mathbf{r}, \mathbf{r}', \omega)$ are connected with the functions $G_{\gtrless}(\mathbf{r}, t, \mathbf{r}', t')$, $F_{\gtrless}(\mathbf{r}, t, \mathbf{r}', t')$ by the following relations:

$$G_{\lessgtr}(\mathbf{r}, t, \mathbf{r}', t') = \int_{-\infty}^{+\infty}\frac{d\omega}{2\pi i}e^{-i\omega(t-t')}g_{\lessgtr}(\mathbf{r}, \mathbf{r}', \omega),$$
$$F_{\lessgtr}(\mathbf{r}, t, \mathbf{r}', t') = \int_{-\infty}^{+\infty}\frac{d\omega}{2\pi i}e^{-i\omega(t-t')}f_{\lessgtr}(\mathbf{r}, \mathbf{r}', \omega). \tag{6}$$

We substitute (5) in (4) and transform to the momentum representation. We then get for the effective current

$$J_{\mathrm{eff}}(\omega_0) = \frac{c_{nn}\lambda_{\alpha\beta}u_\alpha^0 q_\beta}{\sqrt{2}(2\pi)^2}\int d\omega\iint d\xi_1 d\xi_2\iint\frac{do_1 do_2}{(4\pi)^2}$$
$$\times [A^{(2)}(\mathbf{p}_1+\mathbf{q}, \omega)A^{(2)}(\mathbf{p}_1, \omega-\omega_0) + B(\mathbf{p}_1, \omega-\omega_0)B^+(\mathbf{p}_1+\mathbf{q}, \omega)]$$
$$\times A^{(1)}(\mathbf{p}_2, \omega-\omega_0)f_2^+(\omega-\omega_0)[f_1^-(\omega-\omega_0) - f_2^-(\omega)]. \tag{7}$$

Here $c_{nn} = (2\pi e v_F|T|^2/\hbar)N_{1n}(0)N_{2n}(0)$ is the conductivity of the normal tunnel current, $N_{1n}(0)$, $N_{2n}(0)$ are the currents of the electron states of the metals to the left and to the right of the barrier, u_α^0 is the component of the displacement vector, and

$$f_{1,2}^{\pm}(\omega) = \frac{1}{\exp[\mp(\mu_{1,2}-\omega)/T] - 1}$$

The functions $A(\mathbf{p}, \omega)$ and $B(\mathbf{p}, \omega)$ can easily be determined if we equate the imaginary parts of the Green's function, found from (6), with the imaginary part of the retarded Green's function. It is easy to establish the fact that the following equations hold:

$$A(\mathbf{p}, \omega) = 2\,\mathrm{Im}\,G^R(\mathbf{p}, \omega), \quad B(\mathbf{p}, \omega) = 2\,\mathrm{Im}\,F^R(\mathbf{p}, \omega). \qquad (8)$$

If we take it into account that for the normal metal $A(\mathbf{p}, \omega) = 2\pi\delta(\omega - \xi_{\mathbf{p}} - \mu)$, then in place of Eq. (7), we get, after carrying out the integration over ξ_1,

$$J^{\mathrm{eff}}(\omega_0) = \frac{c_{nn}\lambda_{\alpha\beta}u_\alpha{}^0 q_\beta}{\sqrt{2}\cdot 2\pi}\int d\omega \int d\xi_{\mathbf{p}} \int \frac{do_{\mathbf{p}}}{4\pi} \times [A(\mathbf{p}, \omega - \omega_0)A(\mathbf{p}+\mathbf{q}, \omega) + B(\mathbf{p}, \omega-\omega_0)B^+(\mathbf{p}+\mathbf{q}, \omega)] \times f_2^+(\omega - \omega_0)[f_1^-(\omega-\omega_0) - f_2^-(\omega)]. \qquad (9)$$

The resultant expression describes the tunnel current of the normal metal—dielectric—superconductor structure, connected with the presence of a sound field of frequency ω_0. We note that Eq. (9) has a very general form and is valid both for ideal superconductors and for superconductors containing impurities.

2. Let us now consider the case in which we have an ideal superconductor to the right of the barrier. Using the expression for the retarded Green's function of superconductors,[6] we get, in accord with what was pointed out above,

$$A(\mathbf{p}, \omega) = \pi\left[\left(1 + \frac{\xi}{\sqrt{\xi^2+\Delta^2}}\right)\delta(\omega - \sqrt{\xi^2+\Delta^2}) + \left(1 - \frac{\xi}{\sqrt{\xi^2+\Delta^2}}\right)\delta(\omega + \sqrt{\xi^2+\Delta^2})\right],$$

$$B(\mathbf{p}, \omega) = \frac{i\pi}{\sqrt{\xi^2+\Delta^2}}[\delta(\omega - \sqrt{\xi^2+\Delta^2}) - \delta(\omega + \sqrt{\xi^2+\Delta^2})]. \qquad (10)$$

We substitute (10) in (9) and carry out integration over ξ and over the angles. Without account of the anisotropy of the electron-phonon interaction, we have, after several transformations, (11)

$$J^{\mathrm{eff}}(\omega_0) = \frac{c_{nn}\lambda_{\alpha\beta}u_\alpha{}^0 n_\beta}{8\sqrt{2}\,\hbar v_F}\int_{-\infty}^{+\infty} d\omega\, K(\omega, \omega_0) f_2^+(\omega-\omega_0)[f_1^-(\omega-\omega_0) - f_2^-(\omega)],$$

where

$$K(\omega, \omega_0) = \frac{\omega(\omega-\omega_0)}{\sqrt{(\omega-\omega_0)^2-\Delta^2}\sqrt{\omega^2-\Delta^2}} \times \left\{\left[1 + \frac{\sqrt{(\omega-\omega_0)^2-\Delta^2}\sqrt{\omega^2-\Delta^2}}{\omega(\omega-\omega_0)} - \frac{\Delta^2}{\omega(\omega-\omega_0)}\right]\varepsilon(1-|\Phi^-|) + \left[1 - \frac{\sqrt{(\omega-\omega_0)^2-\Delta^2}\sqrt{\omega^2-\Delta^2}}{\omega(\omega-\omega_0)} - \frac{\Delta^2}{\omega(\omega-\omega_0)}\right]\varepsilon(1-|\Phi^+|)\right. \qquad (12a)$$

$$\Phi^\pm = \frac{\sqrt{\omega^2-\Delta^2} \mp \sqrt{(\omega-\omega_0)^2-\Delta^2}}{qv_F}, \quad \varepsilon(x) = \begin{cases} 1, & x > 0 \\ 0, & x < 0 \end{cases} \qquad (12b)$$

and $n_\beta = q_\beta/|\mathbf{q}|$ is a vector in the direction of propagation of the sound.

Equations (11) and (12) for the tunnel current are greatly simplified if we make some assumption on the quantity $\mathbf{q}$—the wave vector of the sound. We limit ourselves to the practically most interesting case, when $|\mathbf{q}|v_F \gg \Delta$. Then we get from Eq. (11) for $T = 0$,

$$J^{\mathrm{eff}}(\omega_0) = \frac{c_{nn}\lambda_{\alpha\beta}u_\alpha{}^0 n_\beta}{4\sqrt{2}\,\hbar v_F}\int_\Delta^V \frac{\omega(\omega+\omega_0) - \Delta^2}{\sqrt{(\omega+\omega_0)^2-\Delta^2}\sqrt{\omega^2-\Delta^2}}\,d\omega \qquad (13)$$

for $V > \Delta$. For $V < \Delta$, we have $J^{\mathrm{eff}}(\omega_0) = 0$ (V is the shift in the Fermi level of the superconductor from that in the normal metal). Equation (13) can be expressed in terms of the elliptic integral; however, we shall present some limiting cases:

$$J^{\mathrm{eff}}(\omega_0) = \frac{c_{nn}\lambda_{\alpha\beta}u_\alpha{}^0 n_\beta}{4\sqrt{2}\,\hbar v_F} \times \begin{cases} \sqrt{\dfrac{\omega_0}{2\Delta}}\left(\sqrt{V^2-\Delta^2} - \dfrac{\omega_0}{4}\ln\dfrac{V+\sqrt{V^2-\Delta^2}}{\Delta}\right), & \omega_0 \gg V - \Delta \\ 2\sqrt{V^2-\Delta^2} + \dfrac{\Delta^2}{\omega_0}\ln\dfrac{V+\sqrt{V^2-\Delta^2}}{\Delta}, & \omega_0 \gg \Delta \end{cases}$$

3. Let us consider paramagnetic impurities. Taking into account the relation (8) and the fact that the retarded Green's function is the analytic continuation of the temperature Green's function in the upper complex plane ω, we can rewrite Eq. (9) in the following form:

$$J^{\mathrm{eff}}(\omega_0) = \frac{\sqrt{2}}{(2\pi)^2}c_{nn}\,\mathrm{Re}\int_{-\infty}^{+\infty} d\omega \int_{-\infty}^{+\infty} d\xi_{\mathbf{p}} \int do_{\mathbf{p}}\,\langle\lambda_{\alpha\beta}(\mathbf{p})u_\alpha{}^0 q_\beta \times \{G(\mathbf{p}, -i(\omega-\omega_0))G(\mathbf{p}+\mathbf{q}, -i\omega) + F(\mathbf{p}, -i(\omega-\omega_0))F^+(\mathbf{p}+\mathbf{q}, -i\omega)\}\rangle_{\mathrm{av}} \cdot f_2^+(\omega-\omega_0)[f_1^-(\omega-\omega_0) - f_2^-(\omega)]. \qquad (14)$$

In the latter formula, the angle brackets denote averaging over the positions of the randomly distributed impurities. We carry out the averaging over the impurities by analogy with[7,8,6]. Without repeating the calculations of[6], we can write down the following expression for the averaged values of the effective tunnel current:

$$J^{\mathrm{eff}}(\omega_0) = \sqrt{2}\,\frac{\overline{\lambda_{\alpha\beta}u_\alpha{}^0 n_\beta}\,\omega_0\tau'}{\hbar s}\int_{\omega'}^{V} d\omega\left[\mathrm{Im}\frac{u}{\sqrt{1-u^2}}\,\mathrm{Im}\frac{u_+}{\sqrt{1-u_+^2}} - \mathrm{Im}\frac{1}{\sqrt{1-u^2}}\,\mathrm{Im}\frac{1}{\sqrt{1-u_+^2}}\right]. \qquad (15)$$

Here $u_+ = u(\omega + \omega_0)$,

$$\frac{1}{\tau'} = \frac{1}{\tau_1} - \frac{1}{\tau_1'},$$

$$\frac{1}{\tau_1} = \frac{nmp_F}{2\pi^2}\int\left[|U_1(\theta)|^2 + \frac{1}{3}S(S+1)|U_2(\theta)|^2\right]do,$$

$$\frac{1}{\tau_1'} = \frac{nmp_F}{2\pi^2\bar{\varphi}}\int\left[|U_1(\theta)|^2 + \frac{1}{3}S(S+1)|U_2(\theta)|^2\right]\varphi(\mathbf{p})\,do,$$

$$\overline{\varphi(\mathbf{p})} = \overline{\lambda_{\alpha\beta}(\mathbf{p})u_\alpha{}^0 n_\beta} = \frac{1}{4\pi}\int \lambda_{\alpha\beta}(\mathbf{p})u_\alpha{}^0 n_\beta\,do, \qquad (16)$$

while the following relation holds between ω and u:

$$\frac{\omega}{\Delta} = u\left(1 - \frac{1}{\tau_s\Delta}\frac{1}{\sqrt{1-u^2}}\right), \quad \frac{1}{\tau_s} = \frac{7}{24}\frac{nmp_F S(S+1)}{(2\pi)^2}\int|U_2(\theta)|^2\,do, \qquad (17)$$

where $U_1(\theta)$, $U_2(\theta)$ are the amplitudes of the exchange and non-exchange scattering of the electrons on the impurities, and S is the spin of the impurity.

In Eq. (15), $\omega' = \Delta[1 - (\tau_S\Delta)^{-2/3}]^{3/2}$ represents a gap in the energy spectrum of the superconductor. To find the expression in the curly brackets, it is necessary to know the values of $\mathrm{Im}(u/\sqrt{1-u^2})$, $\mathrm{Im}(1/\sqrt{1-u^2})$, which always depend in significant fashion on the parameter $1/\tau_S\Delta$. Taking it into account that the quantities differ from zero when $\omega > \omega'$, we can expand them in series in $u - u'$ (u' is the value of u at $\omega = \omega'$). With the aid of the relations (17), we obtain for $\tau_S\Delta \gg 1$

$$\operatorname{Im}\frac{1}{\sqrt{1-u_+^2}} = -\sqrt{\frac{2}{3}}[1-(\tau_s\Delta)^{-2/3}]^{1/4}(\tau_s\Delta)^{2/3}\left(\frac{\omega+\omega_0-\omega'}{\Delta}\right)^{1/2}\varepsilon(\omega+\omega_0-\omega'),$$
$$\operatorname{Im}\frac{u_+}{\sqrt{1-u_+^2}} = -\sqrt{\frac{2}{3}}[1-(\tau_s\Delta)^{-2/3}]^{-1/4}(\tau_s\Delta)^{2/3}\left(\frac{\omega+\omega_0-\omega'}{\Delta}\right)^{1/2}\varepsilon(\omega+\omega_0-\omega'). \tag{18}$$

Considering the small shift V in (15) from the gap ω', and taking the last equations into account, we have

$$J_{\text{eff}}(\omega_0) = \frac{2\sqrt{2}}{3}c_{nn}\overline{\lambda_{\alpha\beta}(\mathbf{p})u_\alpha{}^0 n_\beta}\frac{\omega_0\tau'}{\hbar s}J(\omega_0)\left[1-\left(\frac{\omega'}{\Delta}\right)^{2/3}\right]^{-1}\left(\frac{\omega'}{\Delta}\right)^{-1/3}; \tag{19}$$
$$J(\omega_0) = \frac{(V-\omega')^2}{2\Delta} - \frac{\omega_0^2}{8\Delta}\ln\frac{4(V-\omega')}{\omega_0}, \quad V-\omega' \gg \omega_0,$$
$$J(\omega_0) = \frac{\omega_0}{4\Delta}\sqrt{\omega_0(V-\omega')} - \frac{\omega_0^2}{8\Delta}\ln\left(1+2\sqrt{\frac{V-\omega'}{\omega_0}}\right), \quad V-\omega' \ll \omega_0.$$

We shall consider the gap-free case. As is easy to see, for $1/\tau_S\Delta = 1$, the quantity $\omega' = 0$, i.e., the gap in the spectrum disappears. By using Eq. (17), we can easily see that in this case,

$$\operatorname{Im}\frac{1}{\sqrt{1-u_+^2}} = \frac{\sqrt{3}}{4}\left[\frac{2(\omega+\omega_0)}{\Delta}\right]^{2/3},$$
$$\operatorname{Im}\frac{u_+}{\sqrt{1-u_+^2}} = \frac{\sqrt{3}}{2}\left[\frac{2(\omega+\omega_0}{\Delta}\right]^{1/3}. \tag{20}$$

By substituting the latter formula in (15) and considering small frequencies and shift, we get

$$J_{\text{eff}}(\omega_0) \approx 0{,}2\overline{\lambda_{\alpha\beta}(\mathbf{p})u_\alpha{}^0 n_\beta}\frac{\omega_0\tau'}{\hbar s}\left(\frac{2V}{\Delta}\right)^{5/3}. \tag{21}$$

For sufficiently high concentrations of paramagnetic impurities, when $\tau_S\Delta \ll 1$, Eq. (17) gives

$$\operatorname{Im}\frac{1}{\sqrt{1-u_+^2}} = \frac{(\tau_s\Delta)^2}{\sqrt{1-(\tau_s\Delta)^2}}\left(\frac{\omega+\omega_0}{\Delta}\right)^2,$$
$$\operatorname{Im}\frac{u_+}{\sqrt{1-u_+^2}} = \tau_s\Delta + \frac{3}{2}(\tau_s\Delta)^4[1-(\tau_s\Delta)^2]^{-3/2}\left(\frac{\omega+\omega_0}{\Delta}\right)^2 \tag{22}$$

and for small frequencies and displacements, we get

$$J_{\text{eff}}(\omega_0) \approx \sqrt{2}\,\overline{\lambda_{\alpha\beta}(\mathbf{p})u_\alpha{}^0 n_\beta}\frac{\omega_0\tau'}{\hbar s}V[1+(\tau_s\Delta)^2].$$

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