THE SYSTEM WHOSE HAMILTONIAN IS A TIME-DEPENDENT QUADRATIC FORM IN $\hat{x}$ AND $\hat{p}$

N. A. CHERNIKOV

Joint Institute for Nuclear Research

Submitted March 22, 1967

Zh. Eksp. Teor. Fiz. 53, 1006–1017 (September, 1967)

The solution of the Schrödinger equation can be expressed algebraically in terms of the solution of the classical Hamiltonian equations in the case in which the Hamiltonian is a quadratic form in the coordinates and momenta and the coefficients of the form are arbitrary functions of the time. The Fock representation is introduced for such a system. The results obtained are useful for the formulation of quantum field theory in an arbitrary space-time world.

1. INTRODUCTION

In the present paper we consider the Schrödinger equation

$$ih\,\partial\Psi/\partial t = H(t, \hat{x}, \hat{p})\Psi \qquad (1.1)$$

for a system with ν degrees of freedom and with Hamiltonian an arbitrarily time-dependent quadratic form in the coordinates $\hat{x}$ and the momenta $\hat{p}$:

$$H(t, \hat{x}, \hat{p}) = {}^1/_2\{A_{\alpha\beta}(t)\hat{x}^\alpha\hat{x}^\beta + B_\alpha{}^\beta(t)(\hat{x}^\alpha\hat{p}_\beta + \hat{p}_\beta\hat{x}^\alpha) + C^{\alpha\beta}(t)\hat{p}_\alpha\hat{p}_\beta\}. \qquad (1.2)$$

Here and in what follows it is understood that repeated Greek indices are summed from 1 to ν. The problem of field quantization in an arbitrarily prescribed pseudo-riemannian space-time world reduces to this very problem, except that the number of degrees of freedom is infinite. The present problem, however, is obviously interesting quite apart from its applications in quantum field theory.

It is, of course, impossible to solve this problem in the general case in terms of already studied functions. This is impossible even for the corresponding classical Hamiltonian equations

$$\frac{dx^\alpha}{dt} = \frac{\partial H}{\partial p_\alpha} = B_\beta{}^\alpha(t)x^\beta + C^{\alpha\beta}(t)p_\beta, \qquad (1.3a)$$

$$\frac{dp_\alpha}{dt} = -\frac{\partial H}{\partial x^\alpha} = -A_{\beta\alpha}(t)x^\beta - B_\alpha{}^\beta(t)p_\beta. \qquad (1.3b)$$

We shall proceed to show, however, how to construct a complete system of solutions of Eq. (1.1), as soon as a general solution of Eqs. (1.3),

$$x^\alpha = K_\beta{}^\alpha(t)x_0{}^\beta + L^{\alpha\beta}(t)p_\beta{}^0, \quad x^\alpha|_{t=t_0} = x_0{}^\alpha, \qquad (1.4a)$$

$$p_\alpha = M_{\beta\alpha}(t)x_0{}^\beta + N_\alpha{}^\beta(t)p_\beta{}^0, \quad p_\alpha|_{t=t_0} = p_\alpha{}^0. \qquad (1.4b)$$

is known. Accordingly, it will be shown here that the behavior of the quantum system with the Hamiltonian (1.2) is actually known as soon as one knows the behavior of the corresponding classical system.

In the next section an attempt is made to satisfy the Schrödinger equation with a function of the form

$$\Psi = \sqrt{\rho}e^{i\sigma/h}, \qquad (1.5)$$

by equating the coefficients of all powers of h to zero. It turns out that this is possible for the Hamiltonian (1.2). In Section 3 the general solution of this form is found. We shall call it the fundamental solution of the Schrödinger equation. Each fundamental solution is characterized by $\nu + 1$ complex parameters $u_0, u_1, \ldots, u_\nu$ and $\nu(\nu + 1)/2$ complex parameters $S_{\alpha\beta} = S_{\beta\alpha}$. The parameters $u_0, u_1, \ldots, u_\nu$ are arbitrary. There are no conditions imposed on the real part of the matrix $S_{\alpha\beta} = R_{\alpha\beta} + iQ_{\alpha\beta}$, except the condition that it be symmetric. The matrix $Q_{\alpha\beta}$ is assumed to be positive definite. This is necessary in order that the fundamental solution possess a norm. The function ρ satisfies the equation of continuity and is given by

$$\rho = \rho_0\|K_\beta{}^\alpha(t) + L^{\alpha\gamma}(t)S_{\gamma\beta}\|^{-1}, \qquad (1.6)$$

where ρ_0 is a normalization constant. The function σ satisfies the Hamilton-Jacobi equation

$$\frac{\partial\sigma}{\partial t} + \frac{1}{2}\left\{A_{\alpha\beta}(t)x^\alpha x^\beta + 2B_\alpha{}^\beta(t)x^\alpha\frac{\partial\sigma}{\partial x^\beta} + C^{\alpha\beta}(t)\frac{\partial\sigma}{\partial x^\alpha}\frac{\partial\sigma}{\partial x^\beta}\right\} = 0 \qquad (1.7)$$

and the initial condition

$$\sigma|_{t=t_0} = u_0 + u_\alpha x^\alpha + {}^1/_2 S_{\alpha\beta}x^\alpha x^\beta. \qquad (1.8)$$

In relation to the parameters $u_0, u_1, \ldots, u_\nu$ the function σ is a complete integral of Eq. (1.7).

By differentiating the fundamental solution of the Schrödinger equation with respect to the parameters u_α we can get more and more new solutions of the equa tion. This has enabled us in Sec. 4 to introduce a generating Ψ function, which is itself a solution of the Schrödinger equation and gives a complete orthogonal system of solutions. These last differ from the fundamental solution by factors which are Hermite-Chebyshev polynomials of certain linear combinations of the coordinates x^α with coefficients which are functions of the time.

Owing to this, we have been able in Section 5 to introduce the Fock representation,[1] which is so convenient in quantum field theory, for our present case of the system with the Hamiltonian (1.2). According to the terminology used in quantum field theory, the fundamental solution of the Schrödinger equation is called the vacuum state; the other solutions given by the generating function are states with prescribed numbers of particles. The operators $\hat{x}^\alpha$ and $\hat{p}_\alpha$ then appear in the role of field operators. Operators for annihilation and creation of particles can be expressed in terms of the $\hat{x}^\alpha$ and $\hat{p}_\alpha$. Owing to the finite number of degrees of freedom of the system considered here, the configuration space of the particles introduced in this way consists of ν points in all.

This result is nontrivial even for the simplest system, with A = E, B = 0, C = E; the well known result here applies only to the extremely special case u = 0, R = 0, Q = E.

Having defined the vacuum state as the above-mentioned fundamental solution with the parameters S, u, we have considerably broadened this concept, because the usually accepted practice is to associate the vacuum state with the smallest value of the energy. In the general case of the time-dependent Hamiltonian (1.2) an analog of the energy could be its mean vacuum value. In our treatment, however, the Hamiltonian (1.2) does not necessarily have to be positive definite. Therefore in the general case we cannot raise the question of its lowest mean value. But even if we stipulate that the Hamiltonian (1.2) is positive definite at all times, its vacuum average will depend on the time, and it is not possible to minimize it at each value of the time by varying the constants S and u. Indeed, in this way we can arrive at the equation u = 0. It can in fact be shown that

$$\langle 0|H(t,\hat{x},\hat{p})|0\rangle = H(t,\bar{x},\bar{p}) + \frac{h}{4}\operatorname{Sp} Q^{-1}(K'+S^*L'M+S^*N)\begin{pmatrix} A & B' \\ B & C\end{pmatrix}\begin{pmatrix} K+LS \\ M'+N'S\end{pmatrix}, \tag{1.9}$$

where

$$\bar{x} = \langle 0|\hat{x}|0\rangle = \frac{i}{2}[(K+LS^*)Q^{-1}u' - (K+LS)Q^{-1}u^+],$$
$$\bar{p} = \langle 0|\hat{p}|0\rangle = \frac{i}{2}[uQ^{-1}(M+S^*N) - u^*Q^{-1}(M+SN)]. \tag{1.10}$$

The matrix notations used here are explained in Sec. 3. The vacuum averages $\bar{x}$, $\bar{p}$ obey the classical Hamiltonian equations (1.3). If the Hamiltonian is positive definite at all times, then $H(t, \bar{x}, \bar{p}) > 0$ whenever not all $\bar{x}$ and $\bar{p}$ are equal to zero, and H(t, 0, 0) = 0. But if $\bar{x} = 0$, $\bar{p} = 0$, then u = 0. The trace of the matrix in the second term of (1.9) can depend on the time, and then we cannot minimize it at each value of the time with any choice of the parameters S.

The minimum principle also does not work in the case of a Hamiltonian which does not depend on the time but is not positive definite. For example, for A = 0, B = 0, C = E, we would get from the minimum principle $\bar{p} = 0$, S = 0. This last equation, however, contradicts the condition that the matrix Q be positive definite. Moreover, $\bar{x}$ remains entirely arbitrary. Nevertheless the Fock representation can be used in the most general case of the Hamiltonian (1.2), and so also can our proposed broad interpretation of the vacuum. If indeed there exists some positive definite integral of the motion, an optimal choice of the parameters can be prescribed by the condition that its vacuum average be a minimum.

2. ATTEMPT TO SOLVE THE SCHRÖDINGER EQUATION

We shall solve the Schrödinger equation in the x representation, setting

$$\Psi = \Psi(t, x^1, \ldots, x^\nu), \quad \hat{x}^\alpha = x^\alpha, \quad \hat{p}_\alpha = -ih\frac{\partial}{\partial x^\alpha}. \tag{2.1}$$

If we represent Ψ in the form (1.5), where ρ and σ are functions of r, $x^1, \ldots, x^\nu$ (we do not require that they be real!), Eq. (1.1) can be rewritten in the form

$$\frac{\partial\sigma}{\partial t} + H\left(t, x, \frac{\partial\sigma}{\partial x}\right) = \frac{ih}{2\rho}\left[\frac{\partial\rho}{\partial t} + \frac{\partial}{\partial x^\alpha}(\rho v^\alpha)\right] + \frac{h^2}{2\sqrt{\rho}}C^{\alpha\beta}(t)\frac{\partial^2\sqrt{\rho}}{\partial x^\alpha\partial x^\beta} \tag{2.2}$$

where

$$v^\alpha = B_\beta{}^\alpha(t)x^\beta + C^{\alpha\beta}(t)\frac{\partial\sigma}{\partial x^\beta}. \tag{2.3}$$

We shall try to solve Eq. (2.2) by equating the coefficients of powers of h to zero. The zeroth order gives the Hamilton-Jacobi equation (1.7), the first order, the equation of continuity

$$\frac{\partial\rho}{\partial t} + \frac{\partial}{\partial x^\alpha}(\rho v^\alpha) = 0, \tag{2.4}$$

and, finally, the second order gives the equation

$$C^{\alpha\beta}(t)\frac{\partial^2\sqrt{\rho}}{\partial x^\alpha\partial x^\beta} = 0. \tag{2.5}$$

Suppose that the function $\sigma = \sigma(t, x^1, \ldots, x^\nu)$ satisfies the Hamilton-Jacobi equation (1.7) and that its value at $t = t_0$ is

$$\sigma|_{t=t_0} = \varphi(x^1, \ldots, x^\nu).$$

By the use of the well known Liouville-Lindelöf theorem in the theory of ordinary differential equations, it is not hard to find the function ρ from (2.4). Because the point is an important one we give the detailed calculation.

If we substitute $p_\beta = \partial\sigma/\partial x^\beta$ in (1.3a), we get a system of ν ordinary differential equations

$$\frac{dx^\alpha}{dt} = B_\beta{}^\alpha(t)x^\beta + C^{\alpha\beta}(t)\frac{\partial}{\partial x^\beta}\sigma(t, x^1, \ldots, x^\nu) = v^\alpha. \tag{2.6}$$

We find the general solution of this system if we substitute $p^0_\beta = \partial\varphi(x_0^1, \ldots, x_0^\nu)/\partial x_0^\beta$ in (1.4a)

$$x^\alpha = K_\beta{}^\alpha(t)x_0{}^\beta + L^{\alpha\beta}(t)\partial\varphi(x_0{}^1, \ldots, x_0{}^\nu)/\partial x_0{}^\beta. \tag{2.7}$$

From this we can get the first integrals $x_0^\alpha = x_0^\alpha(t, x^1, \ldots, x^\nu)$ of the system (2.6). They obey the partial differential equation

$$\frac{\partial x_0{}^\alpha}{\partial t} + v^\beta\frac{\partial x_0{}^\alpha}{\partial x^\beta} = 0. \tag{2.8}$$

Differentiating (2.8) with respect to x^γ, we get

$$\left(\frac{\partial}{\partial t} + v^\beta\frac{\partial}{\partial x^\beta}\right)\frac{\partial x_0{}^\alpha}{\partial x^\gamma} = -\frac{\partial v^\beta}{\partial x^\gamma}\frac{\partial x_0{}^\alpha}{\partial x^\beta}. \tag{2.9}$$

Consequently,

$$\left(\frac{\partial}{\partial t} + v^\beta\frac{\partial}{\partial x^\beta}\right)\frac{\partial(x_0{}^1, \ldots, x_0{}^\nu)}{\partial(x^1, \ldots, x^\nu)} = -\frac{\partial(x_0{}^1, \ldots, x_0{}^\nu)}{\partial(x^1, \ldots, x^\nu)}\frac{\partial v^\beta}{\partial x^\beta}. \tag{2.10}$$

And therefore we have found the solution of Eq. (2.4):

$$\rho = \rho_0\frac{\partial(x_0{}^1, \ldots, x_0{}^\nu)}{\partial(x^1, \ldots, x^\nu)} = \rho_0\left\|K_\beta{}^\alpha(t) + L^{\alpha\gamma}(t)\frac{\partial^2}{\partial x_0{}^\gamma\partial x_0{}^\beta}\varphi(x_0{}^1, \ldots, x_0{}^\nu)\right\|^{-1}. \tag{2.11}$$

In order to satisfy (2.5) it is sufficient to prescribe the initial function φ in the form of the second-degree polynomial (1.8). When this is done ρ does not depend on $x^1, \dots, x^\nu$ at all and is given by (1.6).

Accordingly, our attempt to solve Eq. (2.2), and thus also the Schrödinger equation, has been justified. The only thing remaining is to solve the Hamilton-Jacobi equation (1.7) with the initial function (1.8).

3. THE FUNDAMENTAL SOLUTION

For brevity and, we may add, expressivity in the writing we use matrix calculations. We lay out sets of quantities of the forms x^α, $A_{\alpha\beta}$, B^β, $C^{\alpha\beta}$ in the following way:

$$x = \begin{pmatrix} x^1 \\ \vdots \\ x^\nu \end{pmatrix}, \quad A = \begin{pmatrix} A_{11} & \cdots & A_{1\nu} \\ \vdots & & \vdots \\ A_{\nu 1} & \cdots & A_{\nu\nu} \end{pmatrix},$$

$$B = \begin{pmatrix} B_1{}^1 & \cdots & B_\nu{}^1 \\ \vdots & & \vdots \\ B_1{}^\nu & \cdots & B_\nu{}^\nu \end{pmatrix}, \quad C = \begin{pmatrix} C^{11} & \cdots & C^{1\nu} \\ \vdots & & \vdots \\ C^{\nu 1} & \cdots & C^{\nu\nu} \end{pmatrix}.$$

We arrange a set of quantities of the form p_α in a row $p = (p_1, \dots, p^\nu)$. The derivative operators $\partial/\partial x^\alpha$ form a row $\partial/\partial x$, and the operators $\partial/\partial p_\alpha$ form a column $\partial/\partial p$. We shall regard columns and rows of the types x, p as square matrices. Transposition will be denoted by a prime. Thus the transposed column x becomes a row x′, and the transposed row p is a column p′. The matrices A and C which appear in the Hamiltonian are symmetric, i.e., $A' = A$, $C' = C$. The matrix S which appears in the initial condition (1.8) is also symmetric.

In matrix notation the Hamiltonian equations can be written in the form

$$\frac{dx}{dt} = \frac{\partial H}{\partial p} = Bx + Cp', \quad \frac{dp}{dt} = -\frac{\partial H}{\partial x} = -x'A - pB, \tag{3.1}$$

and their solution can be written in the form

$$\begin{aligned} x &= Kx_0 + Lp_0', \quad x|_{t=t_0} = x_0, \\ p &= x_0'M + p_0N, \quad p|_{t=t_0} = p_0. \end{aligned} \tag{3.2}$$

We need to solve the Hamilton-Jacobi equation

$$\frac{\partial\sigma}{\partial t} + \frac{1}{2}\left\{x'Ax + 2\frac{\partial\sigma}{\partial x}Bx + \frac{\partial\sigma}{\partial x}C\frac{\partial\sigma}{\partial x'}\right\} = 0 \tag{3.3}$$

with the initial condition

$$\sigma|_{t=t_0} = \varphi(x) = u_0 + ux + {}^1\!/_2 x'Sx. \tag{3.4}$$

By the method of characteristics we have

$$\frac{d\sigma}{dt} = -H + p\frac{\partial H}{\partial p} = \frac{1}{2}\{pCp' - x'Ax\} = \frac{1}{2}\frac{d}{dt}px. \tag{3.5}$$

From this we get

$$\sigma = \varphi(x_0) - \frac{1}{2}\frac{\partial\varphi(x_0)}{\partial x_0}x_0 + \frac{1}{2}px = u_0 + \frac{1}{2}(ux_0 + px). \tag{3.6}$$

In order to express x_0 and p in terms of x we must substitute

$$p_0 = \frac{\partial}{\partial x_0}\varphi(x_0) = u + x_0'S \tag{3.7}$$

in the solution (3.2) of the Hamiltonian equations (3.1):

$$x = (K + LS)x_0 + Lu', \quad p = x_0'(M + SN) + uN. \tag{3.8}$$

From this we can find x_0, and then p, under the condition that $\|K + LS\| \neq 0$. We shall prove this last inequality, assuming that the Ψ function (1.5) has a norm. To do so we need some information about the matrices K, L, M, N.

If the phase-space vectors $\begin{pmatrix} y \\ q' \end{pmatrix}$ and $\begin{pmatrix} x \\ p' \end{pmatrix}$ both satisfy the Hamiltonian equations (3.1), then

$$d(qx - py)/dt = 0, \text{ i.e. } qx - py = q_0x_0 - p_0y_0. \tag{3.9}$$

Consequently, the solution (3.2) of the Hamiltonian equations (3.1) gives a symplectic transformation of phase space. From (3.9) there follows directly the matrix equation

$$\begin{pmatrix} K' & M \\ L' & N \end{pmatrix}\begin{pmatrix} 0 & -E \\ E & 0 \end{pmatrix}\begin{pmatrix} K & L \\ M' & N' \end{pmatrix} = \begin{pmatrix} 0 & -E \\ E & 0 \end{pmatrix}. \tag{3.10}$$

From this we get

$$\begin{pmatrix} N & -L' \\ -M & K' \end{pmatrix}\begin{pmatrix} K & L \\ M' & N' \end{pmatrix} = \begin{pmatrix} E & 0 \\ 0 & E \end{pmatrix} = \begin{pmatrix} K & L \\ M' & N' \end{pmatrix}\begin{pmatrix} N & -L' \\ -M & K' \end{pmatrix}, \tag{3.11}$$

i.e.,

$$\begin{aligned} &MK = K'M', \quad KL' = LK', \quad KN - LM = N'K' - M'L' = E, \\ &NL = L'N', \quad M'N = N'M, \quad NK - L'M' = K'N' - ML = E. \end{aligned} \tag{3.12}$$

It can also be shown that the determinant of any symplectic transformation is equal to unity. The group properties of symplectic transformations are obvious. Equation (3.11) enables us to solve the equations (3.2) for x_0 and p_0:

$$x_0 = Nx - L'p', \quad p_0 = -x'M' + pK. \tag{3.13}$$

This amount of information about the matrices K, L, M, N will be enough for our purposes.

We now resolve the matrix S into real and imaginary parts

$$S = R + iQ. \tag{3.14}$$

In order for the Ψ function (1.5) to have a norm at $t = t_0$, the quadratic form $x'Qx$ must be positive definite. We shall show that in this case the matrix $K + LS$ has an inverse. First we note that the matrix

$$\begin{pmatrix} K + LR & L \\ M' + N'R & N' \end{pmatrix} = \begin{pmatrix} K & L \\ M' & N' \end{pmatrix}\begin{pmatrix} E & 0 \\ R & E \end{pmatrix} \tag{3.15}$$

is a symplectic one, since both factors in this product are symplectic matrices. In view of the fact that the determinant of a symplectic matrix is not zero, the rows of the rectangular matrix $(K + LR\ L)$ are linearly independent. If in the space of these rows we define a scalar product by means of the matrix $\begin{pmatrix} Q^{-1} & 0 \\ 0 & Q^{-1} \end{pmatrix}$, then the product

$$\begin{aligned} &(K + LR\ L)\begin{pmatrix} Q^{-1} & 0 \\ 0 & Q \end{pmatrix}\begin{pmatrix} K' + RL' \\ L' \end{pmatrix} \\ &= (K + LR)Q^{-1}(K' + RL') + LQL' = G \end{aligned} \tag{3.16}$$

will be the Gram determinant for these rows. Owing to this the quadratic form $x'Gx$ is positive definite and $\|G\| \neq 0$. But the matrix G can be resolved into the product

$$G = (K + LS)Q^{-1}(K' + S^*L') = (K + LS^*)Q^{-1}(K' + SL'). \tag{3.17}$$

To verify this one must use (3.12). It follows from (3.17) that the matrix K + LS has an inverse.

We can now solve Eq. (3.8). It is, however, more convenient to use a different equation for the determination of p, namely

$$p(K+LS)-x'(M'+N'S)-u=0, \qquad (3.18)$$

which follows from (3.13) is we substitute (3.7) in that equation. From (3.8) we find

$$x_0=(K+LS)^{-1}x-(K+LS)^{-1}Lu'. \qquad (3.19)$$

and from (3.18)

$$p=x'(M'+N'S)(K+LS)^{-1}+u(K+LS)^{-1}. \qquad (3.20)$$

Substituting the last two expressions in (3.6), we finally find the function σ:

$$\sigma=u_0-\frac{1}{2}u\Gamma Lu'+u\Gamma x+\frac{i}{2}x'\Omega x, \qquad (3.21)$$

where

$$\Gamma=(K+LS)^{-1},\quad i\Omega=(M'+N'S)(K+LS)^{-1}. \qquad (3.22)$$

With the use of (3.12) it is not hard to show that $\Omega'=\Omega$, $\Gamma L=L'\Gamma'$.

Combining the formulas (1.5), (1.6), and (3.21), we get the fundamental solution of the Schrödinger equation:

$$\Psi_0(u)=\sqrt{\rho_0\|\Gamma\|}\exp\left\{-\frac{x'\Omega x}{2h}+\frac{iu\Gamma x}{h}+\frac{iu_0}{h}-\frac{iu\Gamma Lu'}{2h}\right\}. \qquad (3.23)$$

We note that the matrix Ω satisfies a Riccati equation

$$i\frac{d\Omega}{dt}+A+i\Omega B+iB'\Omega-\Omega C\Omega=0,\quad i\Omega|_{t=t_0}=S. \qquad (3.24)$$

This follows from the Hamiltonian equations (1.3), which give

$$\frac{d}{dt}\begin{pmatrix}K & L\\ M' & N'\end{pmatrix}=\begin{pmatrix}B & C\\ -A & -B'\end{pmatrix}\begin{pmatrix}K & L\\ M' & N'\end{pmatrix},\quad \begin{pmatrix}K & L\\ M' & N'\end{pmatrix}_{t=t_0}=\begin{pmatrix}E & 0\\ 0 & E\end{pmatrix}. \qquad (3.25)$$

If the matrix Ω is known, it is not necessary to solve (1.3). In fact, in this case we can find the matrix Γ from the equation

$$\frac{d\Gamma}{dt}=-\Gamma(B+iC\Omega),\quad \Gamma|_{t=t_0}=E, \qquad (3.26)$$

which also follows from (3.25). Knowing Ω and Γ, we can find ΓL by an algebraic procedure. Namely, from (3.17) we find

$$\Gamma L=-\frac{i}{2}\Gamma[(K+LS)-(K+LS^*)]Q^{-1}=-\frac{i}{2}Q^{-1}+\frac{i}{2}\Gamma G\Gamma' \qquad (3.27)$$

But it follows from (3.17) and (3.12) that

$$\Omega=G^{-1}-i[(M'+N'R)Q^{-1}(K'+RL')+N'QL']G^{-1}. \qquad (3.28)$$

Accordingly

$$\Gamma L=-\frac{i}{2}Q^{-1}+i\Gamma[\Omega+\Omega^*]^{-1}\Gamma'. \qquad (3.29)$$

If the matrices A, B, C do not depend on the time and S is a root of the equation

$$A+SB+B'S+SCS=0, \qquad (3.30)$$

then, according to (3.24), iΩ = S. In this case we have from (3.26) and (3.29)

$$\Gamma=e^{(B+CS)(t_0-t)},\quad \Gamma L=-\frac{i}{2}Q^{-1}+\frac{i}{2}\Gamma Q^{-1}\Gamma'. \qquad (3.31)$$

It may also be, however, that Eq. (3.30) has no root with positive definite imaginary part Q, and then the matrix Ω cannot be regarded as constant, as, for example, in the case[1)] A = 0, B = 0, C = E. But if the Hamiltonian (1.2) is positive definite Eq. (3.30) does have such a root. For this value of the root the trace of the matrix which appears in (1.9) is a minimum.

4. THE GENERATING Ψ FUNCTION AND A COMPLETE SYSTEM OF SOLUTIONS

Since the fundamental solution (3.23) satisfies the Schrödinger equation for all values of $u_0, u_1, \ldots, u_\nu$, all of its derivatives with respect to these parameters also satisfy the same equation, and the derivatives with respect to $u_1, \ldots, u_\nu$ are linearly independent. This follows from the fact that

$$(-ih)^s\frac{\partial^s}{\partial u_{\alpha_1}\ldots\partial u_{\alpha_s}}\Psi_0(u)=\{x_0^{\alpha_1}\ldots x_0^{\alpha_s}+P_{s-1}\}\Psi_0(u), \qquad (4.1)$$

where the x_0^α are defined by Eq. (3.19) and P_{s-1} is a polynomial of degree s − 1 in $x_0^1, \ldots, x_0^\nu$. The functions (4.1) are inconvenient, however, because they are not orthogonal. In order to construct an orthogonal system of solutions, let us consider scalar products of the functions (4.1). Such a product is obviously equal to a factor $(-ih)^{r+s}$ times the derivative with respect to $u_{\alpha_1}, \ldots, u_{\alpha_r}, v^*_{\beta_1}, \ldots, v^*_{\beta_s}$ of the integral

$$\int_{-\infty}^{\infty}\Psi_0(u)\Psi_0^*(v)\,dx=\frac{(\pi h)^{\nu/2}\sqrt{\rho_0\rho_0^*}}{\sqrt{\|Q\|}}\times\exp\left\{\frac{i(u_0-v_0^*)}{h}-\frac{1}{4h}(u-v^*)Q^{-1}(u'-v^+)\right\} \qquad (4.2)$$

for $v^*_\beta=u^*_\beta$. It follows from (4.2) that the fundamental solution will be normalized to unity if we set

$$\rho_0=(\pi h)^{-\nu/2}\sqrt{\|Q\|},\quad i(u_0-u_0^*)={}^1/_4(u-u^*)Q^{-1}(u'-u^+). \qquad (4.3)$$

It can now be shown that the probability density for positions of the system in the ground state is given by

$$\Psi_0(u)\Psi_0^*(u)=\frac{1}{(\pi h)^{\nu/2}\sqrt{\|G\|}}\exp\left\{-\frac{\Delta x'G^{-1}\Delta x}{h}\right\}, \qquad (4.4)$$

where $\Delta x=x-\bar{x}$ and $\bar{x}$ is given by (1.10).

Let us now introduce a function $\Psi(u,v)=\Psi_0(u-iv)$, under the condition that

$$v_0=-\frac{1}{4}vQ^{-1}v'-\frac{i}{2}vQ^{-1}(u'-u^+), \qquad (4.5)$$

i.e.,

$$\Psi(u,v)=\Psi_0(u)\exp\left\{\frac{v\Gamma\Delta x}{h}-\frac{1}{4h}v\Gamma G\Gamma'v'\right\}. \qquad (4.6)$$

---

1)In the case A = 0, B = 0, C = E a solution of the form (3.23) has been obtained by Fock [2] by means of extremely intuitive ideas.

It follows from (4.2) and (4.3) that

$$\int_{-\infty}^{\infty} \Psi(u, v)\Psi^*(u, w)\,dx = \exp\left\{\frac{1}{2h} vQ^{-1} w^+\right\}. \qquad (4.7)$$

It can be seen from this that the derivatives of $\Psi(u, v)$ with respect to the v_α, of different orders, taken at $v = 0$, are orthogonal to each other. In order to orthogonalize the derivatives of equal orders it is necessary to represent the matrix Q in the form $Q = \Lambda'\Lambda$. This is accomplished in the process of reducing the quadratic form $x'Qx$ to a sum of squares $y'y$ by means of a linear transformation $y = \Lambda x$. If we replace v by $(2h)^{1/2} v\Lambda$, we get as our result a function

$$\tilde{\Psi}(u, v) = \Psi(u, \sqrt{2h}\, v\Lambda) \qquad (4.8)$$

$$= \Psi_0(u) \exp\left\{\frac{\sqrt{2}}{\sqrt{h}} v\Lambda\Gamma\,\Delta x - \frac{1}{2} v\Lambda\Gamma G\Gamma'\Lambda' v'\right\},$$

all of whose derivatives with respect to the v_α, taken at $v = 0$, are orthogonal to each other and normalized to unity. These derivatives can be expressed in an obvious way in terms of Hermite-Chebyshev polynomials and form a basis in the space of the Ψ functions.

All of these derivatives satisfy the Schrödinger equation under consideration, since the function (4.8) satisfies it for all values of v. We accordingly get a complete system of solutions

$$\Psi_0(u) H^{\alpha_1 \ldots \alpha_s}(u) = \frac{\partial^s}{\partial v_{\alpha_1} \ldots \partial v_{\alpha_s}} \tilde{\Psi}(u, v)\,|_{v=0} \qquad (4.9)$$

of the Schrödinger equation with the Hamiltonian (1.2). Owing to this the function (4.8) is to be called the generating Ψ function. We note, however, that another convenient form of generating Ψ function is

$$\tilde{\tilde{\Psi}}(u, v) = \Psi(u, \sqrt{2h}\, v) = \Psi_0(u) \exp\left\{\frac{\sqrt{2}}{\sqrt{h}} v\Gamma\Delta x - \frac{1}{2} v\Gamma G\Gamma' v'\right\}. \quad (4.10)$$

Let us expand an arbitrary Ψ function in a series of the functions (4.9):

$$\Psi = \Psi_0(u) \sum_{s=0}^{\infty} \frac{1}{\sqrt{s!}} c_{\alpha_1 \ldots \alpha_s} H^{\alpha_1 \ldots \alpha_s}(u). \qquad (4.11)$$

The coefficients of this series are to be regarded as symmetric in the indices α. If these coefficients do not depend on the time, the function (4.11) represents the general solution of the Schrödinger equation.

Let us calculate the norm of the Ψ function (4.11). We have

$$\int_{-\infty}^{\infty} \tilde{\Psi}(u, v) \tilde{\Psi}^*(u, w)\,dx = e^{vw^+}. \qquad (4.12)$$

Differentiating this integral with respect to v, we get

$$\int_{-\infty}^{\infty} \Psi_0(u) H^{\alpha_1 \ldots \alpha_s}(u) \tilde{\Psi}^*(u, w)\,dx = w_{\alpha_1}{}^* \ldots w_{\alpha_s}{}^*. \qquad (4.13)$$

Consequently, for the Ψ function (4.11) we find

$$\Phi(w^*) = \int_{-\infty}^{\infty} \Psi \tilde{\Psi}^*(u, w)\,dx = \sum_{s=0}^{\infty} \frac{1}{\sqrt{s!}} c_{\alpha_1 \ldots \alpha_s} w_{\alpha_1}{}^* \ldots w_{\alpha_s}{}^*. \qquad (4.14)$$

Differentiating (4.14) with respect to w^*, we get

$$\int_{-\infty}^{\infty} \Psi [\Psi_0(u) H^{\alpha_1 \ldots \alpha_s}(u)]^*\,dx = \sqrt{s!}\, c_{\alpha_1 \ldots \alpha_s}. \qquad (4.15)$$

From this we find the square of the norm of the Ψ function (4.11)

$$\int_{-\infty}^{\infty} \Psi\Psi^*\,dx = \sum_{s=0}^{\infty} c_{\alpha_1 \ldots \alpha_s} c^*_{\alpha_1 \ldots \alpha_s}. \qquad (4.16)$$

Equations (4.11) and (4.15) enable us to solve an arbitrary Cauchy problem for the Schrödinger equation with the Hamiltonian (1.2).

5. THE FOCK REPRESENTATION

The results of the preceding section show that the coefficients c_0, c_α, $c_{\alpha\beta}$, ... form a Fock column.[1] In the terminology used in quantum field theory we shall call the fundamental solution the vacuum state, and the solution (4.9) an s-particle state. The configuration state of a "particle" here consists of only ν points. Equation (4.14) gives the Fock functional.[1]

Let us now find the operators for annihilation and creation of particles. For this purpose we note that besides the Schrödinger equation the fundamental solution also satisfies the system of equations

$$\{\hat{p}(K + LS) - \hat{x}'(M' + N'S) - u\}\Psi_0(u) = 0. \qquad (5.1)$$

Equation (5.1) can easily be verified directly. It corresponds to the classical equation (3.18). One further classical equation holds, namely

$$(K' + S^*L')p' - (M + S^*N)x - u' = 2iQx_0. \qquad (5.2)$$

We can arrive at this very simply, if from the left side of this equation we subtract the transposed form of (3.18) and then use (3.13). The quantum equation corresponding to the classical equation (5.2) is

$$\{(K' + S^*L')\hat{p}' - (M + S^*N)\hat{x} - u'\}\Psi_0(u) = 2iQx_0\Psi_0(u), \qquad (5.3)$$

which can also be easily verified directly.

From (5.1) and (5.3) there follow at once analogous equations for the generating Ψ function (4.8), which by definition is equal to $\Psi_0(u - i(2h)^{1/2} v\Lambda)$. On the other hand when we differentiate with respect to x the condition (4.5) is quite without effect. Accordingly we have, first,

$$\{\hat{p}(K + LS) - \hat{x}'(M' + N'S) - u\}\tilde{\Psi}(u, v) = -i\sqrt{2h}\, v\Lambda\tilde{\Psi}(u, v), \qquad (5.4)$$

and second,

$$\begin{aligned}\{(K' + S^*L')\hat{p}' - (M + S^*N)\hat{x} - u' + i\sqrt{2h}\,\Lambda' v'\}\tilde{\Psi}(u, v) \\ = 2iQ\Gamma(x - Lu' + i\sqrt{2h}\,L\Lambda' v')\tilde{\Psi}(u, v).\end{aligned} \qquad (5.5)$$

When we substitute (3.27) here we get

$$\begin{aligned}\{(K' + S^*L')\hat{p}' - (M + S^*N)\hat{x} - u^+\}\tilde{\Psi}(u, v) \\ = \{2iQ\Gamma\Delta x - i\sqrt{2h}\,Q\Gamma G\Gamma'\Lambda' v'\}\tilde{\Psi}(u, v),\end{aligned} \qquad (5.6)$$

i.e., finally

$$\begin{aligned}\{(K' + S^*L')\hat{p}' - (M + S^*N)\hat{x} - u^+\}\tilde{\Psi}(u, v) \\ = i\sqrt{2h}\,\Lambda' \frac{\partial}{\partial v}\tilde{\Psi}(u, v).\end{aligned} \qquad (5.7)$$

Already it can be seen from this that the desired operators for annihilation of particles form the row

$$z = \frac{i}{\sqrt{2h}}\{\hat{p}(K + LS) - \hat{x}'(M' + N'S) - u\}\Lambda^{-1}, \qquad (5.8)$$

and the operators for creation of particles form the Hermitian-adjoint column

$$z^+ = -\frac{i}{\sqrt{2h}}\Lambda'^{-1}\{(K' + S^*L')\hat{p}' - (M + S^*N)\hat{x} - u^+\}. \qquad (5.9)$$

Let us verify this assertion. According to (5.4) and (5.7)

$$z\tilde{\Psi}(u, v) = v\tilde{\Psi}(u, v), \quad z^+\tilde{\Psi}(u, v) = \frac{\partial}{\partial v}\tilde{\Psi}(u, v). \tag{5.10}$$

Let us consider the series

$$\Psi = \sum_{s=0}^{\infty}\frac{1}{\sqrt{s!}}c_{\alpha_1\ldots\alpha_s}\frac{\partial^s}{\partial v_{\alpha_1}\ldots\partial v_{\alpha_s}}\tilde{\Psi}(u, v), \tag{5.11}$$

which is more general than (4.11). The latter is obtained from (5.11) if we set v = 0. On the basis of (5.10) we conclude that when the operators are applied to (5.11) we get

$$z_\beta\Psi = v_\beta\Psi + \sum_{s=0}^{\infty}\frac{\sqrt{s+1}}{\sqrt{s!}}c_{\beta\alpha_1\ldots\alpha_s}\frac{\partial^s}{\partial v_{\alpha_1}\ldots\partial v_{\alpha_s}}\tilde{\Psi}(u, v), \tag{5.12}$$

$$z_\beta^+\Psi = \sum_{s=0}^{\infty}\frac{1}{\sqrt{s!}}c_{\alpha_1\ldots\alpha_s}\frac{\partial^{s+1}}{\partial v_\beta\,\partial v_{\alpha_1}\ldots\partial v_{\alpha_s}}\tilde{\Psi}(u, v). \tag{5.13}$$

It follows from this that the operator z_β converts the Fock column c_0, c_{α_1}, $c_{\alpha_1\alpha_2}$, ... into the Fock column $\underset{\sim}{c}_0$, $\underset{\sim}{c}_{\alpha_1}$, $\underset{\sim}{c}_{\alpha_1\alpha_2}$, ..., where

$$\underset{\sim}{c}_{\alpha_1\ldots\alpha_s} = \sqrt{s+1}\,c_{\beta\alpha_1\ldots\alpha_s}, \tag{5.14}$$

and that the operator z_β^+ converts this same Fock column into the column $\tilde{c}_0$, $\tilde{c}_{\alpha_1}$, $\tilde{c}_{\alpha_1\alpha_2}$, ..., where $\tilde{c}_0 = 0$ and

$$\tilde{c}_{\alpha_1\ldots\alpha_s} = \frac{1}{\sqrt{s}}\{\delta_{\beta\alpha_1}c_{\alpha_2\ldots\alpha_s} + \ldots + \delta_{\beta\alpha_k}c_{\alpha_1\ldots\alpha_{k-1}\alpha_{k+1}\ldots\alpha_s} + \ldots + \delta_{\beta\alpha_s}c_{\alpha_1\ldots\alpha_{s-1}}\}. \tag{5.15}$$

Accordingly, the operators z_β and z_β^+ are indeed the operators for annihilation and creation of a particle at the point β.

It is also interesting to give a direct derivation of the result of the action of the operators z and z^+ on the Fock functional (4.14). By (4.12) we have for the series (5.11)

$$\int_{-\infty}^{\infty}\tilde{\Psi}^*(u, w)\Psi\,dx = \sum_{s=0}^{\infty}\frac{1}{\sqrt{s!}}c_{\alpha_1\ldots\alpha_s}\frac{\partial^s}{\partial v_{\alpha_1}\ldots\partial v_{\alpha_s}}e^{vw^+} = \Phi(w^*)e^{vw^+} \tag{5.16}$$

From this and (5.10) we get

$$\int_{-\infty}^{\infty}\tilde{\Psi}^*(u, w)z_\beta\Psi\,dx = \sum_{s=0}^{\infty}\frac{1}{\sqrt{s!}}c_{\alpha_1\ldots\alpha_s}\frac{\partial^s}{\partial v_{\alpha_1}\ldots\partial v_{\alpha_s}}v_\beta e^{vw^+} = \frac{\partial}{\partial w_\beta^*}[\Phi(w^*)e^{vw^+}], \tag{5.17}$$

$$\int_{-\infty}^{\infty}\tilde{\Psi}^*(u, w)z_\beta^+\Psi\,dx = \frac{\partial}{\partial v_\beta}[\Phi(w^*)e^{vw^+}] = w_\beta^*\Phi(w^*)e^{vw^+}. \tag{5.18}$$

Consequently, as must be the case, the operator z_β converts the Fock functional $\Phi(w^*)$ into the functional $\partial\Phi(w^*)/\partial w_\beta^*$, and the operator z_β^+ converts it into $w_\beta^*\Phi(w^*)$.

Let us now consider the general Schrödinger equation

$$ih\frac{\partial\Psi}{\partial t} = \{H + V(t, \hat{x}, \hat{p})\}\Psi, \tag{5.19}$$

where as before H is given by (1.2), and V is some operator added to the Hamiltonian H; this is a problem typical of perturbation theory.

For the series (5.11) we have

$$\int_{-\infty}^{\infty}\tilde{\Psi}^*(u, w)\left[ih\frac{\partial}{\partial t} - H\right]\Psi\,dx = ih\frac{\partial}{\partial t}\Phi(w^*)e^{vw^+}, \tag{5.20}$$

i.e., the operator $ih\partial/\partial t - H$ converts the Fock functional $\Phi(w^*)$ into $ih\partial\Phi(w^*)/\partial t$. Consequently, in the Fock representation Eq. (5.19) can be written in the form

$$ih\partial\Phi/\partial t = V(t, \hat{x}, \hat{p})\Phi, \tag{5.21}$$

where $\hat{x}$ and $\hat{p}$ must be expressed in terms of z and z^+.

For this purpose we note that

$$\begin{pmatrix}-M - SN & K' + SL' \\ -M - S^*N & K' + S^*L'\end{pmatrix}\begin{pmatrix}K + LS^* & -K - LS \\ M' + N'S^* & -M' - N'S\end{pmatrix} = \begin{pmatrix}-2iQ & 0 \\ 0 & -2iQ\end{pmatrix}. \tag{5.22}$$

The first of these matrices is taken from (5.8) and (5.9). The second is constructed from the first according to the model of (3.11). In calculating their product we must use (3.12). By means of (5.22) it is not hard to show that

$$\hat{x} = \bar{x} + \frac{\sqrt{h}}{\sqrt{2}}[(K + LS^*)\Lambda^{-1}z' + (K + LS)\Lambda^{-1}z^+],$$
$$\hat{p} = \bar{p} + \frac{\sqrt{h}}{\sqrt{2}}[z\Lambda'^{-1}(M + S^*N) + z'^+\Lambda'^{-1}(M + SN)], \tag{5.23}$$

where $\bar{x}$, $\bar{p}$ are given by (1.10). We now have only to substitute (5.23) in (5.21) and replace z by $\partial/\partial w^*$ and z^+ by w^*. We can also omit doing the latter, by the way, if we use for the Fock function the notation

$$\Phi = \sum_{s=0}^{\infty}\frac{1}{\sqrt{s!}}c_{\alpha_1\ldots\alpha_s}z_{\alpha_1}^+\ldots z_{\alpha_s}^+|0\rangle \tag{5.24}$$

and employ the commutation relations $z_\alpha z_\beta^+ - z_\beta^+ z_\alpha = \delta_{\alpha\beta}$ and the normalization of the vacuum $\langle 0|0\rangle = 1$.

---

[1] V. A. Fock, Raboty po kvantovoĭ teoriya polya (Papers on Quantum Field Theory), LGU (Leningrad State University), 1957.

[2] V. A. Fock, Kvantovaya fizika i stroenie materii (Quantum Physics and the Structure of Matter), LGU (Leningrad State University), 1965.

Translated by W. H. Furry
115