CERENKOV RADIATION OF ALFVEN WAVES

V. K. DOKUCHAEV

Radiophysics Institute, Gorkiĭ State University

Submitted March 17, 1967

Zh. Eksp. Teor. Fiz. 53, 723–731 (August, 1967)

Cerenkov radiation of Alfven waves produced by a source moving uniformly along a straight line forming an arbitrary angle with the magnetic field is discussed. Use is made of the analogy of this phenomenon with the emission of transverse waves in a system of parallel elastic strings with a source of perturbations moving across. The simplest systems of electric currents which excite Alfven waves are listed and expressions for the fields are given. The forces of radiation reaction are determined.

THEORETICAL investigations of electromagnetic Vavilov–Cerenkov radiation in anisotropic plasmas encounter tremendous computational difficulties, owing to the complicated character of the dispersion of normal waves in the medium, and due to the complicated relation between the phase velocities of these waves and their angle relative to the external magnetic field. Therefore one usually restricts one's attention to obtaining and discussing the expressions for the energy and radiation reaction force in the form of integrals over the spectrum of emitted frequencies, or over the angle spectrum.[1] If one is interested in the emission of magnetosonic waves in magnetohydrodynamics, there is no dispersion, but the complicated character of the dependence of the phase velocities of these waves on their angle with the magnetic field remains.[2] Because of this, one usually considers only Cerenkov radiation produced by a source moving along the magnetic field.[3, 4]

A particularly simple example of Cerenkov radiation in an anisotropic medium is the emission of Alfven waves in a magnetoactive plasma. In this case it is easy and simple to determine the fields of magnetohydrodynamic excitations and simple expressions for the radiation reaction can be derived. This example is useful also from a methodological standpoint, since Alfven waves in a plasma have a simple analogy with transverse wave excitations in a mechanical system consisting of parallel elastic strings. This analogy was first pointed out by Alfven, and he made extensive use of it in analyzing various aspects of magnetohydrodynamics. The essence of this analogy consists in the fact that each magnetic field line in a plasma is can be represented in a model by an elastic string of given density and tension.[5]

We shall show below that Cerenkov radiation of Alfven waves occurs for arbitrary velocities of the motion of the source of perturbations, if the velocity vector forms an angle with the direction of the external magnetic field. The case when the source moves along the field lines is degenerate. In this case the perturbations have the character of entropy waves and move together with the source with the velocity of its motion.[2]

From the well known linearized system of equations of magnetohydrodynamics,[2] in the presence of external force sources $\mathbf{f}$ in the equation of motion, and of mass sources Q in the continuity equation, one can derive the following equation for perturbations $\mathbf{v}$ of the velocity vector:

$$\frac{\partial^2\mathbf{v}}{\partial t^2} = c_s^2\,\mathrm{grad\,div}\,\mathbf{v} + c_A^2[\mathrm{rot\,rot}[\mathbf{v}\mathbf{e}_z]\,\mathbf{e}_z] + \frac{1}{\rho_0}\frac{\partial\mathbf{f}}{\partial t} - \frac{c_s^2}{\rho_0}\mathrm{grad}\,Q, \quad (1)^*$$

where c_s is the velocity of sound, $c_A = (H_0/4\pi\rho_0)^{1/2}$ is the Alfven velocity, $\mathbf{e}_z$ is the unit vector along the magnetic field $\mathbf{H}_0$, ρ_0 is the unperturbed density of the plasma. If an external electric current density $\mathbf{j}_0$ is given, then the density of the ponderomotive force of this current is

$$\mathbf{f} = c^{-1}[\mathbf{j}_0\mathbf{H}_0]. \quad (2)$$

The dispersion rule for excitations of the form $\exp i(\omega t - \mathbf{k}\cdot\mathbf{r})$ can be derived from Eq. (1) and has the form

$$(\omega^2 - c_A^2k_z^2)\{\omega^4 - [(c_A^2 + c_s^2)\,\omega^2 - c_A^2c_s^2k_z^2]\,k^2\} = 0. \quad (3)$$

This equation is discussed in detail in [2]. It implies that the medium can propagate three types of normal waves: a fast and slow magnetosonic wave, and a magnetohydrodynamic Alfven wave. The phase velocities of the magnetosonic waves are obtained from the condition that the expression in curly brackets in (3) vanish, and the phase velocity of the Alfven wave is determined by the condition $\omega = \pm c_A k_z$. We recall that in the Alfven wave there are no perturbations of the density, pressure and of the projection of the velocity on the magnetic field. The only nonvanishing excitations are those of the vector $\mathbf{v}_\perp$ and of the magnetic field $\mathbf{h}_\perp$, both perpendicular to the external magnetic field $\mathbf{H}_0$.

It is easy to show that the mass sources Q excite only magnetosonic waves, and as can be seen from Eq. (1), they are equivalent to potential force sources. On the other hand if the following conditions are fulfilled

$$\mathbf{f}\mathbf{H}_0 = 0, \quad \mathrm{div}\,\mathbf{f} = 0 \quad (4)$$

the force sources $\mathbf{f}$ do not excite magnetoacoustic waves and generate only Alfven waves. If the conditions (4) are satisfied and $Q = 0$, then Eq. (2) becomes

$$\frac{\partial^2\mathbf{v}_\perp}{\partial t^2} = c_A^2\frac{\partial^2\mathbf{v}_\perp}{\partial z^2} + \frac{1}{\rho_0}\frac{\partial\mathbf{f}_\perp}{\partial t}. \quad (5)$$

*$[\mathbf{v}\mathbf{e}_z] \equiv \mathbf{v}\times\mathbf{e}_z$.

It is a well-known fact that the same equation describes the small transverse oscillations of an elastic string under tension T_0, having a density ρ_0, under the action of a transverse force of density $\mathbf{f}_\perp$, with $c_A = (T_0/\rho_0)^{1/2}$.[6] In this connection we are of the opinion that it is worthwhile to consider briefly the following mechanical problem of radiation of transverse waves in a system of parallel elastic strings.

We assume that a perturbing force density $\mathbf{f}_\perp$ moves with constant velocity $\mathbf{V}_0$ over a system of elastic strings which are all parallel to the z axis. For greater generality we assume that each of the strings exhibits a resistance to transverse motions proportional to its velocity, i.e., we assume that a friction force $-2\alpha \mathbf{v}_\perp$ is present. Then the equation describing the transverse displacements $\xi_\perp$ of each of the strings, related to the velocity by $\mathbf{v}_\perp = \partial \xi_\perp/\partial t$, has the form[6]

$$\frac{\partial^2 \xi_\perp}{\partial t^2} + 2\alpha \frac{\partial \xi_\perp}{\partial t} - c_A^2 \frac{\partial^2 \xi_\perp}{\partial z^2} = \frac{\mathbf{F}_\perp}{\rho_0} \delta(x - V_{0x}t)\,\delta(z - V_{0z}t). \tag{6}$$

From the way the right hand side is written, it can be seen that the source acts only on those strings which are situated in the (x, z) plane and moves under an angle $\varphi = \arctan(V_{0x}/V_{0z})$ to the z axis—the direction along which the strings are strung. By means of a Fourier transform, the solution of Eq. (6) can be written either in the form of an integral over wave numbers

$$\xi_\perp = \frac{\mathbf{F}_\perp}{4\pi^2 \rho_0} \int_{-\infty}^{+\infty}\!\!\int \frac{\exp\{-ik_x(x - V_{0x}t) - ik_z(z - V_{0z}t)\}}{(\mathbf{k}\mathbf{V}_0)^2 - 2i\alpha \mathbf{k}\mathbf{V}_0 - c_A^2 k_z^2}\, dk_x\, dk_z \tag{7}$$

or, making use of the convolution theorem, in the form of an integral over the source distributions:

$$\xi_\perp = \frac{\mathbf{F}_\perp}{\rho_0} \int_{-\infty}^{+\infty}\!\!\int \delta(x - V_{0x}t')\,\delta(z' - V_{0z}t')\, G(z, t\,|\,z', t')\, dz'\, dt'. \tag{8}$$

The Green's function for Eq. (6) occurring in the integrand of (8) is

$$G(z, t\,|\,z', t') = \frac{H(t - t')\, H[c_A(t - t') - |z - z'|]}{2c_A} e^{-\alpha(t - t')} \times I_0[\alpha\sqrt{(t - t')^2 - (z - z')^2/c_A^2}], \tag{9}$$

where I_0 is the modified Bessel function and $H(x) = 0$ for $x < 0$, $H(x) = 1$ for $x \geq 0$ is the Heaviside step function. For $\alpha = 0$ the Green's function (9) becomes the well known Green's function for the one-dimensional wave equation (5).

A computation of either (7) or (8) yields

$$\xi_\perp = \frac{\mathbf{F}_\perp H(V_{0x}t - x - |M_x z - M_z x|)}{2\rho_0 c_A V_{0x}} \exp\left\{-\frac{\alpha(V_{0x}t - x)}{V_{0x}}\right\} \times I_0\left[\frac{\alpha}{V_{0x}}\sqrt{(V_{0x}t - x)^2 - (M_x z - M_z x)^2}\right], \tag{10}$$

where for convenience we have introduced the "Mach numbers" for the projections of the velocity of the source: $M_x = V_{0x}/c_A$, $M_z = V_{0z}/c_A$.

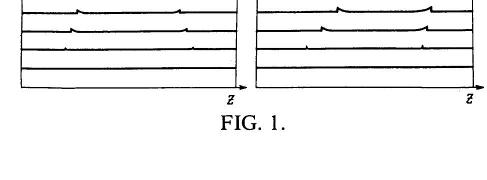

FIG. 1.

The case $V_{0x} = 0$ is degenerate. In order to study it, it is convenient to start directly with the expression (7); setting $f = F\delta(z - V_0 t)$ we obtain

$$\zeta = \frac{\partial \xi}{\partial z} = \begin{cases} \dfrac{FH(z - V_0 t)}{2\pi\rho_0(V_0^2 - c_A^2)} \exp\left\{-\dfrac{\alpha V_0(z - V_0 t)}{V_0^2 - c_A^2}\right\}, & V_0 > c_A \\[2ex] \dfrac{FH(V_0 t - z)}{2\pi\rho_0(c_A^2 - V_0^2)} \exp\left\{-\dfrac{\alpha V_0(V_0 t - z)}{c_A^2 - V_0^2}\right\}, & V_0 < c_A \end{cases} \tag{11}$$

where ζ is the slope of the displacements of the strings with respect to the z axis.

Figure 1 illustrates the transverse displacements ξ_x in a system of strings situated in the (x, z) plane, when the perturbing force moves across the strings perpendicularly, with $M_x = 0$, and under an angle, with $M_z > 1$.

By analyzing the poles in the integrand of (7), or directly from Fig. 1 one can see that waves are radiated for any velocity of the motion of the source of perturbations. All perturbations are included inside a Mach angle, analogous to the Cerenkov cone for three dimensions. It is easy to see from (10) and Fig. 1 that the velocity of propagation of the energy (the group velocity) is c_A and is always directed along the strings. The group velocity differs both in magnitude and direction from the phase velocity $V_{ph} = V_0/(1 + M_x^2)^{1/2}$, which is directed along the normal to the line bounding the perturbed region (we have in mind the case $V_{0z} = 0$). In the degenerate case $V_{0x} = 0$ it follows from (11) that the perturbations move along the string with the velocity of the source V_0, i.e., behave in the same manner as entropy perturbations.[2] If $V_0 < c_A$, the region which is perturbed is in front of the source ("precursor") and for $V_0 > c_A$ all perturbations are only behind the source, forming a wake. In what follows we do not discuss this degenerate case.

The displacements $\xi_\perp$ are finite on the Mach "cone," inside it, and even in the region where the source acts. This permits us to solve the inverse problem: determine the force $\mathbf{F}_\perp$ which it is necessary to apply at the point $x = V_{0x}t$, $z = V_{0z}t$, in order to produce a given displacement $\xi_{\perp 0}$ at that point.[1)] Thus we obtain from (10)

$$\mathbf{F}_\perp = 2\rho_0 c_A V_{0x}\, \xi_{\perp 0}. \tag{12}$$

This force, taken with the opposite sign is essentially equal to the force of radiation reaction, since the work done by it is completely spent on radiation of waves. We remark the physically obvious fact that in an anisotropic medium the radiation reaction force can form an angle with the velocity of motion of the source of perturbations. This also follows from (12).

We now go over to the case of magnetohydrodynamics. The majority of the results of the preceding problem carries over to this case, if one bears in mind that each magnetic field line of the field H_0 can be considered as a string of density ρ_0, equal to the mass density of the plasma, and subject to a tension $T_0 = H_0^2/4\pi$. However, several differences should not be forgotten.

[1)]Such a formulation of the problem, when the perturbations are given near the source and one is required to determine the forces of hydrodynamical reaction acting on the source, is frequently encountered in ordinary hydrodynamics [7].

Firstly, the sources which excite the Alfven waves must satisfy the condition (4), i.e., they must have no divergence and lie in a plane perpendicular to the magnetic field. If these conditions are not satisfied, the source will also excite the other two normal waves in the medium, which complicates matters considerably. Secondly, one cannot take into consideration the absorption of Alfven waves in the same manner as for the case of the strings, since the character of dispersion in the plasma is more complex, owing to Joule and viscous losses.[5] In the presence of losses it was impossible to compute the Cerenkov radiation of Alfven waves exactly; therefore we consider below a lossless medium and make use of the Green's function (9) with $\alpha = 0$ for the initial equation (5). Thus the solution of Eq. (5), written in terms of the displacement $\xi_\perp$, and taking into account all these remarks, will have the form

$$\xi_\perp = \frac{1}{2\rho_0 c_A} \int\limits_{-\infty}^{+\infty}\int \mathbf{f}_\perp(\mathbf{r}, t') H(t-t') H[c_A(t-t') - |z-z'|] dz'\, dt'. \quad (13)$$

Of special interest is the radiation of Alfven waves produced by an external electric current with density $\mathbf{j}_0$, mechanically interacting with the plasma according to Eq. (2). The first of the conditions (4) is then automatically satisfied, and the second becomes

$$\mathrm{rot}_z\, \mathbf{j}_0 = 0. \quad (14)$$

The simplest case is the two-dimensional problem when $\mathbf{j}_0$ depends only on the coordinates (x, z). It is easy to check that one of the simplest plane currents satisfying condition (14) is a moving solenoid of rectangular cross section, with the axis oriented perpendicularly to the external magnetic field. The current density in this case can be expressed in terms of the following combinations of the Heaviside and Dirac generalized functions:

$$\begin{gathered} j_x = I_0 H(l_x^2 - x'^2)[\delta(z'+l_z) - \delta(z'-l_z)], \quad j_y = 0, \\ j_z = I_0 H(l_z^2 - z'^2)[\delta(x'-l_x) - \delta(x'+l_x)], \end{gathered} \quad (15)$$

where $x' = x - V_{0x}t$, $z' = z - V_{0z}t$, $2l_x$ and $2l_z$ are the lengths of the sides of the rectangular cross section of the solenoid, I_0 is now the current per unit length in the y-direction. Another simple example of a system of currents bounded in space and satisfying the condition (14) is a solenoid of rectangular cross section bent into a torus (i.e., a torus of rectangular cross section) with the torus axis along the z axis. In a cylindrical coordinate system (r, z, φ) the current density in the torus is defined as follows:

$$\begin{gathered} j_r = I_0 \frac{H(a-r) - H(b-r)}{2\pi r}[\delta(z'+l) - \delta(z'-l)], \\ j_z = I_0 \frac{\delta(r-a) - \delta(r-b)}{2\pi r} H(l^2 - z'^2), \quad j_\varphi = 0, \end{gathered} \quad (16)$$

where a and b are the external and internal radii of the torus, respectively. $2l$ its height (along the z axis), $z' = z - V_{0z}t$, $r = ((x - V_{0x}t)^2 + (y - V_{0y}t)^2)^{1/2}$.

However, the use of sources of the forms (15) or (16) leads to complicated expressions for the fields. Therefore we approximate (15) by a thin solenoid, with l_x and l_y vanishing

$$\begin{gathered} j_x = I_0 l_x l_z \delta(x - V_{0x}t)\delta'(z - V_{0z}t), \quad j_y = 0, \\ j_z = -I_0 l_x l_z \delta'(x - V_{0x}t)\delta(z - V_{0z}t), \end{gathered} \quad (17)$$

where $I_0 l_x l_z = cm_y$, with m_y the magnetic moment per unit length of the solenoid. The corresponding force density is

$$f_y = m_y H_0 \delta(x - V_{0x}t)\delta'(z - V_{0z}t). \quad (18)$$

Substituting this expression into (13) and integrating we obtain

$$\xi_y = \frac{m_y H_0}{2\rho_0 c_A V_{0x}} \frac{\partial}{\partial z} H[V_{0x}t - x - |M_x z - M_z x|] H(V_{0x}t - x). \quad (19)$$

In order to analyze the radiation reaction it is necessary to know the perturbations of the magnetic field $\mathbf{h} = H_0(\partial\xi/\partial z)$:

$$h_y = \frac{m_y H_0^2 M_x}{2\rho_0 c_A^2} \delta_z'[V_{0x}t - x - |M_x z - M_z x|] H(V_{0x}t - x). \quad (20)$$

Figure 2 gives an approximate representation of the perturbations of the magnetic field h_y in the plane y = 0. The singular generalized function δ' has been regularized[8] graphically in this figure, i.e., it has been replaced by a similarly shaped nonsingular function.

We determine the force of radiation reaction $\mathbf{R}$, acting on the unit length of the thin solenoid. We start from the expression of the force density

$$\mathbf{f}_r = -\frac{1}{c}[\mathbf{j}_0\mathbf{h}] = \frac{h_y}{c}(j_{0z}\mathbf{e}_x - j_{0x}\mathbf{e}_z), \quad (21)$$

where $\mathbf{e}_x$ and $\mathbf{e}_z$ are unit vectors and, as before, j_{0x} and j_{0z} are defined by (17). The force $\mathbf{R}$ is given by the integrals

$$\begin{gathered} R_x = \int\limits_{-\infty}^{+\infty}\int f_{rx}\, dx\, dz = 2\pi m_y^2 M_{0x}[\delta_{xz}''(V_{0x}t - x - |M_x z - M_z x|]_{\substack{x\to V_{0x}t, \\ z\to V_{0z}t,}} \\ R_z = \int\limits_{-\infty}^{+\infty}\int f_{rz}\, dx\, dz = 2\pi m_y^2 M_{0x}[\delta_{zz}''(V_{0x}t - x - |M_x z - M_z x|]_{\substack{x\to V_{0x}t, \\ z\to V_{0z}t.}} \end{gathered} \quad (22)$$

Thus the components of the force $\mathbf{R}$ are expressed in terms of the singular generalized functions δ_{xz}'' and δ_{zz}'', with the argument vanishing, and hence the functions themselves unbounded. Using integral representations for these functions one can show that the spectral densities $R_\omega \sim \omega^2$ and $R_k \sim k^2$. This means that the integrals over the spectrum of emitted frequencies or wave numbers diverge for large ω (short wavelengths). Consequently, in the thin solenoid approximation, both the field perturbations h_y and the radiation reactions are expressed in terms of singular generalized functions, which yield a qualitative idea about the character of perturbations and spectral density of the forces.

A completely similar calculation of the fields has been carried out for the more complicated case of a

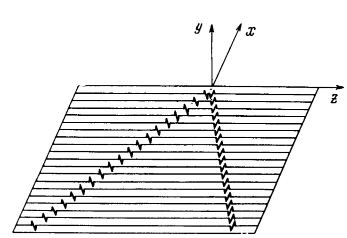

FIG. 2

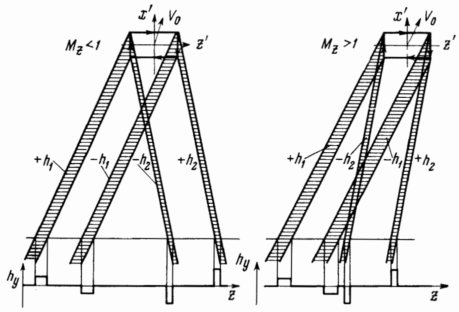

FIG. 3

solenoid of finite dimensions (15). Omitting a series of lengthy intermediate calculations we give only the final result:

$$\begin{aligned} h_y = h_1 \Big\langle & H\Big\{l_x^2 - \Big[x' - \frac{M_x(z'+l_z)}{1+M_z}\Big]^2\Big\} H(-l_z - z') \\ & - H\Big\{l_x^2 - \Big[x' - \frac{M_x(z'-l_z)}{1+M_z}\Big]^2\Big\} H(l_z - z')\Big\rangle \\ & + h_2 \Big\langle H\Big\{l_x^2 - \Big[x' + \frac{M_x(z'-l_z)}{1-M_z}\Big]^2\Big\} H\Big(\frac{z'-l_z}{c_A - V_{0z}}\Big) \\ & - H\Big\{l_x^2 - \Big[x' + \frac{M_x(z'+l_z)}{1-M_z}\Big]^2\Big\} H\Big(\frac{z'+l_z}{c_A - V_{0z}}\Big)\Big\rangle, \end{aligned} \tag{23}$$

where $h_1 = 2\pi I_0/c(1+M_Z)$, $h_2 = 2\pi I_0/c\,|1-M_Z|$. In deriving (23), use has been made of known properties of generalized functions.[8] Thus, the perturbations h_y are expressed in terms of a combination of Heaviside functions and are everywhere finite, with the exception of the degenerate case $M_X = 0$, $M_Z = 1$. Despite its complicated form, Eq. (23) expresses very simple results. We discuss several particular situations.

For $M_X = 0$ it follows from (23) that h_y is always inside the solenoid and has the expressions

$$h_y = \begin{cases} -\dfrac{4\pi I_0}{c(1-M_z^2)} H(l_x^2 - x^2) H(l_z^2 - z'^2) & \text{for } M_z < 1 \\ \dfrac{4\pi I_0}{c(M_z^2 - 1)} H(l_x^2 - x^2) H(l_z^2 - z'^2) & \text{for } M_z > 1 \end{cases} \tag{24}$$

As expected, for $M_X = M_y = 0$ we obtain from (24) that $h_y = -4\pi I_0/c$, i.e., h_y goes over into the unperturbed field inside the solenoid.

Cerenkov radiation of Alfven waves occurs for $M_X > 0$. In this case the character of the perturbations described by Eq. (23) becomes clear from Fig. 3. The shaded regions are those where according to Eq. (23) the magnetic field is perturbed by the motion of the solenoid. The lower part of each of the two situations in Fig. 3 represents $h_y = h_y(z')$ for $x' = \text{const}$, as represented by the horizontal cross sections in the figures. The rectangles at the top of the figures represent the turns of the solenoid and the arrows indicate the direction of flow of the current. These figures illustrate clearly the character and magnitude of the perturbations for different M_X and M_Z. In conclusion of the analysis of Eq. (23) we indicate that by taking the limits $l_X \to 0$ and $l_Z \to 0$ one obtains the expression (20) for the field h_y of the thin solenoid.

We now compute the reaction force $\mathbf{R}$, which acts on the solenoid. For this we substitute h_y from (23) into (21) and (22). Due to the large number of independent parameters on which the solution depends, the final results are tabulated. The form used in the table to express the dependence of R_X and R_Z on the parameters M_Z, M_X, l_X, and l_Z seems to be the most convenient one.

As a simple example illustrating the general formulas of the table we consider the case $M_Z = 0$

$$R_x = \begin{cases} 0 & \text{for} \quad M_x = 0 \\ -8\pi I_0^2 l_z/c^2 & \text{for} \quad 0 < M_x \leqslant l_x/l_z. \\ -8\pi I_0^2 l_x/M_x c^2 & \text{for} \quad l_x/l_z \leqslant M_x \end{cases} \tag{25}$$

The increase of the absolute value of the force R_X (which brakes the motion of the solenoid) from zero to a finite value occurs suddenly. If one takes into account dissipative processes produced by viscosity or finite electrical conductivity of the medium at low velocities V_0, this singularity disappears.

The intensity of Alfven wave emission by a moving source can be computed from the equation

$$d\mathscr{E}/dt = \mathbf{RV}. \tag{26}$$

In the case $M_Z = 0$ it follows from (25), (26) that the intensity increases linearly with the growth of the velocity until the constant value $8\pi I_0^2 l_X/c^2 c_A$. We note also that in addition to the radiation reaction forces, the solenoid is also subject to torques which are produced by the nonuniform distribution of these forces along the sides of the rectangular turns of the solenoid. We do not compute these torques here.

The limits of applicability of the linear approximation used here can be obtained from the condition $\max|h_y| \ll H_0$. Equation (23) for h_y implies that the inequality $2\pi I_0/c\,|1-M_Z| \ll H_0$ must hold in this case. For $M_Z = 0$ this means that the proper magnetic field inside the solenoid, $H_1 = 4\pi I_0/c$ must be smaller than the field H_0.

A last remark has to do with the circumstance that in addition to electromagnetic interactions between the solenoid and the medium there also exists a purely mechanical interaction. Indeed, we have assumed throughout that the individual turns of the solenoid, which have in its cross section the form of rectangular frames, are spaced along the y axis by a certain interval a, with $r_0 < a < \min(l_X, l_y)$, where r_0 is the radius of the cross section of the individual turn. This condition is necessary in order that the plasma and the magnetic field penetrate inside the solenoid without significant distortions. Nevertheless, even in this case the solid conductors making up the turns of the solenoid will experience in their motion through the plasma a frontal resistive force, which for a wide range of velocities has the expression $F \sim V_0^2 r_0 l$, where l is one of the dimensions l_X or l_Z. Obviously, the mechanical interaction of the solenoid with the medium is negligible compared to the electromagnetic interaction whenever this force is small compared to the radiation reaction force. In the case $M_Z = 0$ we find from (25) that this condition becomes $\rho_0 V_0^2 (r_0/a) < I_0^2/c^2$, i.e. if the energy of the magnetic field of the solenoid is larger than the kinetic energy of the medium multiplied by a coefficient r_0/a smaller than unity.

In conclusion I would like to thank B. Ya. Éidman for useful remarks.

Table

M_x	$R_x \Big/ \frac{4\pi I_0^2}{c^2}$	$R_z \Big/ \frac{4\pi I_0^2}{c^2}$
	$M_z < 1$	
0	0	0
$0 < \frac{M_x l_z}{l_x} \leqslant 1 - M_z$	$-\frac{2l_z}{1-M_z^2}$	$-\frac{2M_z l_x}{1-M_z^2}\left[1+\frac{2M_x l_z}{l_x(1-M_z^2)}\right]$
$1 - M_z \leqslant \frac{M_x l_z}{l_x} \leqslant 1 + M_z$	$-\frac{l_x}{M_x}\left[1+\frac{M_x l_z}{l_x(1+M_z)}\right]$	$-\frac{4M_z l_x}{1-M_z^2} - \frac{l_x}{1+M_z}\left[1-\frac{M_x l_z}{l_x(1+M_z)}\right]$
$1 + M_z \leqslant \frac{M_x l_z}{l_x}$	$-\frac{2l_x}{M_x}$	$-\frac{4l_x M_z}{1-M_z^2}$
	$M_z > 1$	
$0 < \frac{M_x l_z}{l_x} \leqslant M_z - 1$	$\frac{2l_z}{M_z^2-1}$	$-\frac{2l_x}{M_z^2-1}\left[1+\frac{2M_x M_z l_z}{l_x(M_z^2-1)}\right]$
$M_z - 1 \leqslant \frac{M_x l_z}{l_x} \leqslant M_z + 1$	$\frac{l_x}{M_x}\left[1-\frac{M_x l_z}{l_x(1+M_z)}\right]$	$-\frac{4l_x}{M_z^2-1} - \frac{l_x}{M_z+1}\left[1-\frac{M_x l_z}{l_x(M_z+1)}\right]$
$M_z + 1 \leqslant \frac{M_x l_z}{l_x}$	0	$-\frac{4l_x}{M_z^2-1}$

[1] B. M. Bolotovskiĭ, Usp. Fiz. Nauk **62**, 201 (1957).

[2] S. I. Syrovatskiĭ, Usp. Fiz. Nauk **62**, 247 (1957).

[3] A. I. Morozov, Fizika plazmy i problema upravlyaemykh termo yadernykh reaktsiĭ (Plasma Physics and the Problem of Controllable Thermonuclear Reactions) **4**, 331, AN SSSR, 1958

[4] V. P. Dokuchaev, Zh. Eksp. Teor. Fiz. **48**, 587 (1965) [Sov. Phys.-JETP **21**, 287 (1965)].

[5] H. Alfven, Cosmic Electrodynamics, Oxford U. Press, 1948 (Russian Transl. IIL, 1952).

[6] P. M. Morse and H. Feshbach, Methods of Theoretical Physics, vol. II, McGraw-Hill, New York, 1953.

[7] V. P. Dokuchaev, Prikl. Mat. Mekh. **30**, 1006 (1966).

[8] I. M. Gel'fand and G. E. Shilov, Obobshennya lunktsii i deistviya nad nimi (Generalized functions and operations with them) Fizmatgiz, 1958. Engl. Transl. by E. Saletan, Generalized Functions, vol. I, Academic Press, New York, 1953.

Translated by M. E. Mayer
82