EMF PRODUCED BY A SHOCK WAVE MOVING IN A DIELECTRIC

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The current flowing in a short-circuited capacitor subjected to shock compression is determined (approximately and exactly) by assuming that a surface charge exists on the wave front when the shock wave passes through the dielectric and electric. The matter in front of the shock wave is assumed to be an insulator and that behind the wave front a conductor. The exact solution is obtained on the assumption that the matter is not compressed and no changes of the dielectric constant occur behind the shock wave front.

STUDIES of the current flowing in a short-circuited solid-dielectric capacitor subjected to shock compression have been reported in a number of recent papers.[1)] We determine here the emf in such a circuit (both approximately and exactly) for a dielectric that behaves as an insulator prior to compression and becomes a conductor after compression. The dielectric is assumed to be isotropic.

We assume that a surface charge with density σ exists on the shock wave front (SWF) as a result of the instantaneous compression and asymmetry (uncompressed matter ahead of the SWF and compressed behind it). The value of σ depends on the material and on the amplitude of the wave, but is constant during the propagation of the SWF through the dielectric. This is equivalent to the same assumption as was made in[1,2] concerning the polarization of the dielectric by the SWF, where it was found that in insulators, in the absence of relaxation processes behind the SWF, the density of the polarization current i in the circuit of Fig. 1 is

$$i = \sigma\varkappa T[\varkappa T + (1-\varkappa)t]^{-2}, \qquad (1)$$

where $\kappa = \epsilon_2\delta/\epsilon_1$, ϵ_1 and ϵ_2 are the dielectric constants of the substance ahead and behind the SWF, δ is the compression, and $T = a/D$ is the time of travel of the SWF with velocity D through the initial thickness of the dielectric a.

Approximate solution. The qualitative differences resulting from conductivity occur when $\Theta \ll T$, where Θ is the characteristic time necessary for the charges to leak through the capacitor by conduction: $\Theta = \rho\epsilon_2/4\pi$. (In our problem ρ is the resistivity behind the SWF.) Then a layer of compensating charge, with volume density ν, is produced behind the SWF.

In a coordinate frame connected with the SWF, with the y coordinate representing the direction of material flow, the current i will contain, besides the term proportional to the field E, also a term describing the transport of charges with the material:

$$i = E/\rho + \nu D/\delta. \qquad (2)$$

The stationarity conditions (i = 0) make it possible to relate E and ν:

$$\nu = -E\delta(D\rho)^{-1}. \qquad (3)$$

Equation (3), in conjunction with the Poisson equation

$$\varepsilon_2(dE/dy) = 4\pi\nu \qquad (4)$$

yields

$$dE/dy = -4\pi\delta/D\rho\varepsilon_2. \qquad (5)$$

Integration of (5) under the condition $E|_{y=0} = E_0 = 4\pi\sigma\epsilon_2^{-1}$ makes it possible to determine

formula

that is, a jump of potential on the order of μ takes place in a layer y_0 behind the SWF, where

$$\mu = \int_0^\infty E dy = E_0 y_0 = D\rho\sigma\delta^{-1}. \qquad (6)$$

If the circuit is closed and ρ is small, then μ falls on a layer a − x (x is the path covered by SWF

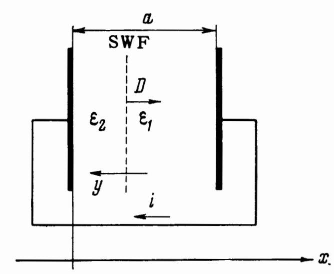

FIG. 1. Circuit diagram of shorted capacitor with dielectric. x – Fixed coordinate axis, y – coordinate axis connected with the SWF.

[1)]In particular, we point to the paper by Ivanov et al. [1], in connection with which the present calculations were made.

in the dielectric) and produces a field $E = \mu/(a - x)$, corresponding to a charge $S = \epsilon_1 \mu[4\pi(a - x)]^{-1}$ and a current density in the circuit

$$i = dS/dt = \varepsilon_1 \sigma \rho D^2 \cdot 4\pi (a - x)^{-2} \delta^{-1}. \quad (7)$$

It follows from (7) that $i \to \infty$ as $x \to a$.

Actually $\rho \neq 0$, and the circuit includes as a rule a certain load resistance R. Therefore formula (7) is limited by the condition that the potential difference on the layer of compressed matter behind the SWF does not exceed μ when $x \to a$:

$$\rho a i/\delta \leqslant D\rho\sigma/\delta, \quad (8)$$

or $i_{max} = D\sigma/a$, that is, i_{max} does not exceed the value of i obtained without allowance for the conductivity and the relaxation processes. Solving (7) and (8) simultaneously, we obtain the limit of applicability of formula (7):

$$a - x \geqslant (aD\rho\varepsilon_1/4\pi\delta)^{1/2} \approx (ay_0)^{1/2}.$$

Accordingly

$$V_{max} = i_{max} R = D\sigma R/a.$$

Formula (7) must be corrected also at the start of the process, during the transient stage. In fact, at t = 0 we have in a real circuit i = 0, that is, the value of i obtained from (7) does not occur instantaneously. It is necessary to take into account here, besides the time Θ, also the relaxation time of the circuit $\tau = RC$, where C is the initial value of the capacitance. Different V(t) curves are obtained at the start of the process, depending on whether $\tau > \Theta$ or $\tau < \Theta$.

If $\Theta < \tau$, then the quantity V(t) grows when $t < \Theta$ and reaches a value $V = \mu$ at $t \sim \Theta$, after which a decrease occurs, within a time τ, to a value determined by (7). If $\Theta > \tau$, a rise takes place at $t < \tau$, up to $t \sim \tau$, after which a decrease takes place, continuing to $t \sim \Theta$. The predicted curve is shown in Fig. 2.

The initial growth of $V = V_{max} tD/a$ continues for a time τ or Θ. A maximum is inevitable in this case. In order of magnitude, $V_{max} \sim Va^2 D^{-2} \Theta^{-1} \tau^{-1}$, where V is the voltage obtained in accordance with (7).

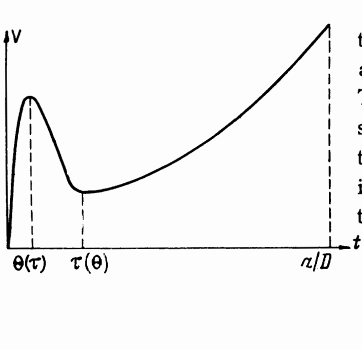

FIG. 2. Approximate time variation of the voltage across the load resistance. The quantities Θ and τ outside the parentheses pertain to the case $\Theta < \tau$, those in the parentheses pertain to $\Theta > \tau$.

Exact solution. Let us consider a short circuited parallel-plate capacitor of thickness a, such that the potential $\varphi(0) = \varphi(a) = 0$. A surface parallel to the electrodes, with bound-charge surface density σ moves from left to right in the capacitor at a velocity D. We have ϵ = const throughout, and the material is at rest ($\delta = 1$). When $x > Dt$, the resistance is $\rho = \infty$, and for $x < Dt$ the value of ρ is finite. The material is assumed isotropic.

In the region $x < Dt$ we have a space-charge density distribution $\nu(x, t)$[2], a potential distribution $\varphi(x, t)$, and a current density distribution $j(x, t)$. The measured quantity is the current flowing through the wire joining the plates of the capacitor and expressed in terms of $\varphi(x, t)$:

$$i = \frac{dS}{dt} = \frac{\varepsilon}{4\pi} \frac{d}{dt} \left(\frac{\partial \varphi}{\partial x} \right) \Big|_{x=a},$$

where S is the charge density the electrode at x = a.

The Poisson equation can be readily integrated under the given boundary conditions with the aid of the Green's function, which gives the potential produced at the point x by a charge situated at the point y,

$$G(x, y) = \begin{cases} 4\pi x (a - y)/\varepsilon a, & x < y \\ 4\pi y (a - x)/\varepsilon a, & x > y \end{cases}; \quad (9)$$

we get

$$\varphi(x, t) = \frac{4\pi\sigma x (a - Dt)}{\varepsilon a} + \frac{4\pi x}{\varepsilon a} \int_x^{Dt} \nu(y, t)(a - y)\, dy + \frac{4\pi(a - x)}{\varepsilon a} \int_0^x \nu(y, t)\, y\, dy, \quad x < Dt, \quad (10)$$

$$\varphi(x, t) = \frac{4\pi\sigma Dt (a - x)}{\varepsilon a} + \frac{4\pi(a - x)}{\varepsilon a} \int_0^{Dt} \nu(y, t)\, y\, dy, \quad x > Dt. \quad (11)$$

In particular, the following expressions will be found useful

$$-\frac{\partial\varphi}{\partial x}\Big|_{x=a} = \frac{4\pi}{\varepsilon a} \left[\sigma Dt + \int_0^{Dt} \nu(y, t)\, y\, dy \right], \quad (12)$$

$$E\Big|_{x=Dt-0} = -\frac{\partial\varphi}{\partial x}\Big|_{x=Dt-0} = \frac{4\pi}{\varepsilon a} \left[-\sigma(a - Dt) + \int_0^{Dt} \nu(y, t)\, y\, dy \right]. \quad (13)$$

We turn to the equation for the continuity of the current

[2] ν does not include the charge of the plane x = Dt, which is equal to σ.

$$\frac{\partial\nu}{\partial t} = -\frac{\partial j}{\partial x} \tag{14}$$

the conduction equation

$$j = \frac{E}{\rho} = -\frac{1}{\rho}\frac{\partial\varphi}{\partial x} \tag{15}$$

and the Poisson equation

$$\varepsilon\frac{\partial E}{\partial x} = -\varepsilon\frac{\partial^2\varphi}{\partial x^2} = 4\pi\nu. \tag{16}$$

We get the relation

$$\frac{\partial\nu}{\partial t} = -\frac{4\pi}{\rho\varepsilon}\nu = -\frac{\nu}{\Theta} \qquad \Theta = \frac{\rho\varepsilon}{4\pi}, \tag{17}$$

from which it follows that the solution in the region $0 < x < Dt$ must be sought in the form

$$\nu = f(x)e^{-t/\Theta}. \tag{18}$$

A special analysis is necessary for the line $x = Dt$. During the time dt, the current brings into the region $dx = Ddt$ a charge $dq = jdt$ from the left, there is no conduction on the right, and consequently

$$\nu = \frac{dq}{dx} = \frac{jdt}{Ddt} = \frac{j}{D} = \frac{E|_{x=Dt-0}}{\rho D}. \tag{19}$$

On the line $x = D$ itself we have a concentrated charge.

We introduce a new system of variables[3]. The time is in units Θ, $t = t'/\Theta$. The length is in units of $D\Theta$, $x = x'/D\Theta$, so that now the front is $x = t$. The entire length of the capacitor is $b = a/D\Theta$ and is dimensionless. The unit surface charge density is taken to be and the unit volume density is the quantity $\sigma/D\Theta$, $\nu = \nu' D\Theta/\sigma$. The unit of potential is $4\pi\sigma D\Theta/\epsilon$, the unit field is $4\pi\sigma/\epsilon$, and the unit current density is σ/Θ. In terms of the new variables we have

$$\varphi = \frac{x(b-t)}{b} + \frac{x}{b}\int_x^t \nu(y,t)(b-y)\,dy + \frac{b-x}{b}\int_0^x \nu(y,t)\,y\,dy, \quad x < t, \tag{10a}$$

$$\varphi = \frac{t(b-x)}{b} + \frac{b-x}{b}\int_0^t \nu(y,t)\,y\,dy, \quad x > t; \tag{11a}$$

$$E|_{x=t-0} = -\frac{b-t}{b} + \frac{1}{b}\int_0^t \nu(y,t)\,y\,dy. \tag{13a}$$

Equation (19) yields

$$\nu(x=t,t) = -\frac{b-t}{b} + \frac{1}{b}\int_0^t \nu(y,t)\,y\,dy. \tag{20}$$

We seek a solution in the form

$$\nu = Z(x)e^{-t}. \tag{21}$$

Substituting (21) in (20), we get

$$Z(t)e^{-t} = -1 + \frac{t}{b} + \frac{e^{-t}}{b}\int_0^t Z(y)\,y\,dy. \tag{22}$$

Multiplying (22) by e^t and taking the derivative, we obtain the differential equation for Z. We again rename the variable x in lieu of t, and then

$$\frac{dZ(x)}{dx} = -e^x + \frac{x}{b}e^x + \frac{1}{b}e^x + \frac{x}{b}Z(x). \tag{23}$$

We obtain the initial conditions by turning to the integral equation (22) and substituting $t = 0$; we get $Z(0) = -1$. Then (23) yields

$$Z = -\exp\left(\frac{x^2}{2b}\right)\left[1 + \int_0^x\left(1 - \frac{y}{b} - \frac{1}{b}\right)\exp\left(y - \frac{y^2}{2b}\right)dy\right]$$
$$= -\exp\left(\frac{x^2}{2b}\right)\left[\exp\left(x - \frac{x^2}{2b}\right) - \frac{1}{b}\int_0^x \exp\left(y - \frac{y^2}{2b}\right)dy\right]. \tag{24}$$

Substituting this in (21), we get

$$\nu(x,t) = e^{-t}Z(x) = -e^{x-t} + \frac{1}{b}\exp\left(-t + \frac{x^2}{2b}\right) \times \int_0^x \exp\left(y - \frac{y^2}{2b}\right)dy. \tag{25}$$

Thus, for $b \gg 1$ we have asymptotically $\nu = -e^{x-t}$, corresponding to a space charge that compensates σ in a double layer of thickness $y_0 = D\Theta$.

Let us find the current through the right-side plate. In dimensionless units we have

$$i = \frac{\partial}{\partial t}E|_{x=b,t} = \frac{1}{b}\frac{\partial}{\partial t}\left[t + \int_0^t \nu(x,t)\,x\,dx\right]. \tag{26}$$

We substitute here (25) and get

$$i = \frac{1}{b} - \frac{d}{dt}\left[e^{-t}\int_0^t e^x x\,dx\right] + \frac{1}{b^2}\frac{d}{dt}\left\{e^{-t}\int_0^t \exp\left(\frac{x^2}{2b}\right)x\left[\int_0^x \exp\left(y - \frac{y^2}{2b}\right)dy\right]dx\right\}. \tag{27}$$

We transform the integral by parts

$$\int_0^t \exp\left(\frac{x^2}{2b}\right)x\,dx\left[\int_0^x \exp\left(y - \frac{y^2}{2b}\right)dy\right]$$
$$= b\int_0^t\left[\int_0^x \exp\left(y - \frac{y^2}{2b}\right)dy\right]d\left[\exp\left(\frac{x^2}{2b}\right)\right]$$
$$= b\exp\left(\frac{t^2}{2b}\right)\int_0^t \exp\left(y - \frac{y^2}{2b}\right)dy$$
$$- b\int_0^t \exp\left(\frac{x^2}{2b} + x - \frac{x^2}{2b}\right)dx$$

[3] The old-dimensional-quantities are primed, and the new-dimensionless-are without an index.

$$= b\exp\left(\frac{t^2}{2b}\right)\int_0^t \exp\left(y-\frac{y^2}{2b}\right)dy - b[e^t-1].$$

Substituting in (27), we get after cancellation

$$i = \frac{1}{b}\frac{d}{dt}\left[\exp\left(-t+\frac{t^2}{2b}\right)\int_0^t \exp\left(y-\frac{y^2}{2b}\right)dy\right] \tag{28}$$

$$= \frac{1}{b}\left[1-\left(1-\frac{t}{b}\right)\exp\left(-t+\frac{t^2}{2b}\right)\int_0^t \exp\left(y-\frac{y^2}{2b}\right)dy\right].$$

It follows from (28) that

$$i(t=0)=1/b, \quad i(t=b)=1/b. \tag{29}$$

This value corresponds to a zero conduction effect, that is, in the dimensional form the expression $i = 1/b$ in (29) yields $i' = \sigma D/a$ in accord with (1).

We now find the intermediate asymptotic value. To this end we assume that $t \gg 1$ and $b - t \gg 1$, and neglect the exponentially small terms. We introduce the variable

$$u = (t-y)(1-t/b). \tag{30}$$

It is easy to verify that

$$I = \left(1-\frac{t}{b}\right)\exp\left(-t+\frac{t^2}{2b}\right)\int_0^t \exp\left(y-\frac{y^2}{2b}\right)dy$$

$$= \int_0^{t(1-t/b)} \exp\left[-u-\frac{bu^2}{2(b-t)^2}\right]du. \tag{31}$$

Putting $b(b-t)^{-2} \ll 1$, we expand the exponential and neglect the exponentially small terms, replacing the limit $t(1-t/b) \to \infty$. We get

$$I = \int_0^\infty e^{-u}\left[1-\frac{bu^2}{2(b-t)^2}\right]du = 1-\frac{b}{(b-t)^2}. \tag{32}$$

It follows therefore that in this approximation

$$i = \frac{1}{b}[1-I] = \frac{1}{(b-t)^2} \tag{33}$$

In dimensional form, (33) yields for the current an expression proportional, given by (7), to $(a - Dt)^{-2}$ in the middle of the interval.

The general expression (28) thus contains different limiting cases. The case $b < 1$ denotes small conductivity and a current $i \approx 1/b$ which is constant for all $0 < t < b$; the case $b \gg 1$ corresponds to large conductivity, when a double layer is produced and the variation of i exhibits a deep minimum (33). A plot of $\eta = i/i_0$, where i_0 corresponds to the formula $i_0 = \sigma D/a$, for different values of the conductivity yields the picture shown in Fig. 3. In the general case we have $\eta = W(t/T, b)$, where W is expressed in terms of the error integral.

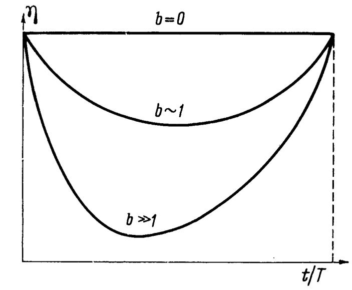

FIG. 3. The dependence of the relative current in the presence of conductivity behind the SWF.

In comparing with Fig. 2, we note that the exact solution shown in Fig. 3 pertains to the case $R = 0$ and $\tau = 0$, so that the left-side maximum disappears.

It is interesting to note that when water is used as the dielectric[3] the qualitative character of the experimentally registered current oscillograms coincides with the curves of Fig. 3. An explanation for this fact is the sharp increase of the conductivity of the water on the SWF.[4,5]

I take the opportunity to thank the authors of[1] for suggesting the problem and for valuable discussions.

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