SPECTRAL PROPERTIES OF LASERS WITH A LARGE ANGULAR DIVERGENCE OF LIGHT

A. M. RATNER

Physico-technical Institute of Low Temperatures, Academy of Sciences, Ukrainian S.S.R.

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The spectral characteristics of a solid-state laser with a large angular beam divergence is considered. The large divergence results in strong spectral overlap of longitudinal modes and hence to degeneracy of the modes generated at a frequency corresponding to the maximum of the amplification band. It is shown that for a sufficiently large divergence angle the main contribution to the integral intensity is from degenerate modes which form a narrow spectral distribution peak. The width of this peak is much smaller than the spectral width of light generated by an ordinary plane-parallel resonator. The conclusions of the theory are confirmed by the experimental data.

1. INTRODUCTION

THE spectral properties of a laser with plane-parallel reflectors were considered by Tang et al.[1] and in greater detail by Livshitz and Tsikunov,[2,3] who showed that the nonlinear interaction between the electromagnetic fields of different longitudinal modes leads to simultaneous generation of a considerable number of these modes, and consequently, to a relatively large spectral width of the generated radiation. This is physically connected with the spatial inhomogeneity of the inverted population. If, for example, only one longitudinal mode were to be generated, then an increased inverted population would be produced near the nodes of this standing wave and would allow simultaneous generation of other standing waves, in which the intensity distribution overlaps the maxima of the inverted population.

The results of the aforementioned authors were obtained for a resonator with infinite cross section, i.e., without actual allowance for the transverse modes. Allowance for the latter, however, does not change these results in practice, provided the number of generated transverse modes is sufficiently small so that modes with different longitudinal indices do not overlap spectrally. This means physically that longitudinal modes of the resonator do not interact in practice with the transverse modes, provided the longitudinal modes remain spectrally separated. We note that this condition is usually satisfied in the case of open plane-parallel resonators.

We consider in this paper the opposite limiting case, when the spectral interval encompassed by the transverse modes is much larger than the interval between the neighboring longitudinal modes, i.e., there is a strong spectral overlap of the longitudinal modes. We shall show that the associated interaction between the longitudinal and transverse modes leads to a considerable narrowing of the spectral line of the laser. Physically this is attributed to the degeneracy of the modes, whose longitudinal and transverse indices can combine in such a way as to make the frequencies fall in a narrow spectral interval in the vicinity of the maximum amplification band; a large gain factor ensures predominant generation of such modes.

It will be seen from what follows that the degree of narrowing of the spectral line is determined essentially by the angular divergence of the light, i.e., by the number of generated transverse modes, and that for a given divergence angle it is practically independent of the resonator shape. We shall carry out the calculations for a plane-parallel resonator with a large angular divergence (due, for example, to total internal reflection of the light from the smooth lateral surface of the active element); the main results, however, can be extended to include the more common case of a resonator with curved reflectors. The analysis will be carried out for the case of stationary generation under the assumption that the luminescence band is homogeneously broadened.

2. EQUATION FOR THE SPECTRAL DISTRIBUTION OF LIGHT

Let us consider the electromagnetic field in a resonator with an active medium. This field constitues a superposition of many natural oscillations (modes) of the resonator. We denote by $J_n(\mathbf{r})$ the intensity of the field of the n-th mode ($\mathbf{r}$—radius vector of the point in the resonator), and by $\kappa(\omega, \mathbf{r})$ the effective absorption coefficient which takes into account both the losses and the amplification of the light in the resonator with the active medium.[4] Then the condition for stationary generation takes the form (see [1-3])

$$\int \varkappa(\omega_n, \mathbf{r}) J_n(\mathbf{r}) d\mathbf{r} = 0 \tag{1}$$

(ω_n—frequency of the n-th mode; integration is carried out over the resonator volume). Equation (1) denotes that the average power drawn by the electromagnetic field of each mode from the active medium and the resonator should be equal to the average power loss, since the intensity of each mode remains constant in time under stationary-generation conditions.[1)]

Inasmuch as the gain depends on the intensity of the electromagnetic field, the condition (1) is a nonlinear equation for the distribution of the intensity of the laser radiation over the modes. In order to consider this equation, it is necessary to represent in explicit form the dependence of κ on J_n. To this end we write down the effective absorption coefficient (confining ourselves for simplicity to the case of a four-level scheme):

$$\varkappa(\omega, \mathbf{r}) = \varkappa_1 - Cn(\mathbf{r})P(\omega) = \varkappa_1 - Bn(\mathbf{r}) \times [1 - (\omega - \omega_0)^2/\bar{\omega}^2 + \ldots]. \tag{2}$$

Here κ_1 is the loss of light energy per unit length, n the number of excited luminescence centers per unit volume, $P(\omega)$ the form of the amplification band (which practically coincides with the luminescence band), C and $B \equiv CP(\omega_0)$ are certain constants, ω_0 is the frequency corresponding to the maximum gain, and $\overline{\omega}$ is a quantity on the order of half-width of the luminescence band. The second term in (2) is the gain; we had expanded it in terms of the frequency in the vicinity of the maximum of the gain, retaining only the quadratic terms.

The generation threshold is reached when the losses are offset by the gain at the frequency ω_0; hence

$$B = \varkappa_1/n^*, \tag{3}$$

where n^* is the threshold value of the number of excited atoms.

In order to express the number of excited atoms in terms of the intensity of the electromagnetic field, we write down the equation for n, assuming the generation to be stationary:

$$\frac{\partial n}{\partial t} = 0 = -\frac{n}{T} - \frac{vBnJ}{\hbar\omega_0} + \frac{N}{\hbar\omega_0}. \tag{4}$$

Here T is the time of spontaneous emission of the excited centers, v the velocity of light in the medium, J the volume energy density of the induced emission, and N the pump power absorbed per unit volume. The first and second terms of (4) describe the spontaneous and stimulated emission of the excited atoms, and the third the excitation of the atoms by the pump. Solving (4) with respect to n and substituting in (2) with allowance for (3), we get

$$\varkappa = \varkappa_1 \left\{ \frac{1 + v\varkappa_1 JT/n^*\hbar\omega_0 - NT/n^*\hbar\omega_0}{1 + v\varkappa_1 JT/n^*\hbar\omega_0} + \frac{n}{n^*} \frac{(\omega - \omega_0)^2}{\overline{\omega}^2} \right\}. \tag{5}$$

In order to simplify this expression, we note that the threshold pump power N^* is equal to the spontaneous-emission power when $n = n^*$, i.e., $N^* = n^*\hbar\omega_0/T$. Further, in the second small term in the curly brackets it is possible to neglect the slight deviation of n from the threshold value during the generation process. Introducing the dimensionless light energy

$$\mathscr{E} = Jv\varkappa_1/(N - N^*), \tag{6}$$

we get

$$\varkappa = \varkappa_1 \left\{ -\frac{1 - \mathscr{E}}{N^*/(N - N^*) + \mathscr{E}} + \left(\frac{\omega - \omega_0}{\overline{\omega}}\right)^2 \right\}. \tag{7}$$

It will be seen from what follows that the spatial distribution of the intensity $\mathscr{E}$ is practically homogeneous,[2)] and that $\mathscr{E}$ differs little from unity. This enables us to simplify (7) somewhat, putting in the denominator of the first term $\mathscr{E} = 1$ (we neglect by the same token the terms which are quadratic in $1 - \mathscr{E}$ and retain the linear ones). Substituting (7) in (1) we obtain an equation for the distribution of the intensity over the modes[3)]

[1)]In the simplest cases, condition (1) can be obtained from the wave equation for the electromagnetic field in the resonator [5].

[2)]This is physically connected with the simultaneous generation of a large number of modes with nearly equal amplitudes and different spatial distribution of the intensity.

[3)]The field intensity $\mathscr{E}(\mathbf{r})$ is equal to the sum of the intensities of the modes, since the terms corresponding to interference between different modes n and n' contain the time-dependent factor $\exp[it(\omega_n - \omega_{n'})]$ and drop out upon averaging with respect to time. A similar equation was used in [1-3] for a system of longitudinal modes with the transverse ones neglected.

$$\int\left\{-\xi+\xi\sum_{n'}\mathscr{E}_{n'}(\mathbf{r})+\left(\frac{\omega_n-\omega_0}{\overline{\omega}}\right)^2\right\}\mathscr{E}_n(\mathbf{r})\,d\mathbf{r}=0. \quad (8)$$

We have introduced here the symbol

$$\xi=1-N^*/N \quad (9)$$

for a four-level scheme.

Omitting the calculations, we assert that Eq. (8) and all the results that follow from it remain in force also for a three-level system, provided we put

$$\xi=1-\frac{N^*}{N\chi+N^*(1-\chi)} \quad (10)$$

for the three-level scheme, where χ is the ratio of the total light losses in the resonator with unexcited active medium to the losses in the same resonator without the active medium.

We note that the pump intensity N enters into the spectral characteristics of the laser only in terms of the parameter ξ, which tends to unity as $N \to \infty$. A similar saturation effect was considered by Livshitz and Tsikunov[3] for a resonator with a small angular divergence of the light.

Equation (8) assumes some concrete form, depending on the shape of the resonator. Let us consider first a closed resonator in the form of a rectangular parallelepiped with a square cross section. If we number the longitudinal modes by means of the index m and the transverse one by the indices p and q, then the spatial distribution of the intensity of the natural oscillations takes the form

$$\mathscr{E}_{mpq}(\mathbf{r})=A_{mpq}\sin^2\frac{\pi mz}{l}\sin^2\frac{\pi px}{a}\sin^2\frac{\pi qy}{a}. \quad (11)$$

Here l is the resonator length, a the side of the cross section, A_{mpq} the positive amplitude of the intensity; the function of the indices A_{mpq} specifies the distribution of the intensity over the modes.

Equation (8) includes not only the intensity of the natural oscillations, but also their spectrum. Let $k_{||} = \pi m/l$ be the longitudinal component and $k_\perp = \pi(p^2+q^2)^{1/2}/a$ the transverse component of the wave vector. Usually the light propagates at a small angle to the optical axis, i.e., $k_\perp \ll k_{||}$. Making use of this fact, we get

$$\omega_{mpq}\equiv vk_{mpq}=v\sqrt{k_{||}^2+k_\perp^2}=v(k_{||}+k_\perp^2/2k_{||}+\ldots) \quad (12)$$

(v is the velocity of light in the medium). The relative change in $k_{||}$ within the limits of the spectral line of the laser is insignificant; we can therefore put in the second term of (12) $k_{||} = \mathrm{const} = \omega_0/v$. Thus,

$$(\omega_{mpq}-\omega_0)/\overline{\omega}=[\alpha(m-m_0)+\beta(p^2+q^2)]\sqrt{\xi}/8, \quad (13)$$

$$\alpha=\frac{8\pi v}{l\overline{\omega}\sqrt{\xi}}\qquad \beta=\frac{4\pi^2v^2}{a^2\omega_0\overline{\omega}\sqrt{\xi}},\qquad m_0=\frac{\omega_0 l}{\pi v}. \quad (14)$$

(The longitudinal index m_0 for $q = p = 0$ corresponds to a "central" frequency ω_0.) Usually[4] $\alpha \ll 1$ and $\beta \ll 1$.

Substitution of (11) and (13) in (8) entails no difficulty if we recognize that

$$4\int_0^a \sin^2(\pi px/a)\sin^2(\pi p'x/a)\,dx=a(1+\delta_{pp'}/2).$$

We ultimately get

$$A_{mpq}=\gamma_{mpq}-[\alpha(m-m_0)+\beta(p^2+q^2)]^2, \quad (15)$$

where

$$\gamma_{mpq}=64-8\sum_{m',p',q'}A_{m'p'q'} -4\Big(\sum_{p',q'}A_{mp'q'}+\sum_{m',q'}A_{m'pq'}+\sum_{m',p'}A_{m'p'q}\Big) -2\Big(\sum_{q'}A_{mpq'}+\sum_{p'}A_{mp'q}+\sum_{m'}A_{m'pq}\Big). \quad (16)$$

We shall assume that the region of summation over the transverse indices is bounded by the inequality

$$p^2+q^2\leqslant\bar{p}^2,\qquad p>0,\qquad q>0. \quad (17)$$

This means that the angular divergence of the radiation is confined to a cone which is coaxial with the laser.

3. CASE OF PLANE-PARALLEL RESONATOR. PEAK OF SPECTRAL DISTRIBUTION

Inasmuch as the gain has a maximum at the frequency ω_0, it follows that the greatest intensity is possessed by the modes with frequency close to ω_0; if the number of such modes is sufficiently large, then the fraction of their energy is appreciable and they form a peak in the spectral distribution of the generated light. The remaining modes form the "wings" of the spectral distribution (see the figure). We shall investigate the spectral width and the ratio of the integral intensities of the "wings" and of the peak under the assumption that this ratio is small compared with unity.

We shall calculate in this section the spectral width of the peak.

It is seen from (13) that if the longitudinal indices m do not go beyond the interval

$$m_0-\beta\bar{p}^2/\alpha\leqslant m\leqslant m_0, \quad (18)$$

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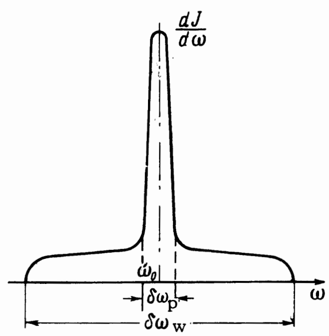
4)Under ordinary experimental conditions we have



then by suitable choice of the transverse indices from the allowed region (17) we can cause the difference $\omega_{mpq} - \omega_0$ to vanish. Thus, the modes from the interval (18) form a peak of the spectral distribution, and the remaining modes form the "wings."

As will be shown later, the peak of the spectral distribution is sufficiently clearly pronounced only if both $\bar{p}$ and the integral (18), which is equal to $\bar{m} = \beta\bar{p}^2/\alpha$, are appreciably larger than unity. Using this, we introduce the small parameters $1/\bar{p}$ and $1/\bar{m}$. If we assume that $\sum_{mpq} A_{mpq}$ is a quantity of zeroth order of smallness with respect to these parameters, then the second line in (16) is a first-order quantity, and the third line is a quantity of second order of smallness; in fact, the sum of the positive quantities A_{mpq} is, roughly speaking, proportional to the number of terms. Further, inasmuch as the second term of formula (15) can vanish in the region (18), it is seen from this formula that the positive quantity A_{mpq} varies in the range from zero to γ_{mpq}; consequently, the quantities γ_{mpq} and A_{mpq} are of the same, third, order of smallness in the region (18).

Retaining the zeroth-order terms in (16) we get, with allowance for (15),[5)]

$$1 - \frac{1}{8}\sum_{m,p,q} A_{mpq} \equiv 1 - \frac{1}{8}\sum_{m,p,q}\{\gamma_{mpq} - [\alpha(m-m_0) + \beta(p^2+q^2)]^2\} = 0. \quad (19)$$

We can restrict the summation here to the region (18), since, by assumption, the integral intensity of the peak is much larger than that of the wings. We shall seek the solution of (19) in the form $\gamma_{mpq} = \text{const} = \gamma$. The sum in (19), which contains a large number of positive terms, can be replaced by the integral

$$\frac{1}{4}\int_{m_0-\beta\bar{p}^2/\alpha}^{m_0} dm \iint dp\,dq\,\{\gamma - [\alpha(m-m_0) + \beta(p^2+q^2)]^2\}. \quad (19a)$$

The region of integration with respect to p and q is replaced here by the inequality

$$\alpha(m_0 - m) - \sqrt{\gamma} \leqslant \beta(p^2+q^2) \leqslant \alpha(m_0 - m) + \sqrt{\gamma}, \quad (20)$$

which follows from the condition that the integrand be non-negative (we have extended the integration to the region of negative p and q, and divided the integral by 4). Actually the inequality (20) also takes into account the limitation (17), since the center of the integration region (20) does not go beyond the limits of the interval (17), and the length of the interval (20), as will be shown by what follows, is much smaller (by a factor $\bar{p}^{-2/3}$) than the length of the interval (17).

The integral (19a) can be readily calculated in cylindrical coordinates, and is equal to $\pi\bar{p}^2\gamma^{3/2}/3\alpha$. Consequently, the solution of (19) takes the form

$$\gamma = (24\alpha/\pi\bar{p}^2)^{2/3}. \quad (21)$$

Substituting it in (15), we get the distribution of the intensity over the modes:

$$A_{mpq} = \left(\frac{24\alpha}{\pi\bar{p}^2}\right)^{2/3} - \frac{64}{\xi}\left(\frac{\omega_{mpq} - \omega_0}{\bar{\omega}}\right)^2. \quad (22)$$

Since the A_{mpq} are positive, we obtain from this the spectral width of the peak:

$$\delta\omega_p = \frac{\bar{\omega}\sqrt{\xi}}{2}\left(\frac{3\alpha}{\pi\bar{p}^2}\right)^{1/3} \equiv \delta\omega_{\parallel}(\pi\bar{p}^2)^{-1/3}. \quad (23)$$

We have introduced here for clarity the spectral interval $\delta\omega_{\parallel}$, which is encompassed by the longitudinal modes of a plane-parallel resonator with small angular divergence of the light; as is well known[1-3]

$$\delta\omega_{\parallel} = (3\pi\xi v\bar{\omega}^2/l)^{1/3}. \quad (24)$$

Thus, with increasing angular divergence of the light, the peak of the spectral distribution becomes narrower and the integral intensity of the peak, as will be shown below, increases. We note that formula (23) is applicable only for sufficiently large angular divergence, when the integral intensity of the peak is larger than that of the "wings."

4. CASE OF PLANE-PARALLEL RESONATOR. "WINGS" OF SPECTRAL DISTRIBUTION

As shown above, the modes which do not fit in the interval (18) are shifted in frequency to the left or to the right relative to ω_0, and form the "wings" of the spectral distribution. Let us consider the spectral width and the integral intensity of the "wings."

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5)We see therefore that in the zeroth approximation the average value of the intensity is $\overline{\mathscr{E}(\mathbf{r})} = \frac{1}{8}\sum_{m,p,q} A_{mpq}$ is equal to unity.

We have seen above that Eq. (16) determines, in the zeroth approximation, the shape of the peak; this is connected with the fact that the triple sum which enters in the zeroth approximation is the integral intensity of the generated light, and is concentrated practically entirely in the peak. In order to consider the "wings" of the spectral distribution, we retain in (16) the terms of first order of smallness. We have

$$\begin{aligned}\gamma_{mpq} &+ 4\Big(\sum_{p',q'} A_{mp'q'} + \sum_{m',q'} A_{m'pq'} + \sum_{m',p'} A_{m'p'q}\Big)\\ &= 64 - 8\sum_{m',p',q'} A_{m'p'q'} = \text{const} = \gamma_{Mpq}\\ &+ 4\Big(\sum_{p',q'} A_{Mp'q'} + \sum_{m',q'} A_{m'pq'} + \sum_{m',p'} A_{m'p'q}\Big).\end{aligned} \tag{25}$$

Here M is a longitudinal index arbitrarily chosen in the interval (18), i.e., in the region of the peaks; the index m, on the other hand, belongs to the region of the wings. We have retained γ_{mpq} in (25) because in the region of the wings, unlike in the region of the peak, the order of magnitude of γ_{mpq} cannot be established beforehand with the aid of (15). On the other hand, the value of γ_{Mpq} in the region of the peak, as shown in the preceding section, is of third order of smallness and can be neglected. Taking this into account and equating the first and last expressions in the chain (25), we get

$$\gamma_{mpq} + 4\sum_{p',q'} A_{mp'q'} = 4\sum_{p',q'} A_{Mp'q'}. \tag{26}$$

The sums involved here can be replaced by integrals. Going over to an integral in cylindrical coordinates, and using (15), we get[6)]

$$\sum_{p',q'} A_{mp'q'} = \frac{\pi}{2}\int \{\gamma_{mpq} - [\alpha(m - m_0) + \beta r^2]^2\}\, r\, dr, \tag{27}$$

with the integration region defined by the inequalities

$$|\alpha(m - m_0) + \beta r^2| \leqslant \sqrt{\gamma_{mpq}}, \tag{28a}$$

$$0 \leqslant r \leqslant \bar{p}. \tag{28b}$$

In Eq. (26) we encounter two sums of the type (27), in one of which the longitudinal index M belongs to the region of the peak (18), and in the other the index m belongs to the region of the wings. In the first case we can disregard the limitation (28b) and use expression (21) for γ. In the second case the upper limit of the integral (27) is equal to $\bar{p}$, and the lower limit is given by the inequality (28a), with γ_{mpq} being the sought unknown. (For concreteness we consider the left wing, i.e., the region adjacent to the interval (18) from the left.) Taking the foregoing into account, the integration can be performed in elementary fashion, and Eq. (26) takes the form

$$\frac{\pi}{3\beta}[2\gamma_{mpq}^{3/2} - 3\gamma_{mpq}\alpha\Delta m + (\alpha\Delta m)^3] + \gamma_{mpq} = \frac{32\alpha}{\beta\bar{p}^2}. \tag{29}$$

Here $\Delta m = m_0 - \beta\bar{p}^2/\alpha - m$ is the distance of the point m from the left boundary of the interval (18).

We have obtained an equation for γ_{mpq} in the region of the wings. Carrying out simple algebraic transformations and omitting the indices, we write this equation in the form

$$\frac{\pi}{3}(\sqrt{\gamma} - \alpha\Delta m)^2(2\sqrt{\gamma} + \alpha\Delta m) + \beta\gamma = 32\alpha\bar{p}^{-2}. \tag{30}$$

Since the right side of (30) contains a very small quantity, and all the terms of the left side are positive, Eq. (30) can be satisfied only if $\sqrt{\gamma} \cong \alpha\Delta m$. Putting in (30) $\sqrt{\gamma} = \alpha\Delta m$ everywhere except in the left factor, we get

$$\sqrt{\gamma} = \alpha\Delta m + \left\{\frac{1}{\pi\alpha\Delta m}\left[\frac{32\alpha}{\bar{p}^2} - \beta(\alpha\Delta m)^2\right]\right\}^{1/2}. \tag{31}$$

Substituting (31) in (15), we obtain the distribution of the intensity over the modes. In order to find the integral intensity of the wings, it is necessary to sum the amplitude of the intensity A_{mpq} over p and q and over the values of m that do not belong to the interval (18). The summation can be carried out without difficulty in the same manner as above. We finally obtain the ratio of the integral intensity of the wings to the integral intensity of the peak:[7)]

$$\frac{\mathscr{E}_w}{\mathscr{E}_p} = \frac{8}{3}\sqrt{\alpha}\left(\frac{2}{\beta\bar{p}^2}\right)^{3/2} = \left(\frac{2\delta\omega_{\|}}{3\delta\omega_{\perp}}\right)^{3/2}. \tag{32}$$

We have used here the notation (24) and introduced the spectral interval

$$\delta\omega_{\perp} = \pi^2 v^2 \bar{p}^2/2a^2\omega_0 = \bar{\omega}\sqrt{\xi}\beta\bar{p}^2/8, \tag{33}$$

which is encompassed by the transverse modes. The ratio (32) decreases with increasing number of generated transverse modes, i.e., with increasing angular divergence of the light.

The result has the following physical meaning. The peak of the spectral distribution is made up by degenerate modes, whose frequencies practically coincide with the maximum of the amplification

6)We make use of the fact that, according to (31), γ_{mpq} is actually independent of p or q.

7)It is easy to verify that the left and right wings of the spectral distribution have the same form and the same integral intensity.

band ω_0. The larger the number of such degenerate modes, the more pronounced the peak is and the smaller the ratio (32). On the other hand, the degree of degeneracy is characterized by the magnitude of the interval $\delta\omega_\perp$. If $\delta\omega_\perp$ reaches the value $\delta\omega_\parallel$, then roughly half the total of the generated modes become degenerate, and the integral intensity of the peak becomes comparable with intensity of the wings. With further increase of $\delta\omega_\perp$, the contribution of the nondegenerate modes tends to zero together with the ratio (32).

Let us consider now the spectral width of the wings. From (31) we can easily obtain the maximum value $\Delta m = \Delta m_{max}$, which causes the radicand to vanish, and also the widths of both wings:

$$\delta\omega_w = 2\Delta m_{max}\frac{\pi v}{l} = \frac{8\pi v}{l}\left(\frac{2}{\alpha\beta\bar{p}^2}\right)^{1/2} = \delta\omega_\parallel\left(\frac{2\delta\omega_\parallel}{3\delta\omega_\perp}\right)^{1/2}. \tag{34}$$

Thus, the spectral widths of both the peak and the wings decrease with increasing degree of degeneracy. When the degeneracy is sufficiently strong, not only the peak but also the wings become narrower than the spectral interval $\delta\omega_\perp$ subtended by the transverse modes. However, there is nothing strange in this, since each individual longitudinal mode subtends only a small fraction of the interval $\delta\omega_\perp$, rather than the entire interval.

5. CASE OF RESONATOR WITH CONCAVE MIRRORS

The formulas (23), (32), and (34) obtained above, which express the spectral characteristics of the laser in terms of the maximum index of the generated transverse modes, are not restricted to a plane-parallel resonator. Apart from numerical factors, these formulas are applicable also for a resonator with concave mirrors, which, as is well known, has a relatively large angular divergence of the light. The maximum transverse index $\bar{p}$ is connected with the angular divergence θ by the relation $\bar{p} \sim ak\theta/\pi$; substituting this in (23), (32), and (34), we express the spectral characteristics in terms of the angular divergence:

$$\frac{\delta\omega_p}{\delta\omega_\parallel} \sim \frac{1}{(ak\theta)^{2/3}}, \quad \frac{\delta\omega_w}{\delta\omega_\parallel} \sim \frac{1}{\theta}\sqrt{\frac{\delta\omega_\parallel}{\omega_0}}, \quad \frac{\mathscr{E}_w}{\mathscr{E}_p} \sim \frac{1}{\theta^3}\left(\frac{\delta\omega_\parallel}{\omega_0}\right)^{3/2}. \tag{35}$$

Formulas (35) are applicable to a resonator of arbitrary shape with sufficiently large angular divergence, satisfying the condition $\theta \gtrsim \sqrt{\delta\omega_\parallel/\omega_0}$.

Let us consider a resonator with spherical mirrors. From Vaĭnshteĭn's theory[6] it follows that in such a resonator the radius of the p-th transverse modes is of the order of $p^{1/2}(lR)^{1/4}k^{-1/2}$, where R is the radius of curvature of the mirror, and k is the wave vector of the light. Obviously the radius of the mode with the maximum transverse index $\bar{p}$ should coincide with the radius of the active element a, hence[8)]

$$\bar{p} \sim ka^2/\sqrt{lR}, \quad \theta \sim a/\sqrt{lR}. \tag{36}$$

According to Vaĭnshteĭn,[6] the neighboring transverse modes of the spherical resonator are separated by a spectral interval of the order of $v\sqrt{2/lR}$, so that the transverse modes subtend the spectral interval

$$\delta\omega_\perp \sim \omega_0 a^2/lR. \tag{37}$$

Substituting (36) and (37) either in formulas (23), (32), and (34), or in the relations (35), we express the spectral characteristics of the spherical resonator in terms of its geometrical parameters:[9)]

$$\frac{\delta\omega_p}{\delta\omega_\parallel} \sim \left(\frac{lR}{k^2a^4}\right)^{1/3}, \quad \frac{\delta\omega_w}{\delta\omega_\parallel} \sim \left(\frac{lR\delta\omega_\parallel}{a^2\omega_0}\right)^{1/2}, \quad \frac{\mathscr{E}_w}{\mathscr{E}_p} \sim \left(\frac{lR\delta\omega_\parallel}{a^2\omega_0}\right)^{3/2}. \tag{38}$$

These formulas can be generalized to include the case of concave reflectors in the form of an arbitrary surface of revolution. We can show[7] that for a resonator with such mirrors $\theta \sim \sqrt{b/l}$, where b is the sag of the reflector within the confines of the generating part of the cross section. Substituting in (35), we get

$$\frac{\delta\omega_p}{\delta\omega_\parallel} \sim \left(\frac{l}{bk^2a^2}\right)^{1/3}, \quad \frac{\delta\omega_w}{\delta\omega_\parallel} \sim \left(\frac{l\delta\omega_\parallel}{b\omega_0}\right)^{1/2}, \quad \frac{\mathscr{E}_w}{\mathscr{E}_p} \sim \left(\frac{l\delta\omega_\parallel}{b\omega_0}\right)^{3/2}. \tag{39}$$

The condition for the applicability of the results $\mathscr{E}_w/\mathscr{E}_p \ll 1$, takes in the case of a spherical resonator the form

$$R \lesssim \bar{R} \equiv a^2\omega_0/l\delta\omega_\parallel. \tag{40}$$

This condition is simultaneously also the condition for an appreciable narrowing of the spectral line of the laser.

The narrowing of the spectral line was investigated experimentally in a ruby laser,[9] using in lieu of a spherical resonator a plane-parallel resonator with two identical positive lenses. Such a resonator is analogous to a spherical one with a radius of curvature R = F, where F is the focal distance of the lenses. Experiment shows that narrowing down of the spectrum is observed only for sufficiently strong lenses, satisfying the condition $F \lesssim \bar{R}$, the width of the spectral line in this region decreasing with decreasing F. When the radius of

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8)This formula can be readily obtained in the geometrical-optics approximation [7,8].

9)Formula (38) or (39) is applicable when $R > l/2$ or $b < a^2/l$.

the generating part of the active element is decreased with the aid of a diaphragm introduced into the resonator, the spectral width of the radiation increases, in accordance with (38). Experiment also shows that narrowing of the spectral line of the laser is accompanied by regular relaxation oscillations of the intensity; this is apparently connected with the fact that the intensity of all the degenerate modes varies with time in accordance with the same law.

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