RADIATION FROM A NON-EQUILIBRIUM PLASMA

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General fluctuation theory is used to compute the intensity and energy density of radiation into a vacuum from a semi-infinite electron plasma characterized by a non-Maxwellian particle distribution, assuming specular reflection of the electrons from the interface.

IT is well known that the intensity of thermal radiation from the surface of an absorbing medium in thermodynamic equilibrium (into a vacuum) is given by the Kirchhoff relation

$$I_\omega = I_{0\omega}(1 - R(\omega, \vartheta)), \tag{1}$$

where $I_{0\omega}$ is the equilibrium radiation flux at a given temperature while R is the reflection coefficient for the energy of the unpolarized radiation incident from vacuum on a plane surface of the body. If the frequencies involved are not too high ($\omega \ll T/\hbar$) the situation is described by the Rayleigh-Jeans law

$$I_{0\omega} = \omega^2 T/8\pi^3 c^2, \tag{2}$$

where T is the temperature (in energy units).[1]

In a plasma (at high thermal velocities or low densities $1/\tau \ll T/\hbar$) there can be a long time period in which a non-equilibrium distribution obtains, in which case the Kirchhoff relation does not hold.[2] It is then of interest to obtain certain general relations, which characterize the radiation from such a non-equilibrium plasma. If the particle distribution is such that the spatial dispersion can be neglected, the Doppler effect causes the dependence on the distribution function to be much more complicated than mere replacement of T in (2) by the mean kinetic energy of the plasma particles $\sim \langle v^2 \rangle$.

1. In the present work, employing the method used for thermal equilibrium,[3] we solve the problem of macroscopic electrodynamics for the radiation of current fluctuations in a concrete non-equilibrium medium, a plasma that occupies the half-space $z < 0$. In the non-equilibrium case the complete solution would also require the solution of the kinetic problem. In this connection we offer the following considerations. For reasons of simplicity we assume that the plasma consists of electrons with a specified stable[1)] nonrelativistic homogeneous and isotropic velocity distribution and (infinitely heavy) fixed ions with a fixed density n. The last assumption imposes a limitation on the frequencies [$\omega \gg (4\pi e^2 n/M)^{1/2}$]. Assuming that the frequency and wave numbers for the radiation in the medium are large compared with the characteristic scale size for the distribution function ($\omega \gg 1/\tau$, $k(\omega) \gg 1/l$, τ is the energy relaxation time and l is the electron mean free path), in computing the radiation we need not take account of the evolution of the non-equilibrium distribution in time and space and can introduce the macroscopic tensors corresponding to this distribution. This procedure in the semi-infinite medium also involves the condition on the reflection of particles from the interface. One can assume that the distribution function depends only on the velocity if it is assumed that the electrons are reflected in specular fashion from the boundary. Under these conditions the basic assumption is the introduction for the fluctuating currents of a dielectric tensor and a correlation function which coincide with the corresponding quantities for an infinite plasma. This procedure also allows us to avoid the solution of a boundary value problem.

Field equations. In the macroscopic approach we introduce the extraneous density $\mathbf{j}$ of the random plasma microcurrents that excite the radiation field, assuming $\bar{\mathbf{j}} = 0$. Under these conditions the mean radiated energy flux will be expressed in terms of an average over the fluctuations (bar over a symbol) of the quadratic quantities $\overline{\mathbf{j}\mathbf{j}}$; the results given by Kadomtsev are used for this calculation.[4]

1)The fact that the distribution is stable means that only damped electromagnetic waves can propagate in the plasma.

The macroscopic field equations in the plasma are expressed as follows:[5]

$$\operatorname{rot}\mathbf{E}_- = -\frac{\partial \mathbf{B}_-}{c\partial t}, \quad \operatorname{rot}\mathbf{B}_- = \frac{\partial \mathbf{D}_-}{c\partial t} + \frac{4\pi}{c}\mathbf{j}_-, \quad z<0. \quad (3)$$

The dependence of the electric induction $\mathbf{D}_-$ on $\mathbf{E}_-$, as well as the correlation function for the fluctuating currents $\overline{\mathbf{j}_-\mathbf{j}_-}$ in the half-space $z < 0$ must be determined from the specified electron distribution by direct solution of the kinetic equations. These relations, together with the homogeneous Maxwell equations in the vacuum $z > 0$ and the usual boundary conditions of continuity on the fields, form the complete system of equations for the problem.

We now introduce the vector

$$\mathbf{E}^p(z) = \theta(-z)\mathbf{E}_-(z) + \theta(z)\mathbf{E}_-^{(-)}(-z), \quad (4)$$

which coincides with the electric field in the plasma at $z < 0$. Here, $\mathbf{E}^{(-)} = (E_1, E_2, -E_3)$ is a vector which undergoes specular reflection at the interface while $\theta(z)$ is a step function ($\theta(z) = 1$, $z>0$; $\theta(z) = 0$, $z<0$). In similar fashion we introduce the vectors $\mathbf{D}^p$ and $\mathbf{j}$; the magnetic induction is defined by the relation

$$\mathbf{B}^p(z) = \theta(-z)\mathbf{B}_-(z) - \theta(z)\mathbf{B}_-^{(-)}(-z), \quad (4')$$

in accordance with the fact that the vector $\mathbf{B}_-$ is an axial vector and changes sign under reflection. It is easy to show that the new vectors which determine the field in the plasma at $z<0$ satisfy the system of equations

$$\operatorname{rot}\mathbf{E}^p = -\frac{1}{c}\frac{\partial \mathbf{B}^p}{\partial t},$$
$$\operatorname{rot}\mathbf{B}^p = \frac{1}{c}\frac{\partial \mathbf{D}^p}{\partial t} + \frac{4\pi}{c}\mathbf{j} - 2\delta(z)[\mathbf{nB}_-], -\infty < z < \infty, \quad (5)^*$$

where $\mathbf{n}$ is a unit vector along the z-axis.

Thus, the problem reduces to the determination, in an infinite plasma, of the electromagnetic field excited by a current whose density is (cf. [6])

$$\mathbf{j} - \frac{c}{2\pi}\delta(z)[\mathbf{nB}_-(z=0)].$$

It is evident that, by virtue of the condition of specular reflection that has been assumed for the electrons at the interface (Fig. 1), the vectors $\mathbf{E}^p$ and $\mathbf{D}^p$ introduced in (4) must be related by the usual formulas for an infinite homogeneous medium, that is to say, we can introduce a dielectric tensor with spatial dispersion[2)] (this requires a correspondingly complicated calculation on the basis of the kinetic equation, cf. [5, 7]).

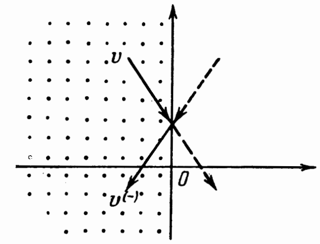

FIG. 1.

Similarly, introducing the quantities

$$\mathbf{E}(z) = \theta(z)\mathbf{E}_+(z) + \theta(-z)\mathbf{E}_+^{(-)}(-z) \quad (6)$$

and analogous quantities for $\mathbf{B}$, we obtain the system

$$\operatorname{rot}\mathbf{E} = -\partial\mathbf{B}/c\partial t,$$
$$\operatorname{rot}\mathbf{B} = \partial\mathbf{E}/c\partial t + 2\delta(z)[\mathbf{nB}_+], \quad (7)$$
$$-\infty < z < \infty,$$

which determines the vacuum field for $z > 0$.

2. Equations (5) and (7) are solved by Fourier transforms. We use the notation

$$\tilde{\mathbf{E}}_{\omega\mathbf{k}} = \frac{1}{(2\pi)^4}\int_{(-\infty)}^{(\infty)} dt\, d^3\mathbf{x}\, e^{-i\omega t + i\mathbf{kx}}\mathbf{E}(t,\mathbf{x}),$$

$$\mathbf{E}(z) \equiv \int_{-\infty}^{\infty} dk_3\, e^{-ik_3 z}\tilde{\mathbf{E}}_{\omega\mathbf{k}}. \quad (8)$$

The equations (5) and (7) of the problem reduce to algebraic equations for the Fourier amplitudes

$$A_{\alpha\beta}\tilde{E}_\beta^p \equiv \left(\frac{\omega^2}{c^2}\varepsilon_{\alpha\beta} + k_\alpha k_\beta - k^2\delta_{\alpha\beta}\right)\tilde{E}_\beta^p = \frac{4\pi i\omega}{c^2}(\tilde{j}_\alpha + J_\alpha(0)),$$
$$A_{\alpha\beta}^{(0)}\tilde{E}_\beta \equiv \left(\left(\frac{\omega^2}{c^2} - k^2\right)\delta_{\alpha\beta} + k_\alpha k_\beta\right)\tilde{E}_\beta = -\frac{4\pi i\omega}{c^2}J_\alpha(0). \quad (9)$$

Here we have used the boundary condition on the continuity of the tangential components of the magnetic induction at the interface

$$\mathscr{P}\mathbf{B}_-(0) = \mathscr{P}\mathbf{B}_+(0), \quad (10)$$

where $\mathscr{P}$ is a projection operator, $\mathscr{P}\mathbf{B} \equiv (B_1, B_2, 0)$ and we have used the notation

$$-2[\mathbf{nB}_-(0)] = \frac{4\pi}{c}\mathbf{J}(0). \quad (11)$$

It follows from the definition (9) that

$$A_{\alpha\beta}^{(0)-1} = \frac{\delta_{\alpha\beta} - k_\alpha k_\beta c^2/\omega^2}{\omega^2/c^2 - k^2}. \quad (12)$$

When the electron velocity distribution is isotropic (no macroscopic currents, external fields, etc.) the dielectric tensor[5]

$$\varepsilon_{\alpha\beta}(\omega,\mathbf{k}) = 1 + \frac{\omega_p^2}{\omega}\left\langle \frac{v_\alpha\,\partial/\partial v_\beta}{\omega - \mathbf{kv} - i/\tau}\middle| f(v^2)\right\rangle \quad (13)$$

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*$[\mathbf{nB}_-] \equiv \mathbf{n}\times\mathbf{B}_-$.

2)We note that if spatial dispersion is taken into account the introduction of the tensor $\epsilon_{\alpha\beta}(\omega,\mathbf{k})$ in an isotropic medium is equivalent to the introduction of two complex scalars the dielectric and magnetic permativities $\epsilon^L(\omega,\mathbf{k})$ and $\mu(\omega,\mathbf{k})$.[5]

must be symmetric[8] and from the covariance requirement we find

$$\varepsilon_{\alpha\beta}(\omega,\mathbf{k})=\left(\delta_{\alpha\beta}-\frac{k_\alpha k_\beta}{k^2}\right)\varepsilon^T(\omega,k)+\frac{k_\alpha k_\beta}{k^2}\varepsilon^L(\omega,k),\quad(13')$$

where $nf(v^2)$ is the electron distribution function and $\omega_p^2=4\pi e^2n/m$ is the plasma frequency; in this case

$$A_{\alpha\beta}^{-1}=\frac{\delta_{\alpha\beta}-k_\alpha k_\beta/k^2}{(\omega^2/c^2)\varepsilon^T(\omega,k)-k^2}+\frac{k_\alpha k_\beta/k^2}{(\omega^2/c^2)\varepsilon^L(\omega,k)}.\quad(14)$$

Now, the system of equations

$$\tilde{\mathbf{E}}^p=\frac{4\pi i\omega}{c^2}A^{-1}\cdot(\tilde{\mathbf{j}}+\mathbf{J}(0)),\quad\tilde{\mathbf{E}}=-\frac{4\pi i\omega}{c^2}A^{(0)-1}\cdot\mathbf{J}(0)\quad(15)$$

and the boundary conditions

$$\mathscr{P}\mathbf{E}^p(0)=\mathscr{P}\mathbf{E}(0)\quad(16)$$

determine the field in the vacuum as a function of j.

The field in the plasma $\mathbf{E}^p(z)$ is determined from the properties of the functions

$$\varepsilon^T=1+\frac{\omega_p{}^2}{\omega}\int d^3\mathbf{v}\frac{v_{k\perp}{}^2}{\omega-kv_k-i/\tau}\frac{df}{dv^2},$$

$$\varepsilon^L=1+2\frac{\omega_p{}^2}{\omega}\int d^3\mathbf{v}\frac{v_k{}^2}{\omega-kv_k-i/\tau}\frac{df}{dv^2},\quad(17)$$

where v_k is the projection of the electron velocity along the vector $\mathbf{k}$ while the projections $v_{k\perp}\perp\mathbf{k}$, ω and k are real. In the plane of (complex) k we note that (17) determines the different analytic branches for

$$\operatorname{Im}\frac{\omega-i/\tau}{k}\gtrless0.$$

Making use of analytic continuation of the function (17) from the lower half-plane of the variable $(\omega-i/\tau)/k$ to the upper half-plane, as in [9], we find

$$\varepsilon^T=1+\frac{\omega_p{}^2}{\omega}\pi\int_{C_+}du\frac{f^I(u^2)}{ku-\omega+i/\tau},$$

$$\varepsilon^L=1+\frac{\omega_p{}^2}{\omega}2\pi\int_{C_+}du\frac{u^2f(u^2)}{ku-\omega+i/\tau}.\quad(18)$$

Here $f^I(u^2)=\int_{u^2}^{\infty}dv^2 f(v^2)$ while C_+ is the contour that goes around the point $(\omega-i/\tau)/k$ from above (Fig. 2). For an entire function $f(u^2)$, the relation in (18) determines ϵ^T and ϵ^L as analytic functions of the complex variable $k=\sqrt{k_1^2+k_2^2+k_3^2}$, which are analytic functions of the complex variable k_3 in the region bounded by the contour C (Fig. 3).

Assuming that perturbation waves can propagate in the plasma in which, for simplicity, we only consider one transverse and one longitudinal type

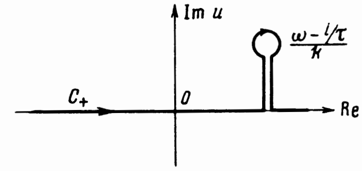

FIG. 2.

[this corresponds to a single transverse $k_T(\omega)$ and a single longitudinal $k_L(\omega)$ branch in the denominator of (14)] we find from (15)

$$z\leqslant0,\quad E_\alpha{}^p(z)=\frac{4\pi i\omega}{c^2}\int_{-\infty}^{\infty}dk_3\,e^{-ik_3z}A_{\alpha\beta}^{-1}(\tilde{j}_\beta+J_\beta(0))$$

$$=\frac{4\pi i\omega}{c^2}\left[\int_{-\infty}^{\infty}dk_3\,e^{-ik_3z}A_{\alpha\beta}^{-1}\tilde{j}_\beta+\int_{C'}dk_3\,e^{-ik_3z}A_{\alpha\beta}^{-1}J_\beta(0)\right]$$

$$=\frac{4\pi i\omega}{c^2}\left[\int_{-\infty}^{\infty}dk_3\,e^{-ik_3z}A_{\alpha\beta}^{-1}\tilde{j}_\beta\right.$$

$$+2\pi i\left(e^{-ik_{T3}z}\frac{\delta_{\alpha\beta}-k_{T\alpha}k_{T\beta}/k_T{}^2}{(\partial T/\partial k_3)_T}\right.$$

$$\left.\left.+e^{-ik_{L3}z}\frac{k_{L\alpha}k_{L\beta}/k_L{}^2}{(\partial L/\partial k_3)_L}\right)J_\beta(0)+\mathfrak{A}_{\alpha\beta}^{-1}(z)J_\beta(0)\right],\quad(19)$$

where T and L are the denominators of (14),

$$(\partial T/\partial k_3)_T\equiv\frac{k_{T3}}{k_T}\frac{\partial(\omega^2c^{-2}\varepsilon^T(\omega,k_T)-k_T{}^2)}{\partial k_T},$$

$$(\partial L/\partial k_3)_L\equiv\frac{k_{L3}}{k_L}\frac{\partial\omega^2c^{-2}\varepsilon^L(\omega,k_L)}{\partial k_L},$$

$$\mathbf{k}_{T,L}\equiv(k_1,k_2,k_{T,L3});\ k_{T,L3}=\sqrt{k_{T,L}{}^2(\omega)-(\mathscr{P}\mathbf{k})^2}$$

is the root with positive imaginary part, C′ is the contour C except for the branch cut (Fig. 3), while $\mathfrak{A}^{-1}$ is an integral along the branch cut of the contour C:

$$\mathfrak{A}_{\alpha\beta}^{-1}(z)=\int_{-i\infty}^{i\infty}\frac{k\,dk}{k_3}e^{-ik_3z}A_{\alpha\beta}^{-1}.\quad(20)$$

Correspondingly, closing the contour of integration in E(z) from below, we have

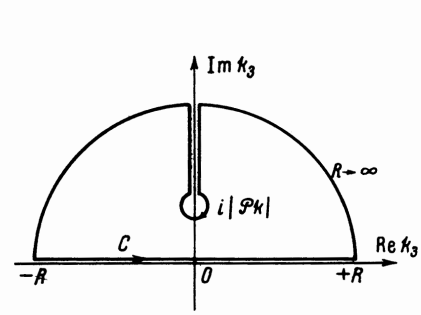

FIG. 3.

$$z \geqslant 0,\quad E_\alpha(z) = -\frac{4\pi i\omega}{c^2}\int_{-\infty}^{\infty} dk_3\, e^{-ik_3 z} A_{\alpha\beta}^{(0)-1} J_\beta(0)$$

$$= \frac{4\pi i\omega}{c^2}\frac{\pi i}{k_{03}} e^{-ik_{03}z}\left(\delta_{\alpha\beta} - \frac{k_{0\alpha}k_{0\beta}}{\omega^2/c^2}\right) J_\beta(0), \tag{21}$$

$$\mathbf{k}_0 \equiv (k_1, k_2, k_{03}),\quad (k_{03} = -[\omega^2/c^2 - i\omega\cdot 0 - (\mathcal{P}\mathbf{k})^2]^{1/2},\quad \operatorname{Im} k_{03} \leqslant 0.$$

Substituting (19) and (21) in (16) taking account of the fact that $\mathcal{P}\mathfrak{A}^{-1}(0)\cdot \mathbf{J}(0)$ can be written

$$\left(\frac{\mathcal{P}_{\alpha\beta} - (\mathcal{P}\mathbf{k})_\alpha(\mathcal{P}\mathbf{k})_\beta/(\mathcal{P}\mathbf{k})^2}{\mathfrak{A}_\perp} + \frac{(\mathcal{P}\mathbf{k})_\alpha(\mathcal{P}\mathbf{k})_\beta/(\mathcal{P}\mathbf{k})^2}{\mathfrak{A}_\parallel}\right) J_\beta(0),$$

$$\frac{1}{\mathfrak{A}_\perp} = \int_{-i\infty}^{i\infty}\frac{k\,dk}{k_3}\frac{1}{T},\quad \frac{1}{\mathfrak{A}_\parallel} = \int_{-i\infty}^{i\infty}\frac{k\,dk}{k_3}\left(\frac{k_3^2/k^2}{T} + \frac{(\mathcal{P}\mathbf{k})^2/k^2}{L}\right), \tag{22}$$

we find

$$\int_{-\infty}^{\infty} dk_3 \mathcal{P} A_{\alpha\beta}^{-1}\tilde{j}_\beta = \left[\beta_\perp\left(\mathcal{P}_{\alpha\beta} - \frac{(\mathcal{P}\mathbf{k})_\alpha(\mathcal{P}\mathbf{k})_\beta}{(\mathcal{P}\mathbf{k})^2}\right.\right.$$

$$\left. + \beta_\parallel \frac{(\mathcal{P}\mathbf{k})_\alpha(\mathcal{P}\mathbf{k})_\beta}{(\mathcal{P}\mathbf{k})^2}\right] J_\beta(0), \tag{23}$$

where we have used the notation

$$\beta_\perp = -\pi i\left[\frac{2}{(\partial T/\partial k_3)_T} + \frac{1}{k_{03}} + \frac{1}{\pi i\mathfrak{A}_\perp}\right],$$

$$\beta_\parallel = -\pi i\left[\frac{2k_{T3}^2/k_T^2}{(\partial T/\partial k_3)_T} + \frac{2(\mathcal{P}\mathbf{k})^2/k_L^2}{(\partial L/\partial k_3)_L} + \frac{k_{03}}{\omega^2/c^2} + \frac{1}{\pi i\mathfrak{A}_\parallel}\right]. \tag{24}$$

Now, $\mathbf{J}(0)$, which determines the surface current and, in accordance with (15), the vacuum field, can easily be expressed in terms of $\tilde{\mathbf{j}}$ (using (23)):

$$J_\alpha(0) = \left(\frac{\mathcal{P}_{\alpha\beta} - (\mathcal{P}\mathbf{k})_\alpha(\mathcal{P}\mathbf{k})_\beta/(\mathcal{P}\mathbf{k})^2}{\beta_\perp}\right.$$

$$\left. + \frac{(\mathcal{P}\mathbf{k})_\alpha(\mathcal{P}\mathbf{k})_\beta/(\mathcal{P}\mathbf{k})^2}{\beta_\parallel}\right)\int_{-\infty}^{\infty} dk_3 \mathcal{P} A_{\beta\gamma}^{-1}\tilde{j}_\gamma. \tag{25}$$

Kadomtsev[4] (cf. also [10]) has found the correlation function for the fluctuations in the distribution $\overline{\varphi\varphi}$ for the case of a weakly interacting ($\omega \gg 1/\tau$) non-equilibrium gas with an isotropic distribution function $nf(v^2)$ and an effective collision integral φ/τ. Knowing this correlation function, one can then compute the correlation for the surface currents $\overline{J_\alpha(0)J_\beta(0)}$. Since the fluctuations in the electron current density $\mathbf{j} = \bar{e}\int \mathbf{v}\varphi d^3\mathbf{v}$, one can easily find the Fourier amplitudes

$$\overline{\tilde{j}_\alpha(\omega,\mathbf{k})\tilde{j}_\beta^*(\omega',\mathbf{k}')} = \frac{m\omega_p^2}{(2\pi)^5}\operatorname{Im}_{\omega\mathbf{k}}\left\langle\frac{v_\alpha v_\beta}{\omega - \mathbf{kv} - i/\tau}\middle| f(v^2)\right\rangle$$

$$\times\,\delta(\omega-\omega')\delta^3(\mathbf{k}-\mathbf{k}') = \left[C^T(\omega,k)\left(\delta_{\alpha\beta} - \frac{k_\alpha k_\beta}{k^2}\right)\right.$$

$$\left. + C^L(\omega,k)\frac{k_\alpha k_\beta}{k^2}\right]\delta(\omega-\omega')\delta^3(\mathbf{k}-\mathbf{k}'), \tag{26}$$

where the imaginary part is computed for real ω and $\mathbf{k}$. Thus

$$C^T(\omega,k) = \frac{m\omega_p^2}{(2\pi)^5}\frac{1}{2}\operatorname{Im}_k\int d^3\mathbf{v}\frac{v_{k\perp}^2}{\omega - kv_k - i/\tau}f(v^2),$$

$$C^L(\omega,k) = \frac{m\omega_p^2}{(2\pi)^5}\operatorname{Im}_k\int d^3\mathbf{v}\frac{v_k^2}{\omega - kv_k - i/\tau}f(v^2). \tag{27}$$

The single-valued analytic continuations in the plane of the variable k can be found by similar expressions for (18).

Now, using (25) we have

$$\overline{J_\alpha(0)J_\beta'^*(0)} = \left[C_\perp\left(\mathcal{P}_{\alpha\beta} - \frac{(\mathcal{P}\mathbf{k})_\alpha(\mathcal{P}\mathbf{k})_\beta}{(\mathcal{P}\mathbf{k})^2}\right)\right.$$

$$\left. + C_\parallel\frac{(\mathcal{P}\mathbf{k})_\alpha(\mathcal{P}\mathbf{k})_\beta}{(\mathcal{P}\mathbf{k})^2}\right]\delta(\omega-\omega')\delta^2(\mathcal{P}\mathbf{k} - \mathcal{P}\mathbf{k}') \tag{28}$$

where we have introduced the notation

$$C_\perp = \frac{1}{|\beta_\perp|^2}\int_{-\infty}^{\infty} dk_3\frac{C^T}{TT^*},$$

$$C_\parallel = \frac{1}{|\beta_\parallel|^2}\int_{-\infty}^{\infty} dk_3\left(\frac{C^T k_3^2/k^2}{TT^*} + \frac{C^L(\mathcal{P}\mathbf{k})^2/k^2}{LL^*}\right). \tag{29}$$

3. We now have the expressions required to find the radiation intensity in the vacuum. From the symmetry of the problem the energy flux must be directed along normal from the interface so that we need only compute the z-component of the Poynting vector (averaged over the fluctuations):

$$\bar{S}_n = \frac{c}{4\pi}\overline{[\mathbf{E}(t,\mathbf{x})\mathbf{B}(t,\mathbf{x})]_3}. \tag{30}$$

Introducing the Fourier expansion with the amplitudes $\tilde{\mathbf{E}}_{\omega,\mathbf{k}}$ from (15) and

$$\tilde{\mathbf{B}}_{\omega\mathbf{k}} = \frac{4\pi i}{c}\frac{[\mathbf{kJ}(0)]}{k^2 - \omega^2/c^2}$$

and going from the calculation of the integrals in the complex plane of k_3 as in (21), using (28) we find

$$z > 0,\ \bar{S}_n = \frac{c}{8\pi}\int d\omega d^3\mathbf{k}\, d\omega' d^3\mathbf{k}' \exp\{i(\omega-\omega')t$$

$$+ i(\mathbf{k}-\mathbf{k}')\mathbf{x}\}\,(\overline{[\tilde{\mathbf{E}}\tilde{\mathbf{B}}'^*]_3} + \overline{[\tilde{\mathbf{E}}'^*\tilde{\mathbf{B}}]_3} = \frac{2\pi^3}{c^2}\int\omega\, d\omega\, dk_1\, dk_2$$

$$\times\frac{\exp\{i(k_{03}^* - k_{03})z\}}{|k_{03}|^2}(k_{03}^* + k_{03})\left(C_\perp + C_\parallel\frac{k_{03}^2}{\omega^2/c^2}\right). \tag{31}$$

Here, as in (21)

$$k_{03} = -\left[\frac{\omega^2}{c^2} - i\omega\cdot 0 - (\mathcal{P}\mathbf{k})^2\right]^{1/2}$$

is expressed in terms of the branch of the root with the positive imaginary part. With the proper choice of sign we formally introduce an imaginary correction to the radical which leads to infinitesimally small damping of the wave in the vacuum. By virtue of the factor $k_{03}^* + k_{03}$, which disappears when $\omega^2/c^2 - k_1^2 - k_2^2 < 0$, the contribution in $\overline{S}_n$ of the nonphysical (not wave-like) region of the variables k_1 and k_2 disappears. Replacing $\int_{-\infty}^{\infty} d\omega$ in (31) by twice the integral taken over the positive frequencies and introducing spherical coordinates

$$k_1 = \frac{\omega}{c}\sin\vartheta\cos\varphi, \quad k_2 = \frac{\omega}{c}\sin\vartheta\sin\varphi, \quad 0 \leqslant \vartheta \leqslant \frac{\pi}{2},$$

$$\int\limits_{\omega^2/c^2-(\mathcal{P}\mathbf{k})^2\geqslant 0} dk_1\,dk_2 = \int\limits_{0\leqslant\vartheta\leqslant\pi/2} \frac{\omega^2}{c^2}\cos\vartheta\,do, \tag{32}$$

by definition of the spectral intensity of the radiation

$$S_n = \int\limits_{\omega>0} d\omega \int\limits_{\vartheta<\pi/2} \sin\vartheta\,d\vartheta\,d\varphi\,I_\omega\cos\vartheta, \tag{33}$$

we find

$$z > 0, \quad \bar{I}_\omega = \frac{8\pi^3\omega^2}{c^3}\left(\frac{C_\perp}{\cos\vartheta} + C_\parallel\cos\vartheta\right) = \frac{8\pi^2\omega^2}{c^3}\operatorname{Re}\left\{\frac{\Lambda_1}{\Lambda_2} + \frac{\Lambda_3}{\Lambda_4}\right\}, \tag{34}$$

where

$$\Lambda_1 = -\frac{C_T{}^T}{(\operatorname{Im}_k\varepsilon^T)_T}\frac{2}{(\partial T/\partial k_3)_T} + \frac{\omega^2}{\pi c^2}\int\limits_{-i\infty}^{i\infty}\frac{k\,dk}{k_3}\frac{C^T}{TT^*},$$

$$\Lambda_2 = \frac{\omega^2}{c^2}\cos\vartheta\left|\frac{2}{(\partial T/\partial k_3)_T} + \frac{c}{\omega\cos\vartheta} + \frac{1}{\pi i}\int\limits_{-i\infty}^{i\infty}\frac{k\,dk}{k_3}\frac{1}{T}\right|^2,$$

$$\Lambda_3 = \cos\vartheta\left[-\frac{C_T{}^T}{(\operatorname{Im}_k\varepsilon^T)_T}\frac{2k_{T3}{}^2/k_T{}^2}{(\partial T/\partial k_3)_T} - \frac{C_L{}^L}{(\operatorname{Im}_k\varepsilon^L)_L}\frac{2(\mathcal{P}\mathbf{k})^2/k_L{}^2}{(\partial L/\partial k_3)_L} + \frac{\omega}{\pi c^2}\int\limits_{-i\infty}^{i\infty}\frac{k\,dk}{k_3}\left(\frac{C^T k_3{}^2/k^2}{TT^*} + \frac{C^L(\mathcal{P}\mathbf{k})^2/k^2}{LL^*}\right)\right],$$

$$\Lambda_4 = \frac{\omega^2}{c^2}\left|\frac{2k_{T3}{}^2/k_T{}^2}{(\partial T/\partial k_3)_T} + \frac{2(\mathcal{P}\mathbf{k})^2/k_L{}^2}{(\partial L/\partial k_3)_L} + \frac{c\cos\vartheta}{\omega} + \frac{1}{\pi i}\int\limits_{-i\infty}^{i\infty}\frac{k\,dk}{k_3}\left(\frac{k_3{}^2/k^2}{T} + \frac{(\mathcal{P}\mathbf{k})^2/k^2}{L}\right)\right|^2,$$

and the quantities $C_T^T \equiv C^T(\omega, k_T)$ as $(\operatorname{Im}_k\epsilon^T)_T$, C_L^L while T and L are defined in (19).

For the particular case of an electron distribution for which these formulas apply, namely a Maxwell-Boltzmann equilibrium distribution, comparison of (17) and (27) shows that

$$C_{MB}{}^{TL} = -\frac{\omega T}{(2\pi)^5}\operatorname{Im}_k\varepsilon_{MB}{}^{TL} \tag{35}$$

Here T is the plasma temperature and in this case (34) leads, as it should, to (1) the Kirchhoff law.

The first term in (34) describes the contribution of transverse electromagnetic waves in a medium with electric vector lying in the plane of the interface; the second divides into two terms corresponding respectively to the contribution of the transverse and longitudinal waves with electric vector coplanar to the wave vector and the normal. Under these conditions, in accordance with (1), the expressions in (34) and (35) yield the possibility of determining (in thermal equilibrium) both Fresnel (energy) reflection coefficients for transverse waves as well as the coefficient for transformation of longitudinal waves into transverse waves at the boundary of a semi-infinite plasma characterized by specular reflection of the electrons.

In completely analogous fashion we can compute the mean energy density of the vacuum field. This quantity is found to be

$$z > 0, \ \overline{\mathscr{E}}(z) = \frac{\overline{\mathbf{E}^2} + \overline{\mathbf{B}^2}}{8\pi} = \frac{4\pi^3}{c^2}\int\limits_{\omega^2/c^2-(\mathcal{P}\mathbf{k})^2\geqslant 0} d\omega\,dk_1\,dk_2\,\frac{\omega^2/c^2}{k_{03}^2} \times\left(C_\perp + C_\parallel\frac{k_{03}^2}{\omega^2/c^2}\right) + \frac{4\pi^3}{c^2}\int\limits_{\omega^2/c^2-(\mathcal{P}\mathbf{k})^2<0} d\omega\,dk_1 dk_2 e^{-2|k_{03}|z}\frac{(\mathcal{P}\mathbf{k})^2}{|k_{03}|^2}\left(C_\perp + C_\parallel\frac{|k_{03}^2|}{\omega^2/c^2}\right). \tag{36}$$

As in the equilibrium case,[3] the energy density $\overline{\mathscr{E}}(z)$ divides into two terms. The first integral, which is independent of z, corresponds to the energy of the transverse wave field with a spectral density $\overline{\mathscr{E}}(\omega, \vartheta) = \bar{I}(\omega, \vartheta)/c$; the second term corresponds to the quasi-stationary field arising as a result of total internal reflection.

Solving the problem for the case of distributions in which we can neglect spatial dispersion [the term $\mathbf{k}\cdot\mathbf{v}$ as compared with ω in the denominators of the integrals of (17) and (27)] i.e., in the case

$$\varepsilon^T(\omega) = \varepsilon^L(\omega) = 1 - \frac{\omega_p{}^2}{\omega(\omega - i/\tau)},$$

$$C^T(\omega) = C^L(\omega) = -\frac{\omega\langle mv^2/3\rangle}{(2\pi)^5}\operatorname{Im}\varepsilon(\omega), \tag{37}$$

we easily obtain the following energy flux for the radiation emitted from the plasma

$$\bar{I}(\omega, \vartheta) = \frac{\omega^2\langle mv^2/3\rangle}{8\pi^3c^2}\left(1 - \frac{1}{2}\left|\frac{\sqrt{\varepsilon - \sin^2\vartheta} + \cos\vartheta}{\sqrt{\varepsilon - \sin^2\vartheta} - \cos\vartheta}\right|^2 - \frac{1}{2}\left|\frac{\sqrt{\varepsilon - \sin^2\vartheta} + \varepsilon\cos\vartheta}{\sqrt{\varepsilon - \sin^2\vartheta} - \varepsilon\cos\vartheta}\right|^2\right). \tag{38}$$

Thus, neglecting the thermal motion of the plasma particles leads, as in the equilibrium case, to the Kirchhoff formula with the replacement of the temperature by an effective temperature $\langle mv^2/3\rangle$.

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