SOVIET PHYSICS JETP VOLUME 25, NUMBER 6 DECEMBER, 1967

ASYMPTOTIC RELATIONS BETWEEN THE TERMS OF ASYMPTOTIC EXPANSIONS OF THE AMPLITUDES OF CROSSED PROCESSES

Yu. LOMSADZE and S. TOKAR'

Uzhgorod State University

Submitted to JETP editor October 24, 1966; resubmitted January 20, 1967

J. Exptl. Theoret. Phys. (U.S.S.R.) 52, 1529–1535 (June, 1967)

We improve the results of previous investigations,[1, 7] in which asymptotic relations between the amplitudes of various crossed processes, binary as well as nonbinary, were obtained. Specifically, asymptotic (for $\omega \to \infty$ along the real axis) equalities between the moduli of the leading as well as all successive corresponding terms of the asymptotic expansions of the amplitudes of crossed reactions are derived, which are also valid in the most general case of arbitrary binary and nonbinary processes.

1. INTRODUCTION

THE situation in the interpretation of the strong interactions of the particles at high energies has become rather critical after the recent falling into disrepute of "reggeism." At present only attempts at its partial resurrection are made. Of particular significance are now the theoretical predictions derived from the fundamental postulates of quantum field theory without recourse to any model considerations or any not-quite-consistent approximation methods. By this we mean such predictions as the Pomeranchuk theorem and various generalizations thereof (cf. the review article[1]). As shown in [2-7], these predictions can be obtained under rather general assumptions directly from the physically justified postulates of quantum field theory without use of any purely mathematical assumptions about the temperate growth of the generalized functions one is dealing with.

Among these postulates, a particular role is played either by the special "principle of the absence of action at a distance" introduced by Meĭman[3, 4] or by the more liberal requirement of the rigorous formulation of the microcausality principle (III-rd basic formulation).[5, 6] Thus the experimental verification of these predictions is at the same time a verification of the validity of the extrapolation of our fundamental ideas about the particles and their interactions into the region of high energies.

Of particular importance among the above-mentioned generalizations of the Pomeranchuk theorem are the asymptotic equalities between the differential cross sections of arbitrary binary and nonbinary crossed processes. A proof of these relations on the basis of the crossing symmetry of the exact amplitudes has been given in [1]. In [7] a proof of these equalities was given without any assumption about the crossing symmetry of the exact amplitudes,[1)] using only the analytic and crossing properties of the asymptotic amplitudes introduced by Meĭman[3, 4] which follow either from the "principle of the absence of action at a distance" or from the basic formulation of the microcausality principle. These amplitudes agree asymptotically (energy $s \to \infty$ along the real axis) with the exact amplitudes under very general assumptions amounting to the requirement that there be no oscillations in the amplitude for $s \to \infty$ along the real axis.[2] However, although it is impossible, in the absence of analyticity, to ascribe a crossing symmetry to the exact amplitudes of arbitrary crossed processes, it is clear that they have "something more" than just the asymptotic crossing symmetry used in [3, 4, 7] for the proof of the Pomeranchuk theorem and various generalizations thereof.

In the present paper we formulate rigorously this "something more" and, on the basis of this, improve the results of [1, 7], i.e., establish the asymptotic equality between the moduli of the leading as well as of all successive corresponding terms in the asymptotic expansions of the amplitudes of crossed processes in the most general case of arbitrary binary and nonbinary reactions. As in [7], no assumptions about the analytic prop-

1)It must be borne in mind that this symmetry has a meaning only if these amplitudes are analytic, which has not been proven for the general case of nonbinary processes.

erties of the exact amplitudes for these processes are used. However, as is always the case in the method of asymptotic amplitudes, we must make the assumption that there are no oscillations in the amplitude at infinity (cf. in this connection [4]) or that some definite restrictions can be imposed on the variation of the "effective perpendicular area of interaction $\sigma^{\perp}$ of the particles" (cf. in this connection [8]).

2. BINARY REACTIONS

For simplicity we restrict ourselves to the scattering of a spinless particle with mass μ_1 on a spinless "nucleon" leading to a spinless particle with mass μ_2 (both spinless particles will be called "pions"):

$$\pi_1^+(q_1) + N_1(p) \to \pi_2^+(q_2) + N_2(p') \qquad (\mathrm{I})$$

with the crossed reaction

$$\pi_1^-(q_1) + N_2(p) \to \pi_2^-(q_2) + N_1(p'), \qquad (\mathrm{II})$$

where the four-momenta of the corresponding particles are indicated in parentheses.

In the axiomatic method of Bogolyubov, Medvedev, and Polivanov[9] the amplitudes (I) and (II) can be written in the Breit system, where $\mathbf{p} + \mathbf{p}' = 0$,

$$T^{\mathrm{I}}(\omega, t) = \int dx \exp\left[-i\left(\omega x_0 - \lambda \mathbf{ex} + \frac{\mu_1^2 - \mu_2^2}{t}\mathbf{px}\right)\right] \times G_{ret}(x; p, p'),$$

$$G_{ret}(x; p, p') \equiv \left\langle 2, p' \left| \frac{\delta j_2(-x/2)}{\delta \varphi_1^H(x/2)} \right| 1, p \right\rangle; \qquad (1)$$

$$T^{\mathrm{II}}(\omega, t) = \int dx \exp\left[-i\left(\omega x_0 - \lambda \mathbf{ex} - \frac{\mu_1^2 - \mu_2^2}{t}\mathbf{px}\right)\right] \times \overset{\bullet}{G}_{ret}(x; p', p),$$

$$\overset{\bullet}{G}_{ret}(x; p', p) \equiv \left\langle 1, p' \left| \frac{\delta j_2^H(-x/2)}{\delta \varphi_1(x/2)} \right| 2, p \right\rangle, \qquad (2)$$

where

$$\lambda^2 = \omega^2 - \alpha^2, \quad t = -4\mathbf{p}^2, \qquad \alpha^2 = {}^1/_2(\mu_1^2 + \mu_2^2) - t/4 - (\mu_2^2 - \mu_1^2)^2/t, \qquad (3)$$

$\mathbf{e}$ is a unit vector, perpendicular to $\mathbf{p}$, and H indicates the Hermitian conjugate. Expanding formally the amplitudes $T^{\mathrm{I}}(\omega, t)$ and $T^{\mathrm{II}}(\omega, t)$ in powers of α^2, we obtain under very general assumptions (see below) the following asymptotic expansions for these amplitudes (for $\omega \to \infty$ along the real axis and fixed t):

$$T^{\mathrm{I,II}}(\omega, t) \underset{\omega\to\infty}{\sim} \sum_{n=0}^{\infty} T_n^{\mathrm{I,II}}(\omega, t), \qquad (4)$$

where

$$T_n^{\mathrm{I}}(\omega, t) = \alpha^{2n} \int dx P_n(i\mathbf{ex}/\omega, \omega^{-2}) \times \exp\left\{-i\left[\omega(x_0 - \mathbf{ex}) + \frac{\mu_1^2 - \mu_2^2}{t}\mathbf{px}\right]\right\} G_{ret}(x; p, p'), \qquad (5)$$

$$T_n^{\mathrm{II}}(\omega, t) = \alpha^{2n} \int dx P_n(i\mathbf{ex}/\omega, \omega^{-2}) \times \exp\left\{-i\left[\omega(x_0 - \mathbf{ex}) - \frac{\mu_1^2 - \mu_2^2}{t}\mathbf{px}\right]\right\} \overset{\bullet}{G}_{ret}(x; p', p), \qquad (6)$$

and $P_n(u, v)$ is a form of n-th degree with real coefficients. We emphasize especially that we have nowhere made any assumptions about the analyticity of the amplitude in the parameter α^2, and hence the series (4) will, in general, diverge.

The expansion (4) becomes really asymptotic for $\omega \to \infty$, i.e., it will give rise to a sequence of ever improving estimates for $T^{\mathrm{I,II}}(\omega, t)$ for $\omega \to \infty$ (cf., e.g., [10]), if the "effective area of interaction of the particle perpendicular to $\mathbf{p}$," $\sigma^{\perp}$, increases less rapidly than ω, if at all, for $\omega \to \infty$. This follows directly from (5) and (6). Here we have used the terminology first introduced rigorously by Gribov, Ioffe, and Pomeranchuk.[8]

Let us now write the exponent in (1) and (2) for $\omega \to \infty$ along the real axis in the form

$$-i\{\omega(x_0 - \mathbf{ex}) + {}^1/_2\alpha^2\mathbf{ex}[\omega^{-1} + O(\omega^2)] \pm (\mu_1^2 - \mu_2^2)t^{-1}\mathbf{px}\}. \qquad (7)$$

From this it is seen that the assumption that $\sigma^{\perp}$ increases less rapidly than ω is closely connected with the assumption that there are no oscillations in the amplitude at infinity in the sense (cf. the corresponding condition in the paper of Meĭman,[4] p. 1970)

$$\lim \frac{T^{\mathrm{I,II}}(\omega, t)}{T^{\mathrm{I,II}}(\omega + c\omega^{-1}, t)} = 1 \quad \text{for } \omega \to \infty. \qquad (8)$$

We thus see that the condition for the validity of the asymptotic expansion (4) is in no way different from the condition that the exact and asymptotic amplitudes are equivalent—a condition which must inevitably be introduced in the proof of the asymptotic equality of the differential cross sections of nonbinary crossed reactions.[1,7] This condition is very weak. Moreover, it is, at least in principle, verifiable in experiment, since it has a rather clear physical meaning. Thus, although we have no proof of whether it is satisfied, we shall assume that the amplitudes do indeed fulfil it. Then the expansion (4) is indeed asymptotic.

The functions $T_n^{\mathrm{I,II}}(\omega, t)$ are naturally called the n-th asymptotic amplitudes of reactions (I) and

(II). The zeroth asymptotic amplitudes $T_0^{I,II}(\omega, t)$ coincide with the asymptotic amplitudes introduced by Meĭman.[3,4]

Since we wish to base our proof only on the physically justified postulates of quantum field theory, we must drop the assumption about the temperate growth of the generalized function $G_{ret}(x)$, which, though standard, is nevertheless a purely mathematical hypothesis. This forces us to give a rigorous mathematical meaning to the principle of microcausality, $G_{ret}(x) = 0$ for $x \lesssim 0$. Because, first, of the polynomial boundedness of the n-th asymptotic amplitude $T_n^I(\omega, t)$ on the real axis and, second, of the III-rd formulation of the principle of microcausality[5,6] for the generalized function $G_{ret}(x)$, this asymptotic amplitude $T_n^I(\omega, t)$ is holomorphic in the lower ω plane except for a neighborhood of the point $\omega = 0$, and satisfies there all conditions of the generalized maximum principle of Phragmén-Lindelöf-Nevanlinna (cf., for example, [4], Sec. 7). Moreover, it is easily seen from (5) and (6) that the following relation exists between the n-th asymptotic amplitudes of the reactions (I) and (II):

$$T_n{}^{I}(-\omega^*, t) = T_n{}^{II*}(\omega, t), \quad \operatorname{Im}\omega \leqslant 0, \tag{9}$$

where the left-hand side is the analytic continuation of $T_n^I(\omega, t)$ from the lower rim of the right-hand cut to the lower rim of the left-hand cut.

Here we must make two remarks. First, the polynomial boundedness of the n-th asymptotic amplitude $T_n^I(\omega, t)$ on the real axis follows from the polynomial boundedness of the amplitude $T^I(\omega, t)$ itself on the real axis. The latter is practically a rigorous consequence (cf. the discussion of this point in [1]) of the results obtained in [11]. Second, instead of the III-rd formulation of the principle of microcausality one could also use the "principle of locality" of Meĭman[3,4] with the same success; however, the requirements of the "principle of locality" are much less liberal than those of point III of the formulation of the principle of microcausality, and the "principle of locality" itself does not have[5,6] such a clear physical content as the principle of microcausality.

Applying the method of Logunov, Nguyen van Hieu, and Todorov[1] to the n-th asymptotic amplitudes $T_n^{I,II}(\omega, t)$, we finally obtain

$$\lim_{\omega\to\infty}[|T_n{}^{I}(\omega, t)|/|T_n{}^{II}(\omega, t)|] = 1. \tag{10}$$

for physical (real) values of ω. This leads in the particular case $n = 0$ to the asymptotic equality of the moduli of the asymptotic amplitudes $T_0^I(\omega, t)$ and $T_0^{II}(\omega, t)$ of the crossed reactions (I) and (II), which was obtained in [1]. If the conditions for the asymptotic equivalence (for $\omega \to \infty$ along the real axis) of the exact, $T^{I,II}(\omega, t)$, and asymptotic amplitudes, $T_\infty^{I,II}(\omega, t)$ in the sense

$$\lim_{\omega\to\infty}[T_\infty^{I,II}(\omega, t)/T^{I,II}(\omega, t)] = 1, \tag{11}$$

are fulfilled, then it also follows that the differential cross sections of these crossed reactions are asymptotically equal.

Let us show that the preceding considerations can be applied with minor modifications to the case of pion scattering on real nucleons, i.e., particles with spin. In this case we must only express the amplitudes T^I and T^{II} in terms of the invariant amplitudes $T_i^I(\omega, t)$ and $T_i^{II}(\omega, t)$ and then introduce the n-th asymptotic amplitudes T_n^I and T_n^{II} and the n-th asymptotic invariant amplitudes $T_{in}^I(\omega, t)$ and $T_{in}^{II}(\omega, t)$. The latter will satisfy all conditions of the generalized maximum principle of Phragmén-Lindelöf-Nevanlinna and of the crossing symmetry of the type (9). This gives us the possibility to prove the asymptotic equality (10) for the case of real nucleons.

3. NONBINARY REACTIONS

Since all of the preceding discussion was based only on the use of the analytic and crossing properties of the n-th asymptotic amplitudes and did not presuppose any analytic properties of the exact amplitudes (moreover, the values of the exact amplitudes for complex ω may even be meaningless), it can be taken over without any special difficulty to the case of arbitrary nonbinary reactions. Let us consider, as an example, the scattering of a pion on a spinless "nucleon" (the spin of the nucleon may be included in the fashion indicated in Sec. 2) leading to the emission of two pions (all pions have the same mass μ):

$$\pi(q) + N_1(p) \to \pi(q') + \pi(q'') + N_2(p') \tag{III}$$

and the crossed reaction

$$\bar{\pi}(q) + N_2(p) \to \bar{\pi}(q') + \bar{\pi}(q'') + N_1(p'). \tag{IV}$$

In the framework of the kinematical description of Logunov, Nguyen van Hieu, and Todorov[1] and in their notation, we can write the amplitudes for the reactions (III) and (IV) in the form

$$T^{III}(\omega; t, t'', w^2, \xi) = \int dx_0 \int dx_1 \exp[i(\omega x_0 - \sqrt{\omega^2 - \omega_0^2}\, x_1)] \times G_{ret}(x_0, x_1; p, p'), \tag{12}$$

$$T^{IV}(\omega; t, t'', w^2, \xi) = \int dx_0 \int dx_1 \exp[i(\omega x_0 - \sqrt{\omega^2 - \omega_0^2}\, x_1)] \times G_{ret}^*(x_0, x_1; p', p). \tag{13}$$

Expanding formally the amplitude (12) in powers of ω_0^2, we obtain, under the very general assumptions discussed in Sec. 2, an asymptotic expansion for this amplitude for $\omega \to \infty$ along the real axis:

$$T^{\mathrm{III}}(\omega) \underset{\omega\to\infty}{\sim} \sum_{n=0}^{\infty} T_n^{\mathrm{III}}(\omega), \tag{14}$$

where

$$T_n^{\mathrm{III}}(\omega) \equiv \omega_0^{2n} \int dx_0 \int dx_1 P_n\left(-\frac{ix_1}{\omega}, \frac{1}{\omega^2}\right) e^{i\omega(x_0-x_1)}$$

$$\times G_{ret}(x_0, x_1; p, p') = \omega_0^{2n} \int_0^\infty d\tau e^{i\omega\tau} T_{ret}(\tau, \omega; p, p') \tag{15}$$

and

$$T_{ret}(\tau, \omega; p, p') \equiv \int dx_1 P_n\left(-\frac{ix_1}{\omega}, \frac{1}{\omega^2}\right) G_{ret}(x_1+\tau, x_1; p, p'). \tag{16}$$

Because, first, of the polynomial boundedness of the n-th asymptotic amplitude $T_n^{\mathrm{III}}(\omega)$ on the real axis[2)] and, second, of point III of the formulation of the principle of microcausality[5, 6] for $G_{ret}(x_0, x_1)$, this amplitude $T_n^{\mathrm{III}}(\omega)$ is holomorphic in the upper ω plane except for a neighborhood of the point $\omega = 0$, and satisfies there all conditions of the generalized maximum principle of Phragmén-Lindelöf-Nevanlinna. Moreover, writing the asymptotic expansion for $\omega \to \infty$ of the amplitude $T^{\mathrm{IV}}(\omega)$ of the reaction (IV) in a form analogous to (12), we easily see that the following relation exists between the corresponding terms $T_n^{\mathrm{III}}(\omega)$ and $T_n^{\mathrm{IV}}(\omega)$:

$$T_n^{\mathrm{III}}(-\omega^*) = T_n^{\mathrm{IV}*}(\omega). \tag{17}$$

Applying further the method of [1] to the n-th asymptotic amplitudes $T_n^{\mathrm{III,IV}}(\omega)$, we obtain finally

$$\lim_{\omega\to\infty} [|T_n^{\mathrm{III}}(\omega)| / |T_n^{\mathrm{IV}}(\omega)|] = 1. \tag{18}$$

This leads in the particular case $n = 0$ to the asymptotic equality of the differential cross sections of the reactions (III) and (IV).

4. CONCLUSION

In this section we show that our asymptotic equalities (10) and (18) for $n \neq 0$ yield information about the relative asymptotic behavior of the amplitudes of the direct and crossed reactions which goes beyond that provided by the asymptotic equalities for $n = 0$. In particular, the equality (18) for $n = 1$ is equivalent to

$$\lim_{\omega\to\infty} \left[\left| \frac{T_0^{\mathrm{III}}(\omega)}{T^{\mathrm{III}}(\omega)} - 1 \right| \Big/ \left| \frac{T_0^{\mathrm{IV}}(\omega)}{T^{\mathrm{IV}}(\omega)} - 1 \right| \right] = 1. \tag{19}$$

The equality (18) for $n = 0$ implies only the asymptotic (for $\omega \to \infty$ on the real axis) equality of the moduli of the asymptotic Meĭman amplitudes $T_0^{\mathrm{III}}(\omega)$ and $T_0^{\mathrm{IV}}(\omega)$ of the direct and crossed reactions, which in turn are asymptotically equal (for $\omega \to \infty$ on the real axis) to the exact amplitudes $T^{\mathrm{III}}(\omega)$ and $T^{\mathrm{IV}}(\omega)$ of these reactions. The equality (19) implies, moreover, that the exact amplitudes of these crossed reactions tend, for $\omega \to \infty$ along the real axis, to the asymptotic Meĭman amplitudes in the same way.

The asymptotic equalities (10) and (18) for arbitrary $n = 0, 1, \ldots$ were obtained by the use of the crossing symmetry properties (9) and (17) of all n-th asymptotic amplitudes, which moreover satisfy all conditions for the generalized maximum principle of Phragmén-Lindelöf-Nevanlinna. These crossing relations (9) and (17) for all corresponding terms of the asymptotic expansions of the amplitudes for arbitrary crossed reactions are what in the Introduction has been called "something more" than just the "asymptotic" crossing symmetry of these amplitudes, and which they possess without any assumptions about their analyticity.

If, for example, the exponentials in (1) and (2) for $T^{\mathrm{I}}(\omega)$ and $T^{\mathrm{II}}(\omega)$ contained different α's, then the asymptotic equality (10) still holds for $n = 0$ but would be violated for all $n \neq 0$. Analogously, if the exponentials in (12) and (13) for $T^{\mathrm{III}}(\omega)$ and $T^{\mathrm{IV}}(\omega)$ would contain different ω_0's, then the asymptotic equality (18) would hold for $n = 0$ but not for other $n \neq 0$. Thus the asymptotic equalities (10) and (18) for arbitrary n make more exhaustive use of the information about the exact amplitudes of arbitrary crossed processes contained in their integral equations.

Note added in proof (April 21, 1967): We note that the n-th asymptotic amplitudes $T_n^{\mathrm{I,II}}(\omega)$ depend on the masses μ_1 and μ_2 of the scattering particles only via $\Delta \equiv \mu_1^2 - \mu_2^2$, whereas the parameter α^2 of the asymptotic expansion (4) depends on $\Sigma \equiv \mu_1^2 + \mu_2^2$. Let us assume that it is possible to study experimentally (cf. the idea of Gribov, Ioffe, and Pomeranchuk [8]) the dependence of the amplitudes $T^{\mathrm{I,II}}(\omega)$ on Σ for fixed Δ and large ω. Then we could reproduce the experimental form of $T_n^{\mathrm{I,II}}(\omega)$ for large ω for different n and hence, verify the asymptotic equality (10).

---

2)As in the case of binary reactions, this polynomial boundedness follows from the polynomial boundedness of the $T^{\mathrm{III}}(\omega)$ itself on the real ω axis. The polynomial boundedness on the real ω axis of the amplitude of an arbitrary nonbinary reaction follows, by the optical theorem, from the polynomial boundedness of the corresponding binary reaction on the real ω axis (private communication from V. I. Fushchich). The latter can at present be regarded as being proven rigorously, as mentioned in Sec. 2.

[1] A. A. Logunov, Nguyen van Hieu, and I. T. Todorov, UFN **88**, 51 (1966), Soviet Phys. Uspekhi **9**, 31 (1966).

[2] N. N. Meĭman, JETP **43**, 2277 (1962), Soviet Phys. JETP **16**, 1609 (1963).

[3] N. N. Meĭman, JETP **46**, 1502 (1964), Soviet Phys. JETP **19**, 1018 (1964).

[4] N. N. Meĭman, JETP **47**, 1966 (1964), Soviet Phys. JETP **20**, 1320 (1965).

[5] Yu. Lomsadze and I. Krivskiĭ, Ukr. fiz. zhurn. **11**, 721 (1966); DAN SSSR **173**, 312 (1967), Soviet Phys. Doklady 12, 239 (1967).

[6] Yu. M. Lomsadze, in Fizika chastits vysokikh energiĭ i teoriya elementarnykh chastits (Particle Physics at High Energies and the Theory of Elementary Particles), Kiev, 1966, p. 791.

[7] N. N. Meĭman and Xu. M. Lomsadze, Nucl. Phys. in press.

[8] V. N. Gribov, B. L. Ioffe, and I. Ya. Pomeranchuk, JNP **2**, 768 (1965), Soviet Phys. JNP **2**, 549 (1966).

[9] N. N. Bogolyubov, B. V. Medvedev, and M. K. Polivanov, Voprosy teorii dispersionnykh sootnosheniĭ (Problems in the Theory of Dispersion Relations), Fizmatgiz, 1958.

[10] N. G. de Bruijn, Asymptotic Methods in Analysis, North-Holland, Amsterdam, 1958; Russ. Transl. IIL, 1961.

[11] J. Bros, in High Energy Physics and Elementary Particles, IAEA, Vienna, 1965, p. 121.

Translated by R. Lipperheide
186