QUASILINEAR THEORY OF RUNAWAY ELECTRONS

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Quasilinear effects due to acceleration of a beam of runaway electrons by an external electric field are studied. It is assumed that initially a low density beam of high energy electrons exists in the plasma, Coulomb collisions between the electrons and the plasma electrons and ions being insignificant. The electric field is assumed to be small compared with the critical Dreicer field at which all plasma electrons begin to accelerate with respect to the ions. A solution of the quasilinear equations is found by taking into account the effect of the external electric field and collisions between the plasma electrons and ions. The time dependence of the noise spectrum excited by the beam is determined. The results of the theory may be useful in interpreting the experiments performed on installations of the "Tokamak" and "Stellarator" type.

1. INTRODUCTION

THE purpose of the present work was to investigate certain collective effects accompanying the formation of a beam of runaway electrons in a plasma.

We consider a plasma situated in an external electric field E which is weak compared with the Dreicer critical field[1] $E_c \sim mv_T\nu(v_T)/e$, where e and m are respectively the charge and mass of the electron, v_T is the thermal velocity of the electrons, and $\nu(v_T)$ is the effective frequency of the electron-ion collisions, corresponding to the thermal velocity v_T. The electric field E leads to the appearance in the plasma of a quasi-stationary current, and the current velocity u can be estimated with the aid of the relation $u \sim eE/m\nu(v_T) \ll v_T$. This is precisely the situation realized in experiments on Joule heating of a plasma with devices of the "Tokamak" and "Stellarator" type.[2,3]

However, even when $E \ll E_c$ part of the electrons in the plasma can go over into the continuous-acceleration mode (the so-called electron "runaway"). The possibility of continuous acceleration is connected with the fact that the frequency of the Coulomb collisions decreases rapidly with increasing electron velocity: $\nu(v) = \nu(v_T)(v_T/v)^3$. Therefore, if the electron velocity exceeds $v_0 \sim v_T[mv_T\nu(v_T)/eE]^{1/2}$ at the initial instant of time, then the electron begins to accelerate[4,5]. The runaway electrons experience practically no pair collisions; only collective processes, connected with the buildup of Langmuir oscillations, can greatly influence the motion of such electrons. The condition for the excitation of Langmuir oscillations is the formation of a second maximum on the plot of the velocity distribution of the electrons (in other words, the formation of a beam of runaway electrons). The concrete mechanisms that lead to the formation of the beam will be considered later. For the moment we focus our attention on the investigation of those collective effects, which become manifest when an electron beam, with velocity greatly exceeding v_0, has already been produced in the plasma for some reason or another.

The evolution of the distribution function of the beam electrons is described by quasilinear equations. We succeeded in carrying through to conclusion only the investigation of the case when we could confine ourselves to the one-dimensional variant of the quasilinear theory[6,7]. Such a situation is realized, for example, if a strong magnetic field[1)] parallel to the electric field exists in the plasma. The equations of the quasilinear theory are then written in the form

$$\frac{\partial f}{\partial t}+\frac{e}{m}E\frac{\partial f}{\partial v}=\frac{4\pi^2e^2}{m^2}\frac{\partial}{\partial v}\frac{1}{v}w\left(\frac{\omega_p}{v}\right)\frac{\partial f}{\partial v}, \quad (1)$$

$$\frac{\partial w(k)}{\partial t}=[\gamma(k)-\nu(v_T)]\,w(k), \quad (2)$$

$$\gamma(k)=\pi\omega_p\frac{n'}{n}\left(v^2\frac{\partial f}{\partial v}\right)\Big|_{v=\omega_p/k}. \quad (3)$$

Here f is the runaway electron distribution function normalized to unity, w(k) the spectral energy den-

[1)]Such that the electron cyclotron frequency greatly exceeds the electronic plasma frequency.

sity of the Langmuir oscillations, n the plasma electron concentration, n′ the beam electron concentration, which is assumed small compared with n, and ω_p the electron plasma frequency; the remaining notation is standard. For convenience, we have chosen the direction of the electric field opposite to the direction of the velocity axis.

Assuming that the beam-electron velocity greatly exceeds v_0, we neglect completely the pair collisions in the equation for the runaway-electron distribution function. On the other hand, allowance for the collision damping in the equation for the spectral energy density of the oscillations is essential. This effect is determined by the collisions between the plasma electrons and the ions, and is described by the last term of (2).

It must be emphasized that we regard the concentration n′ of the runaway electrons to be fixed, neglecting the accumulation effect connected with the presence of a stationary flux of runaway electrons in velocity space[4,5]. Therefore the proposed theory is valid only so long as the increment of the runaway-electron concentration due to the accumulation effect is small compared with their initial concentration.

2. SOLUTION OF QUASILINEAR EQUATIONS

The quasilinear equations for beams in a homogeneous plasma without an external electric field were dealt with in a large number of theoretical papers[6-10]. Our case has a principal singularity connected with the fact that the problem includes continuously acting energy source and energy sink, namely an external electric field and collisions. Therefore the energy conservation law that follows from (1)—(3) is written in the form

$$\frac{mn'}{2}\frac{\partial}{\partial t}\int v^2 f\,dv+\frac{\partial}{\partial t}\int w(k)\,dk=en'E\int vf\,dv-\nu(v_T)\int w(k)\,dk. \qquad (4)$$

The left side of (4) is the rate of change of the energy of the system consisting of the beam electrons and the Langmuir oscillations. The first term in the right side describes the increase in the energy of this system under the influence of the electric field, and the second term the decrease of the energy as a result of the collision damping of the oscillations.

We solve the problem of the temporal evolution of the distribution function under the assumption that the external electric field is small. Let us consider first the case when the electric field is equal to zero. Then the system acquires as the result of quasilinear relaxation a stationary state: The distribution function takes such a form that the instability increment vanishes in a certain interval $k_1 < k < k_2$ of the wave vectors: $\gamma(k) - \nu(v_T) = 0$. The limits of the quasilinear-relaxation region in velocity space are determined by the equations $v_1 = \omega_p/k_2$ and $v_2 = \omega_p/k_1$; outside the velocity interval $[v_1, v_2]$ the distribution function remains unchanged and equal to the initial distribution function $f_0(v)$ (see Fig. 1).

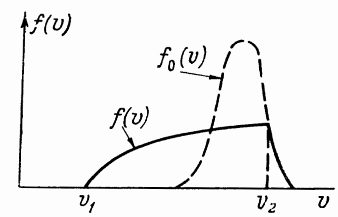

FIG. 1. Relaxation of the distribution function as $E \to 0$.

By means of the substitution $k = \omega_p/v$ we can rewrite the equation $\gamma(k) - \nu(v_T) = 0$ in the form

$$\frac{\partial f}{\partial v}=\frac{\alpha}{v^2},\quad \alpha=\frac{\nu(v_T)}{\pi\omega_p}\frac{n}{n'}, \qquad (5)$$

whence

$$f=\alpha\left(\frac{1}{v_1}-\frac{1}{v}\right),\quad v_1\leqslant v\leqslant v_2. \qquad (6)$$

To determine v_1 and v_2 it is necessary to use the continuity of the distribution function at the point v_2 and the conservation of the total number of particles in the beam:

$$\alpha\left(\frac{1}{v_1}-\frac{1}{v_2}\right)=f_0(v_2), \qquad (7)$$

$$\alpha\left(\frac{v_2}{v_1}-\ln\frac{v_2}{v_1}-1\right)+\int_{v_2}^{\infty}f_0(v)\,dv=1. \qquad (8)$$

Equations (6) and (7) allow us to obtain v_1 and v_2, and by the same token to determine completely the distribution function in the stationary state. The energy of the oscillations in the stationary state will be equal to zero. Indeed, as seen from the energy conservation law (4), the relation $\int w(k)dk = 0$ should be satisfied when $E = 0$ and the time derivatives vanish, thus leading to the statement made above.

Let us proceed to investigate the problem in the presence of an external electric field E. It is clear that if the electric field is sufficiently weak, then the electron distribution function at each instant of time differs little from the stationary distribution defined by formulas (6)—(8). But now the points v_1 and v_2 are no longer stationary, for in the presence of a nonzero electric field, albeit small, the electrons to the right of the point v_2, i.e., to the right of the region of quasilinear relaxation, accelerate freely and the distribution function varies when $v > v_2$ in accord with

$$f(v,t) = f_0\left(v - \frac{e}{m}Et\right).$$

Accordingly, when $E \neq 0$, formulas (7) and (8), which determine v_1 and v_2, are written in the form

$$\alpha\left(\frac{1}{v_1} - \frac{1}{v_2}\right) = f_0\left(v_2 - \frac{e}{m}Et\right), \tag{9}$$

$$\alpha\left(\frac{v_2}{v_1} - \ln\frac{v_2}{v_1} - 1\right) + \int_{v_2}^{\infty} f_0\left(v - \frac{e}{m}Et\right)dv = 1. \tag{10}$$

Formulas (9) and (10) define implicitly the time dependence of v_1 and v_2. In order to trace the general tendency of the variation of v_1 and v_2, we calculate with the aid of these formulas the values of the derivatives $\dot{v}_1$ and $\dot{v}_2$:

$$\dot{v}_1 = \frac{v_1}{v_2}\frac{e}{m}E, \tag{11}$$

$$\dot{v}_2 = \frac{e}{m}E\left[1 + \frac{v_2 f_0(v_2 - em^{-1}Et)}{\alpha - v_2^2\,\partial f_0/\partial v\,|_{v=v_2-em^{-1}Et}}\right]. \tag{12}$$

It is seen from (12) that

$$\dot{v}_2 > eE/m, \tag{13}$$

since

$$\left.\frac{\partial f_0}{\partial v}\right|_{v=v_2-em^{-1}Et} < 0.$$

On the basis of this result we can show that the number of electrons to the right of the region of quasilinear relaxation tends to zero as $t \to \infty$:

$$\lim_{t\to\infty}\int_{v_2}^{\infty} f_0\left(v - \frac{e}{m}Et\right)dv = 0. \tag{14}$$

To prove this, we note that v_1 is an increasing function of the time, and consequently has a limit, whether it be finite or infinite. If v_1 were to tend to a finite limit, then the ratio v_2/v_1 would grow without limit, and (10) could not be satisfied. We must therefore conclude that $v_1 \to \infty$ as $t \to \infty$. We then get from (9)

$$\lim_{t\to\infty} f_0\left(v_2 - \frac{e}{m}Et\right) = 0,$$

from which, taking (13) into account, we get (14). The variation of the distribution function of the runaway electrons is illustrated by Fig. 2a.

We have established that in the long-time limit the integral in the left side of (10) becomes small compared with unity. Actually the transition to such

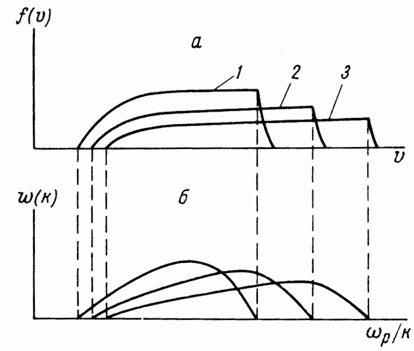

FIG. 2. Time dependence of the distribution function f(v) (a) and of the noise spectral density w(k) (b) in the asymptotic mode; $t_3 > t_2 > t_1$.

a mode, which we shall call asymptotic, is realized within a time on the order of several times ν^{-1} after the start of the two-stream instability. In the asymptotic mode, the ratio v_2/v_1 does not depend on the time and can be obtained from the equation

$$\alpha\left(\frac{v_2}{v_1} - \ln\frac{v_2}{v_1} - 1\right) = 1.$$

We denote the solution of this equation by $\varphi(\alpha)$: $v_2/v_1 = \varphi(\alpha)$. Returning to (11), we find that as $t \to \infty$ we get

$$v_1 = \frac{eE}{m\varphi(\alpha)}t, \quad v_2 = \frac{eE}{m}t. \tag{15}$$

Formulas (6) and (15) solve the problem of the evolution of the distribution function of the "runaway" electrons. It is convenient to use for qualitative estimates the following expressions for the average velocity $\bar{v}$ and the effective thermal velocity $(\overline{\Delta v^2})^{1/2}$ of the beam electrons in the asymptotic mode:

$$\bar{v} = v_1\alpha(\varphi - 1)^2/2,$$

$$\overline{\Delta v^2} = {}^1/_{12}v_1^2\alpha(\varphi - 1)^2[2(2\varphi + 1) - 3\alpha(\varphi - 1)^2]. \tag{16}$$

We note that in the limiting cases $\alpha \gg 1$ and $\alpha \ll 1$ the function $\varphi(\alpha)$ has a very simple form:

$$\varphi(\alpha) = 1 + \sqrt{\frac{2}{\alpha}} + \frac{2}{3\alpha}, \quad \alpha \to \infty,$$

$$\varphi(\alpha) = 1/\alpha, \quad \alpha \to 0.$$

It remains for us to find the spectrum of the Langmuir oscillations in the quasilinear-relaxation region. When $E \neq 0$ there should be established in this region an oscillation level such as to ensure maintenance of the quasistationary state (6) as a result of quasilinear diffusion. w(k) can be calculated explicitly with the aid of Eq. (1), in which it is necessary to substitute the distribution function (6):

$$-\frac{1}{v_1^2}\frac{dv_1}{dt} + \frac{eE}{mv^2} = \frac{4\pi^2e^2}{m^2}\frac{\partial}{\partial v}\frac{1}{v^3}w\left(\frac{\omega_p}{v}\right).$$

Hence

$$-\frac{1}{v_1^2}\frac{dv_1}{dt}\frac{\omega_p}{k}-\frac{eEk}{m\omega_p}=\frac{4\pi^2e^2}{m^2}\frac{k^3}{\omega_p^3}w(k)+C.$$

The value of the integration constant C can be determined on the basis of the following considerations: In wave-vector space, the boundaries of the region of quasilinear relaxation are displaced in the directions of the small wave vectors

$$k_1(t)=\frac{\omega_p}{v_2(t)},\quad k_2(t)=\frac{\omega_p}{v_1(t)}.$$

It is clear that on the left boundary of this region ($k = k_1$) the oscillation level is zero: $w(k_1) = 0$. Hence

$$C=-\frac{\omega_p}{k_1v_1^2}\frac{dv_1}{dt}-\frac{eEk_1}{m\omega_p}.$$

Taking also account of (15), we obtain ultimately

$$w(k)=\frac{neE}{\pi k^4}(k-k_1)(k_2-k). \quad (17)$$

Thus, the function w(k) vanishes on both boundaries of the quasilinear-relaxation region. The time dependence of the oscillation spectrum is shown in Fig. 2b. The total energy of the Langmuir oscillations $W = \int w(k)\,dk$ is

$$W=neE(k_2-k_1)^3/6\pi k_1^2k_2^2.$$

In the asymptotic mode, $k_2 = \varphi(\alpha)k_1$ and

$$W=\frac{neE}{4\pi k_1}\frac{(\varphi-1)^3}{\varphi^2}=E^2(\omega_pt)\frac{(\varphi-1)^3}{\varphi^2}. \quad (18)$$

3. LIMITS OF APPLICABILITY OF THE THEORY

All the calculations in Sec. 2 were based on the weak-field approximation. Namely, we assumed that the time of deformation of the distribution function under the influence of the electric field is long compared with the time of quasilinear relaxation. In calculating the distribution function in the region of the nonlinear relaxation, we have confined ourselves to the zeroth approximation in the electric field, assuming that the distribution function is determined from the quasilinear stationary condition $\gamma = \nu$. To determine the conditions of applicability of this approximation, it is necessary to determine the error introduced into the distribution function by the presence of the electric field. Physically this error is connected with the following circumstance:

When $E \neq 0$ the quasistationary distribution function is maintained by the quasilinear diffusion due to the Langmuir oscillations. As seen from (18), the oscillation energy should increase with time, in order that the diffusion velocity be maintained at the required level. Consequently, when $E \neq 0$ the instability increment should have a certain nonzero value $\gamma - \nu = \delta\gamma$. Accordingly, when $E \neq 0$ the distribution function must be determined not from the condition $\gamma = \nu$, but from the condition $\gamma = \nu + \delta\gamma$. It is clear that the difference between this distribution function and the quasistationary one will be small if $\delta\gamma \ll \nu$. The value of $\delta\gamma$ can be estimated with the aid of (18): $\delta\gamma \sim \dot{W}/W \sim t^{-1}$. From this we get the condition for the applicability of the theory, namely, $t \gg \nu^{-1}$. This means that the weak-field approximation begins to "work" at the instant when the asymptotic mode is attained, i.e., the limitation connected with the use of this approximation is not significant.

One more applicability condition is connected with the possibility of using formula (3) for γ. This formula is valid only in the case of sufficiently "broad" beams, such that $\overline{\Delta v^2}/\overline{v}^2 \gg (n'/n)^{2/3}$. Turning to the formulas in (16), we find that when $\alpha \ll 1$ this inequality is always satisfied, but if $\alpha \gg 1$, then the inequality leads to a lower limit for the beam density: $n'/n > (\nu/\omega_p)^3$.

In conclusion, let us estimate the role of the nonlinear interaction of the Langmuir oscillations. The most essential under our conditions is the scattering of these oscillations by the plasma electrons, since the decay processes are forbidden by the energy and momentum conservation laws.[11] Owing to the scattering effect, a correction dependent on the oscillation energy appears in the expression for the increment (see, for example,[12,13]):

$$\delta\gamma_{scat}=-\frac{3}{2}(2\pi)^{-5/2}\frac{k^2v_T^2}{mn\omega_p^4}\times\int_{k_1}^{k_2}dk'\,w(k')k'^2(k+k')\,\mathrm{sign}(k-k').$$

From this we get on the basis of (17) the following estimate:

$$\delta\gamma_{scat}/\nu \lesssim (v_T/v_1)^4 eE/m\nu(v_T)v_T.$$

Inasmuch as we assume that $E \ll E_c \sim m\nu(v_T)v_T/e$ and $v_1 \gg v_T$, we get $\delta\gamma_{scat}/\nu \ll 1$. Consequently scattering has negligible effect on Eq. (2).

Similar reasoning shows that scattering likewise does not affect Eq. (1).

4. DISCUSSION OF RESULTS

Let us compare now the results of the quasilinear theory with the case of free acceleration of the beam electrons, which would take place if the collective processes were completely absent. The average velocity of the accelerated electrons would then grow like $\bar{v} = eEt/m$, and their effective thermal velocity would remain unchanged: $\overline{\Delta v^2} = \overline{\Delta v^2}|_{t=0}$.

The influence of the collective processes becomes most clearly pronounced when the parameter α is small compared with unity ($n' \gg n\nu/\omega_p$). In this case the average electron velocity increases at half the rate obtained in free acceleration: $\bar{v} = eEt/2m$, and the thermal scatter turns out to be of the same order as the average velocity: $(\overline{\Delta v^2})^{1/2} = eEt/\sqrt{12}m$, i.e., the beam spreads rapidly under the influence of the oscillations (see (16)). The oscillation energy increases linearly:

$$W = E^2(\omega_p t)\pi\omega_p n'/\nu(v_T)n,$$

but remains small compared with the beam energy:

$$2W/mn'(\bar{v}^2 + \overline{\Delta v^2}) \sim (\nu t)^{-1} \ll 1.$$

On the other hand, if the parameter α is much larger than unity ($n' \ll n\nu/\omega_p$), then the average electron velocity increases in practice in the same manner as in free acceleration, and the thermal velocity remains at all times much smaller than the average one: $\overline{\Delta v^2}/\bar{v}^2 = 1/9\alpha \ll 1$. This case is essentially close to free acceleration.

Under typical conditions prevailing in apparatus of the "Tokamak" and "Stellarator" type, namely at 10 eV $\lesssim T_e \lesssim 10^2$ eV and 10^{12} cm$^{-3} \lesssim n \lesssim 10^{13}$ cm^{-3}, the ratio ν/ω_p lies in the range 3×10^{-5}—3×10^{-6}, i.e., the parameter α is equal to $(10^{-5}$—$10^{-6})n/n'$. Thus, as soon as the concentration of the beam electrons exceeds 10^{-5}—10^{-6} of the plasma concentration, the collective processes become important These processes are outwardly manifested by the occurrence of strong electromagnetic radiation in the ω_p range, connected with the transformation of the longitudinal oscillations into transverse ones.[14-18] To check on the proposed variant of the theory, it would therefore be desirable to measure simultaneously the microwave and x-ray emission[2)] in apparatus used for Joule heating of plasma.

Let us stop to discuss the possible mechanism of production of runaway-electron beams. A great variety of such mechanisms can operate in real installations. The most important, in our opinion, are two of them. First, the beam can occur during the initial stages of the discharge, when the plasma density is low and the conditions for "runaway" are greatly facilitated. During this stage, the electrons acquire a considerable energy, after which they experience practically no Coulomb collisions with the produced dense plasma. Second, if the plasma touches the walls surrounding it during the course of the experiment, then large electric fields are produced at the points of tangency and lead to the appearance of electron beam, which can then be injected into the plasma.

Of course, beams can be produced also by other causes: sharp changes of the electric field in the course of time[5], production of local density inhomogeneities in the development of large-scale turbulence, and finally injection of a beam from the outside.

To investigate this aspect of the problem it is evidently necessary to analyze thoroughly the conditions of plasma production and current excitation in each concrete installation.

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[2)] X-ray measurements would yield information on the distribution function of the runaway electrons.

[1] H. Dreicer, Phys. Rev. 115, 238 (1959).
[2] E. P. Grobunov, G. G. Dolgov-Savel'ev, K. B. Kartashev, V. S. Mukhovatov, V. S. Strelkov, M. N. Shepelev, and N. A. Yavlinskiĭ, Nucl. Fus. Suppl. 3, 941 (1962).
[3] W. Bernstein, F. Chen, M. Heald, A. Kranz, Phys. Fluids 1, 430 (1958).
[4] H. Dreicer, Phys. Rev. 117, 329 (1960).
[5] A. V. Gurevich, JETP 39, 1296 (1960), Soviet Phys. JETP 12, 904 (1961).
[6] A. A. Vedenov, E. P. Velikhov, and R. Z. Sagdeev, Nucl. Fus. Suppl. 2, 465 (1962).
[7] W. E. Drummond and D. Pines, ibid. 3, 1049 (1962).
[8] V. D. Shapiro, JETP 44, 613 (1963), Soviet Phys. JETP 17, 416 (1963).
[9] I. B. Bernstein and F. Engelman, Phys. Fluids 9, 937 (1966).
[10] A. A. Ivanov and L. I. Rudakov, JETP 51, 1522 (1966), Soviet Phys. JETP 24, 1027 (1967).
[11] A. A. Vedenov in: Voprosy teorii plazmy (Problems of Plasma Theory), 3, Atomizdat, 1963, p. 203.
[12] L. M. Gorbunov and V. P. Silin, JETP 47, 200 (1963), Soviet Phys. JETP 20, 135 (1965).
[13] L. M. Al'tshul' and V. I. Karpman, JETP 47, 1552 (1963), Soviet Phys. JETP 20, 1043 (1965).
[14] V. L. Ginzburg and V. V. Zhelznyakov, Astronom. zh. 35, 694 (1958), Soviet Astronomy AJ 2, 653 (1959).
[15] D. A. Tidman and G. H. Weiss, Phys. Fluids 4, 703 (1961).
[16] A. Gailitis and V. N. Tsytovich, JETP 47, 1468 (1964), Soviet Phys. JETP 20, 987 (1965).
[17] V. N. Tsytovich, DAN SSSR 154, 76 (1964), Soviet Phys. Doklady 9, 49 (1964).
[18] L. M. Kovrizhnykh, Preprint EUR-CEA-FC-258, Fontenay-aux-Roses, France.

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