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AN AXIOMATIC FORMULATION OF QUANTUM ELECTRODYNAMICS

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A system of coupled equations is obtained for the matrix elements of the currents in quantum electrodynamics. These equations do not involve undetermined quasilocal terms. It was possible to avoid these terms in the equations by including among the basic axioms a so-called "principle of minimal singularity." Various methods of excluding the quasilocal terms are considered, as well as the role of gauge invariance in this procedure. It is shown that within the framework of the assumptions made, only one independent interaction constant is admissible, namely the charge of the electron. An iterative solution of the derived equations leads to the renormalized perturbation series.

I. INTRODUCTION

WITHIN the axiomatic method in quantum field theory[1, 2] several methods have been recently proposed[3, 4] for deriving equations for the current matrix elements. A characteristic feature of the equations derived in [3] for the scalar field was the absence of undetermined quasilocal terms in these equations. These terms were excluded by means of a new axiom, the so-called "principle of minimal singularity," and by taking into account the covariance properties of the matrix elements. Departures from the mass shell in these equations are allowed only in one variable at a time. Soon similar equations were derived for pseudoscalar mesodynamics.[5]

In the present paper[1)] this program is carried through for quantum electrodynamics.[2)] A realization of this program can lead, in particular, to a closed system of equations for the simplest matrix elements at low energies. In this connection the quantum electrodynamics (of strongly interacting particles) is worthy of special attention, first of all, owing to the smallness of the electromagnetic coupling constant, and secondly, owing to the considerable simplification which is introduced if one makes use of gauge invariance (together with the principle of minimal singularity) in determining the quasilocal terms.[6, 7] Finally, the method may be useful in deriving various "sum rules."

In Sec. 2 the fundamental quantities are defined and the principle of minimal singularity is formulated within the framework of electrodynamics. We note that this principle contains the Lagrangian approach with the usual minimal electromagnetic interaction. In Sec. 3 the axiomatic equations for the current matrix elements are derived, equations which do not involve undetermined quasilocal terms. Various methods of eliminating these quasilocal terms are discussed, as well as the role played by gauge invariance in this procedure. The simplest among the equations are analyzed in the last section, where it is shown that only one independent electromagnetic constant—the electron charge e—is admissible in quantum electrodynamics. An iterative solution of the equations leads directly to the renormalized perturbation-theory series. Some relations involving the longitudinal part of the electromagnetic field are discussed in the Appendix.

2. THE PRINCIPLE OF MINIMAL SINGULARITY

Starting from the axiom of the existence of Heisenberg field operators $\psi(x)$, $\bar{\psi}(x)$, and $A_\mu(x)$, we define the current operators:[3)]

$$j_\mu(x) = -\Box A_\mu(x), \quad \frac{\partial}{\partial x_\mu} j_\mu(x) = 0,$$

$$\eta(x) = (\hat{\nabla} + m)\psi(x), \quad \bar{\eta}(x) = \bar{\psi}(x)\ (-\hat{\nabla} + m). \quad (2.1)$$

We further define the in-fields:

[1)]A detailed exposition of a series of related problems can be found in [6].

[2)]A slightly different approach to the same problems is developed in [7].

[3)]All notations and metric conventions are the same as in the book by A. I. Akhiezer and V. B. Berestetskii [8].

$$A_\mu^{in}(x) = A_\mu(x) - \int dx' D_R(x-x') j_\mu(x'),$$

$$\psi_{in}(x) = \psi(x) - \int dx' S_R(x-x') \eta(x'),$$

$$\overline{\psi}_{in}(x) = \overline{\psi}(x) - \int dx' \overline{\eta}(x') \overline{S}_R(x-x'), \qquad (2.2)$$

which satisfy the equations

$$\Box A_\mu^{in}(x) = 0, \quad (\hat{\nabla} + m)\psi_{in}(x) = 0,$$

$$\overline{\psi}_{in}(x)(-\hat{\nabla} + m) = 0 \qquad (2.3)$$

and the commutation relations

$$[A_\mu^{in}(x), A_\nu^{in}(x')]_- = -i\left(\delta_{\mu\nu} + 2M\frac{\partial^2}{\partial x_\mu \partial x_\nu}\right) D(x-x'),$$

$$[\psi_{in,\alpha}(x), \overline{\psi}_{in,\beta}(x')]_+ = -iS_{\alpha\beta}(x-x'), \qquad (2.4)$$

where $M = \frac{1}{2}\int \kappa^{-2}\sigma(\kappa^2)d\kappa^2$, and $\sigma(\kappa^2) = \rho_1(\kappa^2) - \delta(\kappa^2)$ is the spectral density of the photon Green's function.[6] Equations (2.4) are satisfied if the expansion of the in-fields in terms of the usual particle creation and annihilation operators is of the form

$$A_\mu^{in}(x) = \sum_{\mathbf{k},\lambda} \frac{1}{\sqrt{2\omega}} \tilde{e}_\mu^\lambda(\mathbf{k})[c_\lambda(\mathbf{k})e^{ikx} + c_\lambda^+(\mathbf{k})e^{-ikx}],$$

$$\psi_{in}(x) = \sum_{\mathbf{p},r} u_-^r(\mathbf{p})e^{ipx}a_-^r(\mathbf{p}) + u_+^r(\mathbf{p})e^{-ipx}a_+^{r*}(\mathbf{p}),$$

$$\overline{\psi}_{in}(x) = \sum_{\mathbf{p},r} \overline{u}_-^r(\mathbf{p})e^{-ipx}a_-^{r*}(\mathbf{p}) + \overline{u}_+^r(\mathbf{p})e^{ipx}a_+^r(\mathbf{p}), \qquad (2.5)$$

where

$$\tilde{e}_\mu^\lambda(\mathbf{k}) = (\delta_{\mu\nu} - Mk_\mu k_\nu)e_\nu^\lambda, \quad e_\mu^\lambda e_\nu^\lambda = \delta_{\mu\nu}, \quad e_\mu^\lambda e_\mu^{\lambda'} = \delta_{\lambda\lambda'},$$

$$(\pm i\hat{p} + m)u_\mp^r(\mathbf{p}) = 0, \quad u_\mp^r(\mathbf{p})\overline{u}_\mp^r(\mathbf{p}) = \frac{-i\hat{p} \pm m}{2E},$$

$$u_\mp^{r*}(\mathbf{p})u_\mp^{r'}(\mathbf{p}) = \delta_{rr'} \qquad (2.6)$$

The term involving M in (2.4) guarantees the canonical form of the commutation relations for the Heisenberg field operators,[9,10] a fact which will be utilized later.

We supplement the well known set of axioms,[1] including Lorentz-invariance, spectrality, charge-conjugation and gauge invariance, stability of one-particle states, and completeness of the set of in-fields (for details cf., e.g., [6,11]), by a new axiom: the principle of minimal singularity. It was first introduced for the scalar field[3] and in axiomatic language[1] it signifies postulating the character of the singularity of the equal-time commutators among the fields and the currents. This requirement characterizes a sufficiently broad class of interactions between fields (not necessarily in the sense of the existence of an interaction Lagrangian); apparently it encompasses the majority of renormalizable theories. In the same measure this leads to a restriction of the arbitrariness in the selection of the quasilocal terms which occur in various axiomatic equations.

In quantum electrodynamics the formulation of such a principle encounters several difficulties (noncovariant form of the quasilocal terms and their dependence on the choice of a gauge; also the fact that the particles involved have spin) and details are discussed in [6]. Concretely, the principle requires that the singularities of the following equal-time commutators be delta-like:

$$[\eta(x), \overline{\psi}(x')]_{+\atop t=t'} = \delta(\mathbf{x}-\mathbf{x}')\alpha(x),$$

$$[\eta(x), \psi(x')]_{+\atop t=t'} = \delta(\mathbf{x}-\mathbf{x}')\beta(x), \qquad (2.7)$$

$$[\eta(x), A_\mu(x')]_{-\atop t=t'} = 0,$$

$$[\eta(x), \dot{A}_\mu(x')]_{-\atop t=t'} = \delta(\mathbf{x}-\mathbf{x}')a_\mu(x), \qquad (2.8)$$

$$[j_\mu(x), \psi(x')]_{-\atop t=t'} = \delta(\mathbf{x}-\mathbf{x}')\gamma_4 b_\mu(x), \qquad (2.9)$$

$$[j_\mu(x), A_\nu(x')]_{-\atop t=t'} = 0,$$

$$[j_\mu(x), \dot{A}_\nu(x')]_{-\atop t=t'} = \delta(\mathbf{x}-\mathbf{x}')a_{\mu\nu}(x), \qquad (2.10)$$

Here $\alpha(x)$, $\beta(x)$, $a_\mu(x)$, $b_\mu(x)$, and $a_{\mu\nu}(x)$ are arbitrary but nonsingular operators, depending on x, but in general a_μ, b_μ, and $a_{\mu\nu}$ are not covariant in μ and ν. (The same is true for the canonical commutation relations for $A_\mu(x)$.)

One can derive from Eqs. (A.6) and (A.7) (cf. the Appendix)

$$a_\mu(x) = (\delta_{\mu\nu} - \delta_{\mu4}\delta_{\nu4})\alpha_\nu(x) + \delta_{\mu4}e\gamma_4\psi(x), \qquad (2.11)$$

$$b_\mu(x) = (\delta_{\mu\nu} - \delta_{\mu4}\delta_{\nu4})\beta_\nu(x) + \delta_{\mu4}ie\gamma_4\psi(x), \qquad (2.12)$$

$$a_{\mu\nu}(x) = (\delta_{\mu\rho} - \delta_{\mu4}\delta_{\rho4})(\delta_{\nu\sigma} - \delta_{\nu4}\delta_{\sigma4})\alpha_{\rho\sigma}(x), \qquad (2.13)$$

where α_μ, β_μ, and $\alpha_{\mu\nu}$ are covariant in μ and ν.

We note that (2.7)–(2.10) are valid only in so-called "genuine" gauges (involving a term with M in (2.4)[9,10]). In addition (2.10) does not hold for vacuum matrix elements, but these do not occur in the equations (cf. below).

From the standpoint of the Lagrangian approach these relations are satisfied by the minimal electromagnetic interaction, but do not allow nonrenormalizable derivative couplings, and when $\beta(x) \equiv 0$ in (2.7) they also forbid four-fermion interactions.

3. EQUATIONS FOR THE MATRIX ELEMENTS OF THE CURRENTS

Let $\langle m|J(0)|l\rangle$ denote an arbitrary matrix element of the current $J(0) \equiv j_\mu(0)$, $\eta(0)$, $\bar{\eta}(0)$ between states of definite particle number of given momenta and polarizations. The required system of equations is obtained by transforming each of the creation operators of the particles in l (or annihilation operators for the particles in m) into the x-representation with the aid of the appropriate in-field, then expressing the latter in terms of the current operator, and expanding the product of currents so obtained in terms of the complete set of in-states.

We carry out this procedure for the electron states p_-^r, the positron states p_+^r and the photon states k^λ contained in the vector $|l\rangle$. In doing this we assume for simplicity that the states of the particles in $|l\rangle$ and $\langle m|$ are different, in order to avoid disconnected diagrams.

Thus, taking into account (2.5) and (2.6), one can write for an electron state

$$\langle m|J(0)|\mathbf{p}_-^r; l\rangle = \int dx\, e^{ipx}\delta(t)\langle m|[J(0),\bar{\psi}_{in}(x)]_\mp|l\rangle\gamma_4 u_-^r(\mathbf{p}).$$

(The minus sign corresponds to $J(0) \equiv j_\mu(0)$, and the plus sign on the square bracket corresponds to $J(0) \equiv \eta(0)$, $\bar{\eta}(0)$.) Making use of (2.2) and of the relation

$$\int dx\, e^{ipx}\delta(t)\bar{S}_R(x-x')\gamma_4 u_-^r(\mathbf{p}) = -i\theta(-t')e^{ipx'}u_-^r(\mathbf{p}),$$

we obtain

$$\langle m|J(0)|\mathbf{p}_-^r; l\rangle = \langle m|R^-(\mathbf{p}) + K^-(\mathbf{p})|l\rangle u_-^r(\mathbf{p}), \quad (3.1)$$

where the R-function and the quasilocal term are respectively equal to

$$\langle m|R^-(\mathbf{p})|l\rangle u_-^r(\mathbf{p}) = i\int dx\, e^{ipx}\theta(-t)\langle m|[J(0),\bar{\eta}(x)]_\mp|l\rangle u_-^r(\mathbf{p}), \quad (3.2)$$

$$\langle m|K^-(\mathbf{p})|l\rangle u_-^r(\mathbf{p}) = \int dx\, e^{ipx}\delta(t)\langle m|[J(0),\bar{\psi}(x)]_\mp|l\rangle\gamma_4 u_-^r(\mathbf{p}). \quad (3.3)$$

Expanding the right hand side of (3.2) in terms of the base vectors we write the R-function in the form

$$\langle m|R^-(\mathbf{p})|l\rangle u_-^r(\mathbf{p}) = (2\pi)^3\sum_n\Big\{\langle m|J(0)|n\rangle\langle n|\bar{\eta}(0)|l\rangle \times\frac{\delta(\mathbf{p}+\mathbf{p}_l-\mathbf{p}_n)}{E_n-E_l-E-i\varepsilon} \mp\langle m|\bar{\eta}(0)|n\rangle\langle n|J(0)|l\rangle \times\frac{\delta(\mathbf{p}-\mathbf{p}_m+\mathbf{p}_n)}{E_m-E_n-E-i\varepsilon}\Big\}u_-^r(\mathbf{p}). \quad (3.4)$$

For the quasilocal term (3.3) it can be seen from (2.7) and (2.9) that the **p**-dependence is entirely concentrated in the spinor $u_-^r(\mathbf{p})$:

$$\langle m|K^-(\mathbf{p})|l\rangle u_-^r(\mathbf{p}) = \langle m|K^-|l\rangle u_-^r(\mathbf{p}). \quad (3.5)$$

Similarly, making use of the relation

$$\bar{u}_+^r(\mathbf{p})\gamma_4\int dx\, e^{ipx}\delta(t)S_R(x-x') = \bar{u}_+^r(\mathbf{p})i\theta(-t')e^{ipx'},$$

we obtain for the positron state

$$\langle m|J(0)|\mathbf{p}_+^r; l\rangle = \bar{u}_+^r(\mathbf{p})\langle m|R^+(\mathbf{p}) + K^+(\mathbf{p})|l\rangle, \quad (3.6)$$

$$\bar{u}_+^r(\mathbf{p})\langle m|R^+(\mathbf{p})|l\rangle = -i\bar{u}_+^r(\mathbf{p})\int dx\, e^{ipx}\theta(-t)\langle m|[J(0),\eta(x)]_\mp|l\rangle = \bar{u}_+^r(\mathbf{p})(2\pi)^3\sum_n\langle m|J(0)|n\rangle\langle n|\eta(0)|l\rangle \times\frac{\delta(\mathbf{p}+\mathbf{p}_l-\mathbf{p}_n)}{E_l-E_n+E+i\varepsilon} \mp\langle m|\eta(0)|n\rangle\langle n|J(0)|l\rangle\frac{\delta(\mathbf{p}-\mathbf{p}_m+\mathbf{p}_n)}{E_n-E_m+E+i\varepsilon}, \quad (3.7)$$

$$\bar{u}_+^r(\mathbf{p})\langle m|K^+(\mathbf{p})|l\rangle = \bar{u}_+^r(\mathbf{p})\gamma_4\int dx\, e^{ipx}\delta(t)\langle m|[J(0),\psi(x)]_\mp|l\rangle = \bar{u}_+^r(\mathbf{p})\langle m|K^+|l\rangle. \quad (3.8)$$

Finally, considering the photon state

$$\langle m|J(0)|\mathbf{k}^\lambda; l\rangle = \frac{1}{\sqrt{2\omega}}\bar{e}_\mu^\lambda(\mathbf{k})\int dx\, e^{ikx}\delta(t) \times\langle m|[J(0),\omega A_\mu^{in}(x) - i\dot{A}_\mu^{in}(x)]_-|l\rangle,$$

where

$$\bar{e}_\mu^\lambda(\mathbf{k}) = (\delta_{\mu\nu} + Mk_\mu k_\nu)e_\nu^\lambda,$$

and making use of the relation

$$\int dx\, e^{ikx}\delta(t)\left(\omega - i\frac{\partial}{\partial t}\right)D_R(x-x') = -i\theta(-t')e^{ikx'},\ k^2 = 0,$$

we obtain the expression

$$\langle m|J(0)|\mathbf{k}^\lambda; l\rangle = \frac{1}{\sqrt{2\omega}}\bar{e}_\mu^\lambda(\mathbf{k})\langle m|R_\mu(\mathbf{k}) + K_\mu(\mathbf{k})|l\rangle, \quad (3.9)$$

where

$$\frac{1}{\sqrt{2\omega}}\bar{e}_\mu^\lambda(\mathbf{k})\langle m|R_\mu(\mathbf{k})|l\rangle = \frac{i}{\sqrt{2\omega}}\bar{e}_\mu^\lambda(\mathbf{k})\int dx\, e^{ikx}\theta(-t) \times\langle m|[J(0),j_\mu(x)]_-|l\rangle = \frac{(2\pi)^3}{\sqrt{2\omega}}\bar{e}_\mu^\lambda(\mathbf{k})\sum_n\langle m|J(0)|n\rangle \times\langle n|j_\mu(0)|l\rangle\frac{\delta(\mathbf{k}+\mathbf{p}_l-\mathbf{p}_n)}{E_n-E_l-\omega-i\varepsilon} - \langle m|j_\mu(0)|n\rangle\langle n|J(0)|l\rangle\frac{\delta(\mathbf{k}-\mathbf{p}_m+\mathbf{p}_n)}{E_m-E_n-\omega-i\varepsilon}, \quad (3.10)$$

$$\frac{1}{\sqrt{2\omega}}\bar{e}_\mu^\lambda(\mathbf{k})\langle m|K_\mu(\mathbf{k})|l\rangle = \frac{1}{\sqrt{2\omega}}\bar{e}_\mu^\lambda(\mathbf{k})\int dx\, e^{ikx}\delta(t)\langle m|[J(0),\omega A_\mu(x) - i\dot{A}_\mu(x)]_-|l\rangle = \frac{1}{\sqrt{2\omega}}\bar{e}_\mu^\lambda(\mathbf{k})\langle m|K_\mu|l\rangle. \quad (3.11)$$

It is just as simple to derive relations for the states $p_{\mp}^{r}$ and k^{λ} contained in $\langle m|$.

The last step required for deriving the equations is the elimination of the quasilocal terms from (3.1) and similar expressions. Within the framework of the principle of minimal singularity this procedure can be effected by different methods (cf. also [12]).

1. The most natural among these methods is at the same time the most involved, and consists in an extension to the present context of the method employed for the scalar field.[3, 13] It is impossible to apply this method directly, since the quasilocal terms here (cf. (3.5), (3.8), and (3.11)) contain the quantities $u_{\mp}^{r}(p)$, $\bar{u}_{\mp}^{r}(p)$, and $\bar{e}_{\mu}^{\lambda}(k)$, and therefore depend on the invariants related to $\mathbf{p}$ and and $\mathbf{k}$. One can however exploit the specific character of this dependence.

To this end we expand the current matrix element under consideration

$$\langle m|J(0)|\mathbf{q};\, l\rangle \equiv r(q_i,\, q;\, \beta_i),$$

the R-function

$$\langle m|R(\mathbf{q})|l\rangle u(\mathbf{q}) \equiv R(q_i,\, q;\, \beta_i)$$

and the quasilocal term

$$\langle m|K(\mathbf{q})|l\rangle\, u(\mathbf{q}) = \langle m|K|l\rangle\, u(\mathbf{q}) \equiv K(q_i, \beta_i).$$

in terms of the invariant amplitudes. Here q_i are the particle momenta in the states $|l\rangle$ and $\langle m|$; q is the momentum of the distinguished particle (p and k in (3.1), (3.6), and (3.9)); $u(\mathbf{q})$ is a symbolic notation for $u_{\mp}^{r}(p)$, $\bar{u}_{\mp}^{r}(p)$, $\bar{e}_{\mu}^{\lambda}(k)$; β_i are the spin parameters (the quantities $u(\mathbf{q})$ and the 16 irreducible gamma-matrices).

Such an expansion has the form[14]

$$r(q_i, q;\, \beta_i) = \sum_n r_n(s_i, s_q) M_n(q_i, q;\, \beta_i),$$
$$R(q_i, q;\, \beta_i) = \sum_n R_n(s_i, s_q) M_n(q_i, q;\, \beta_i),$$
$$K(q_i;\, \beta_i) = \sum_n K_n(s_i) M_n(q_i;\, \beta_i), \qquad (3.12)$$

where r_n, R_n, and K_n are the invariant amplitudes depending on the scalar products $s_i = \mp q_i q_{i+1}$ (the upper sign is chosen if both q_i and q_{i+1} belong to either $|l\rangle$ or $\langle m|$) and $s_q = \mp q_i q$: the $M_n(q_i,\, q;\, \beta_i)$ are covariant spin structures. It is clear that $M_n(q_i;\, \beta_i)$ is part of $M_n(q_i,\, q;\, \beta_i)$. Therefore the relation

$$r(q_i, q;\, \beta_i) = R(q_i, q;\, \beta_i) + K(q_i;\, \beta_i) \qquad (3.13)$$

can be rewritten in the form

$$\sum_{n_1}[r_{n_1}(s_i, s_q) - R_{n_1}(s_i, s_q) - K_{n_1}(s_i)] M_{n_1}(q_i;\, \beta_i) + \sum_{n_2}[r_{n_2}(s_i, s_q) - R_{n_2}(s_i, s_q)] M_{n_2}(q_i, q;\, \beta_i) = 0, \quad (3.14)$$

where M_{n_2} and M_{n_1} are spin structures which respectively contain or do not contain q (in addition to $u(\mathbf{q})$).

Hence, on the basis of the independence of the $M_n(q_i,\, q;\, \beta_i)$ one obtains relations involving the invariant amplitudes

$$r_{n_1}(s_i, s_q) = R_{n_1}(s_i, s_q) + K_{n_1}(s_i),$$
$$r_{n_2}(s_i, s_q) = R_{n_2}(s_i, s_q). \qquad (3.15)$$

The quasilocal terms can be excluded from (3.15) in the same manner as for the scalar field.

We remark that the noncovariance of $R(q_i,\, q;\, \beta_i)$ and $K(q_i;\, \beta_i)$, which contain j_μ or A_μ is easily taken into account with the help of Eqs. (2.11)–(2.13).

2. The use of gauge invariance simplifies considerably the determination of the quasilocal terms for processes involving photons.[6, 7] We consider as an example the relation (3.9) for $J(0) \equiv \eta(0)$, where

$$\langle m|K_\mu|l\rangle = \int dx\, e^{ikx}\delta(t)\langle m|[\eta(0), \omega A_\mu(x) - i\dot{A}_\mu(x)]_-|l\rangle. \qquad (3.16)$$

From (2.8) and (2.11) we have

$$\langle m|K_4|l\rangle = -ie\gamma_4\langle m|\psi(0)|l\rangle. \qquad (3.17)$$

Computing (cf. (A.7))

$$\mathbf{k}\langle m|\mathbf{K}|l\rangle = i\int dx\, e^{ikx}\delta(t) \times \left\langle m\left|\left[\eta(0), -i\Box\dot{\Lambda}(x) + \frac{\partial^2}{\partial t^2}A_4(x)\right]_-\right| l\right\rangle$$
$$= -e\langle m|\eta(0)|l\rangle - ie\mathbf{k}\boldsymbol{\gamma}\langle m|\psi(0)|l\rangle + i\int dx\, e^{ikx}\delta(t)\langle m|[\eta(0), j_4(x)]_-|l\rangle,$$

we obtain

$$\langle m|\mathbf{K}|l\rangle = -ie\boldsymbol{\gamma}\langle m|\psi(0)|l\rangle + i\frac{\partial}{\partial\mathbf{k}}\int dx\, e^{ikx}\delta(t)\langle m|[\eta(0), j_4(x)]_-|l\rangle. \qquad (3.18)$$

Combining (3.17) and (3.18) we obtain

$$\langle m|K_\mu|l\rangle = -ie\gamma_\mu\langle m|\psi(0)|l\rangle + i(\delta_{\mu\nu} - \delta_{\mu4}\delta_{\nu4})\frac{\partial}{\partial k_\nu} \times \int dx\, e^{ikx}\delta(t)\langle m|[\eta(0), j_4(x)]_-|l\rangle = -ie\gamma_\mu\langle m|\psi(0)|l\rangle - (\delta_{\mu\nu} - \delta_{\mu4}\delta_{\nu4})\frac{\partial}{\partial k_\nu}(k\langle m|R|l\rangle). \qquad (3.19)$$

The derivation of (3.19) involves (3.17) and the gauge invariance condition for $\langle m|\eta(0)|k^{\lambda};\, l\rangle$, but does not require leaving the mass shell $k^2 = 0$.

One can also show[6] that the quasilocal term in (3.6) with $J(0) \equiv j_\mu(0)$ is

$$\langle m|Q_\mu|l\rangle = \gamma_4 \int dx\, e^{ipx}\delta(t)\langle m|[j_\mu(0), \psi(x)]_-|l\rangle$$

$$= -\langle m|K_\mu|l\rangle. \qquad (3.20)$$

4. THE SIMPLEST EQUATIONS. COUPLING CONSTANTS

In this section we consider the equations for the simplest matrix elements mainly in order to determine the number of independent constants which characterize the theory. We first remark that independent constants can occur as threshold values of the invariant amplitudes $r_n(s_i, s_q)$ only for spin structures $M_n(\beta_i)$ which do not depend on the momenta q_i, and q. In the opposite case $r_n(s_i, s_q)$ can be expressed directly in terms of $R_n(s_i, s_q)$. In addition, gauge invariance implies that photon processes are characterized by only one independent constant: the electron charge e.

1. Three-prong diagrams. Consideration of invariance properties allows to reduce the totality of matrix elements with three "prongs" to the vertices:

$$\langle \mathbf{p}_-'^{r'}|j_\mu(0)|\mathbf{p}_-^r\rangle = \bar{u}_-^{r'}(\mathbf{p}')F_\mu(p', p)u_-^r(\mathbf{p}), \qquad (4.1)$$

$$\langle 0|\eta(0)|\mathbf{p}_-^r; \mathbf{k}^\lambda\rangle = \frac{1}{\sqrt{2\omega}} e_\mu^\lambda \Phi_\mu(p, k)u_-^r(\mathbf{p}) \qquad (4.2)$$

for various domains of values of the invariant variables.

One of the two possible relations for (4.1) is a special case of (3.1)–(3.3). Expressing everything in terms of invariant amplitudes and eliminating the quasilocal term, we obtain

$$\bar{u}_-^{r'}(\mathbf{p}')\{\gamma_\mu F_1(s) + \sigma_{\mu\nu}q_\nu F_2(s)\}u_-^r(\mathbf{p})$$

$$= \bar{u}_-^{r'}(\mathbf{p}')\{-ie\gamma_\mu + \gamma_\mu[R_1(s) - R_1(-m^2)]$$

$$+ \sigma_{\mu\nu}q_\nu R_2(s)\}u_-^r(\mathbf{p}), \qquad (4.3)$$

where $q = p' - p$, $s = pp'$, $e = iF_1(-m^2)$, and $R_1(s)$ is the amplitude of $\bar{u}_-^{r'}(\mathbf{p}')\gamma u_-^r(\mathbf{p})$ (or of $\bar{u}_-^{r'}(\mathbf{p}')\gamma_4 u_-^r(\mathbf{p})$) in the expansion of $R_\mu(p', p)$. The anomalous magnetic moment is not an independent constant here, but is determined by

$$\Delta\mu = F_2(-m^2) = R_2(-m^2). \qquad (4.4)$$

We note that in the equation for $\langle 0|j_\mu(0)|\mathbf{p}_+'^{r'}, \mathbf{p}_-^r\rangle$ there appears the constant $F_1(m^2) \neq -ie$, which can also be expressed in terms of e.

It is simpler to derive the equation for (4.2) from (3.19), rather than from (3.14). As a result we obtain

$$\frac{1}{\sqrt{2\omega}} e_\mu^\lambda \Phi_\mu(p, k)u_-^r(\mathbf{p}) = \frac{1}{\sqrt{2\omega}} \bar{e}_\mu^\lambda(\mathbf{k})\Big\{-ie\gamma_\mu$$

$$+ R_\mu(p, k) - (\delta_{\mu\nu} - \delta_{\mu 4}\delta_{\nu 4})\frac{\partial}{\partial k_\nu}(kR)\Big\}u_-^r(\mathbf{p}), \qquad (4.5)$$

where $R_\mu(p, k)$ is defined by the relation (3.10).

2. Four-prong diagrams. Among all possible four-prong matrix elements only

$$\langle \mathbf{p}_-'|\eta_\alpha(0)|\mathbf{p}_-, \mathbf{q}_-\rangle = \bar{u}_{-,\gamma}(\mathbf{p}')F_{\alpha\beta\gamma\delta}(p', p, q)u_{-,\beta}(\mathbf{p})u_{-,\delta}(\mathbf{q}),$$

which describes electron-electron scattering, can contribute an independent constant. Indeed, in the expansion

$$F_{\alpha\beta\gamma\delta}(p', p, q) = \sum_n r_n(s_1, s_2, s_3)M_{n,\alpha\beta\gamma\delta}(p', p, q; \gamma)$$

there are five momentum-independent spin structures

$$M_S = \delta_{\alpha\beta}\delta_{\gamma\delta}, \quad M_P = (\gamma_5)_{\alpha\beta}(\gamma_5)_{\gamma\delta}, \quad M_V = (\gamma_\mu)_{\alpha\beta}(\gamma_\mu)_{\gamma\delta},$$
$$M_A = (\gamma_5\gamma_\mu)_{\alpha\beta}(\gamma_5\gamma_\mu)_{\gamma\delta}, \quad M_T = (\sigma_{\mu\nu})_{\alpha\beta}(\sigma_{\mu\nu})_{\gamma\delta}$$

(antisymmetrized in β and δ) corresponding (in the language of the Lagrangian formalism) to five possible couplings for an effective contact four-fermion interaction. The independent constants λ_S, λ_P, λ_V, λ_A, and λ_T lead to divergences (in perturbation theory) and are eliminated from the theory by means of the requirement $\beta(x) \equiv 0$ in (2.7).

Thus the charge e is the only independent constant involved in the theory. An iteration solution of the equations yields the renormalized perturbation theory series. This is verified in [6] for three- and four-prong diagrams up to the fourth order in e.

Summarizing, one can say that the method of deriving equations for the current matrix elements, based on the principle of minimal singularity is applicable to quantum electrodynamics. It is true that the difficulties related to the infrared divergence have not been taken into account—these difficulties can complicate the problem in a nontrivial way.[14] Nevertheless, one may hope that the axiomatic method in electrodynamics can yield interesting results, in particular for low energy phenomena.

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APPENDIX

The electromagnetic field tensor $F_{\mu\nu}(x)$ is completely independent of the longitudinal part of $A_\mu(x)$. This makes it possible to introduce the Landau gauge $A_\mu^T(x)$. Then, in an arbitrary gauge

$$A_\mu(x) = A_\mu^\tau(x) + \partial\Lambda(x)/\partial x_\mu,$$
$$\psi(x) = \exp\{-ie\Lambda(x)\}\psi^\tau(x). \qquad (A.1)$$

Only the transverse part $A_\mu^T(x)$ describes a real physical photon. The longitudinal part $\partial\Lambda(x)/\partial x$

(or simply $\Lambda(x)$), can be considered an operator acting in a different Hilbert space (than $A_\mu^T(x)$; e.g., $\Lambda(x)$ generates the space of "inadmissible states"). In any event it is necessary that

$$[A_\mu{}^\tau(x),\ \Lambda(x')]_- = 0, \quad [\psi^\tau(x),\ \Lambda(x')]_- = 0. \qquad (A.2)$$

The equations (2.1) and (2.2) and the canonical commutation relations for $A_\mu(x)$ are compatible if

$$[\Lambda(x), \Lambda(x')]_- = i \int d\varkappa^2 \frac{1}{\varkappa^2} \rho_2(\varkappa^2)\,\Delta(x-x';\varkappa^2),$$
$$\rho_2(\varkappa^2) = \delta(\varkappa^2) + 2M\varkappa^2\delta(\varkappa^2). \qquad (A.3)$$

It follows directly from (A.1)–(A.3) that

$$[A_\mu(x), \Box\Lambda(x')]_- = i\frac{\partial}{\partial x_\mu} D(x-x'),$$
$$[\psi(x), \Box\Lambda(x')]_- = eD(x-x')\psi(x). \qquad (A.4)$$

Hence we obtain for the currents

$$[j_\mu(x), \Box\Lambda(x')]_- = 0,$$
$$[\eta(x), \Box\Lambda(x')]_- = eD(x-x')\eta(x) + e\hat{\nabla}D(x-x')\psi(x). \qquad (A.5)$$

In computing the quasilocal terms the following commutators are used

$$[j_\mu(x),\ \Box\Lambda(x')]_{t=t'} = [j_\mu(x),\ \Box\dot{\Lambda}(x')]_{t=t'} = 0, \qquad (A.6)$$
$$[\eta(x),\ \Box\Lambda(x')]_{t=t'} = -ie\gamma_4\delta(\mathbf{x}-\mathbf{x}')\,\psi(x), \qquad (A.7)$$
$$[\eta(x),\ \Box\dot{\Lambda}(x')]_{t=t'} = -e\delta(\mathbf{x}-\mathbf{x}')\,\eta(x) - e\gamma\frac{\partial}{\partial\mathbf{x}}\delta(\mathbf{x}-\mathbf{x}')\,\psi(x).$$

[1] H. Lehmann, K. Symanzik and W. Zimmerman, Nuovo Cimento 1, 205 (1955); 6, 319 (1957). A. S. Wightman, Phys. Rev. 101, 860 (1956). K. Nishijima, Prog. Theor. Phys. 17, 756 (1957).

[2] N. N. Bogolyubov, B. V. Medvedev and M. K. Polivanov, Voprosy teorii dispersionnykh sootnosheniĭ (Problems of the Theory of Dispersion Relations) GTT I, 1958. B. V. Medvedev and M. K. Polivanov, JETP 41, 1130 (1961), Soviet Phys. JETP 14, 807 (1962); DAN SSSR 143, 1071 (1962), Soviet Phys. Doklady 7, 298 (1963).

[3] V. Ya. Faĭnberg, Lectures at the International School, Dubna, 1964; JETP 47, 2285 (1964), Soviet Phys. JETP 20, 1529 (1965).

[4] B. V. Medvedev and M. K. Polivanov, Lectures at the International School, Dubna, 1964.

[5] V. L. Voronov, JETP 49, 1802 (1965), Soviet Phys. JETP 22, 1232 (1966).

[6] V. D. Skarzhinskiĭ, FIAN Preprint (1966).

[7] E. A. Leont'ev, JETP (in press).

[8] A. I. Akhiezer and V. B. Berestetskiĭ, Kvantovaya élektrodinamika (Quantum Electrodynamics) 2nd ed. Fizmatgiz, 1959.

[9] L. Evans, G. Feldman, and P. T. Matthews, Ann. Phys. (N. Y.) 13, 268 (1961).

[10] H. Rollnik, B. Stech, and E. Nunnemann, Z. Physik 159, 482 (1960).

[11] L. D. Solov'ev, Doctoral Dissertation, JINR, 1966.

[12] V. D. Skarzhinskiĭ, JNP (in press).

[13] R. É. Kallosh, V. Ya. Faĭnberg, JETP 49, 1611 (1965), Soviet Phys. JETP 22, 1102 (1966).

[14] A. C. Hearn, Nuovo Cimento 2, 333 (1961).

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