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FINITE SOLUTIONS IN THE THEORY OF NONRENORMALIZABLE INTERACTIONS

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The charge-differential axiomatic method previously proposed[4] is applied to nonrenormalizable interaction models: a nonrelativistic model of the vector type and a relativistic four-fermion model in the two-particle approximation with closed fermion loops. It is shown that besides the usual meaningless solution for the scattering amplitude corresponding to the dynamic statement of the problem, an additional solution appears in the axiomatic theory. This solution is finite, nonanalytic with respect to the coupling constant and increases as $\exp(\mathrm{const}\cdot\sqrt{E})$ along certain directions in the first sheet of the complex energy plane. The solution under discussion cannot be obtained from the dynamic theory since a nonunitary scattering "semi-matrix" $S(t, -\infty)$ corresponds to it. Upon regularization the additional solution disappears for any finite value of the "cut-off" momentum.

1. As is well known, application of the usual dynamic apparatus of field theory to the description of nonrenormalizable interactions leads, at least in perturbation theory, to essential difficulties. Various points of view exist regarding the causes for the appearance of these difficulties. For example, one might think that the reason lies in our inability to solve dynamic equations outside the framework of perturbation theory. This point of view which is subscribed to in the papers concerning the so-called "peratization"[1,2] has not, however, as yet received a sufficiently convincing confirmation.

On the other hand, it is not excluded that the difficulties under discussion simply reflect the inadequacy of those concepts and quantities which lie at the basis of the dynamic description and which have appeared in the relativistic dynamic apparatus as a result of an unjustified extrapolation of nonrelativistic quantum mechanics. We have in mind, for example, such quantities as the scattering "semi-matrix" $S(t, -\infty)$, field operator, etc., which in relativistic quantum theory cannot be regarded as observable quantities. This second possibility which can be traced back to the old ideas of Heisenberg on the necessity to exclude unobservable quantities from the formulation of the scattering problem lies at the basis of the axiomatic method and the S-matrix method which have been developed recently[1)].

Excluding (either completely or partially) from the apparatus of the theory unobservable quantities we go over to equations which are more general than the dynamic equations. The corresponding relaxation of requirements imposed on the scattering matrix[2)] allows us to hope that the new equations will have in addition to the usual divergent solutions also solutions free from the difficulties of the dynamic theory.

An investigation of this problem by means of the axiomatic method proposed in a paper by one of the authors[4] which is differential with respect to the coupling constant shows that at least in the simplest models of a nonrenormalizable interaction there is a definite basis for such hopes. The corresponding axiomatic equations indeed do have an "extra" solution with respect to the dynamic theory which satisfies all the requirements which are necessary from the point of view of the axiomatic approach. But, the scattering semi-matrix which formally corresponds to this solution turns out to be nonunitary in the system of <u>in</u>-states, and this shows the impossibility of obtaining this solution from the

[1)]Landau (cf. [3]) has pointed out the possible inadequacy of the dynamic method in connection with the problem of the vanishing of charge.

[2)]As has been already pointed out in [4] within the framework of the method utilized below the difference between the approaches to scattering theory enumerated above reduces to the fact that the axioms of field theory are satisfied by a scattering matrix with a different choice of the function $g(x)$ describing the switching on of the interaction [5]: the dynamic approach corresponds to an arbitrary $g(x)$ (including a step function), while the axiomatic method corresponds to a $g(x)$ infinitely close to a constant, and the S-matrix method corresponds to $g(x) = \mathrm{const}$.

dynamic equations.

In this paper we consider from a single point of view the following models:

a) a renormalizable nonrelativistic model (cf.[4]):

$$L^{in}(i,f) \equiv \langle \mathbf{k}_i, -\mathbf{k}_i | L^{in}(0) | \mathbf{k}_{f_2}, -\mathbf{k}_f \rangle = 2\pi;$$

b) a nonrenormalizable nonrelativistic model (cf.[6]):

$$L^{in}(i,f) = -2\pi \mathbf{k}_i \mathbf{k}_f;$$

c) a relativistic four-fermion model in the two-particle approximation with closed fermion loops:

$$L^{in}(i,f) = (\bar{u}_i \Gamma v_i)(\bar{v}_f \Gamma u_f), \quad \Gamma = 1, \gamma_5$$

(cf., section 5 below). Here L^{in} is the interaction "Lagrangian" divided by the charge in the in-representation which specifies the type of interaction.

We emphasize at once that the nonrelativistic problems stated above are considered exclusively as models which admit a complete solution of the scattering problem and which can give an indication of the possibilities that could be expected in the relativistic case.

Section 2 of this paper is devoted to the simplification of the equation obtained in[4] for the scattering phase. Section 3 contains the solution of this equation for self-preserving problems. In Sec. 4 a comparison is made of the axiomatic and the dynamic approaches as applied to the models under discussion. Finally, in section 5 the four-fermion model is discussed.

2. The charge-differential equations of the scattering problem formulated previously[4] are significantly simplified when the following two conditions are satisfied:

a) only two-particle intermediate states are taken into account;

b) the zero-order interaction Lagrangian $L^{in}(i,f)$ is split into two factors which depend respectively on k_i and k_f. In such a case the equations referred to above go over into the following equation for the scattering phase[4] 3):

$$\frac{\delta_l''(k)}{\delta_l'(k)} = \frac{4}{\pi} k^2 \int_0^\infty \frac{dp\, \delta_l'(p)}{p(p^2 - k^2)}. \qquad (1)$$

This equation can be brought to a form in which it can be conveniently solved. In order to eliminate the integral term we differentiate both sides of the equation with respect to g taking into account the identity

$$\mathrm{P}\frac{1}{a}\mathrm{P}\frac{1}{b} = \mathrm{P}\frac{1}{a}\mathrm{P}\frac{1}{b-a} - \mathrm{P}\frac{1}{b}\mathrm{P}\frac{1}{b-a} + \pi^2\delta(a)\delta(b),$$

where the symbol P denotes the principal value. This yields

$$\left(\frac{\delta''}{\delta'}\right)' - \frac{1}{2}\left(\frac{\delta''}{\delta'}\right)^2 + 2(\delta')^2 = I,$$

$$I = \frac{8}{\pi} k^2 \left\{ \int_0^\infty dp \int_0^\infty dq - \int_0^\infty dq \int_0^\infty dp \right\} \frac{p\delta'(p)\delta'(q)}{q(p^2-k^2)(q^2-p^2)}.$$

It is possible to interchange the order of integration or to consider the quantity I to be equal to zero only in the case when the integrals are uniformly convergent. This, in turn, requires a sufficiently rapid falling off of the quantity $\delta'(k)$ as k increases. Introducing the notation

$$C = \lim_{k\to\infty} \frac{\delta'(k)}{k}, \qquad (2)$$

we obtain $I = 2C^2k^2$ (cf. Appendix), and we finally obtain[6]

$$\left(\frac{\delta''(k)}{\delta'(k)}\right)' - \frac{1}{2}\left(\frac{\delta''(k)}{\delta'(k)}\right)^2 + 2[(\delta'(k))^2 - C^2k^2] = 0. \qquad (3)$$

A characteristic feature of this equation is its explicit dependence on the limiting value of its solution.

In order to simplify this equation further it is convenient to go over from the scattering phase to the amplitude f(k). Utilizing the well-known relation $\delta(k) = -\frac{1}{2} i \ln[1 + 2ikf(k)]$ we obtain from (3)

$$f'''/f' - {}^3/_2 (f''/f')^2 - 2C^2k^2 = 0.$$

We further introduce the Born value for the amplitude $f_B(k)$, which is related to the matrix element of the zero-order Lagrangian by the equation (cf., formula (19) in[4]):

$$f_B(k) = \frac{g}{2\pi} L^{in}(i,f)\,|_{E_i = E_f}$$

Introducing the new variable u(k) by the relation

$$f'(k) = f_B'(k)/u^2, \qquad (4)$$

we obtain the final equation

$$u''(k) + C^2k^2u(k) = 0. \qquad (5)$$

We indicate the boundary conditions for this equation. From the relation $f'(k) \to f_B'(k)$ it follows for $g \to 0$ 4) that

$$u|_{g=0} = 1. \qquad (6)$$

The second condition follows from (4) and (1):

3)Here primes denote differentiation with respect to g. The index l is omitted in further discussion. We note that in the published text of an earlier paper by the authors [6] formula (1) is erroneously ascribed to Landau.

4)From the renormalization condition (cf. [4]) it also follows that $u|_{k=0} = 1$.

$$u'\Big|_{g\to 0} = -\frac{2}{\pi}k^2\int_0^\infty \frac{dp\,\delta'(p)}{p(p^2-k^2-i\varepsilon)}\Big|_{g\to 0} \qquad (7)$$

For a renormalizable theory in which the integral

$$\int_0^\infty \frac{dp}{p^2} f_B'(p) = \int_0^\infty \frac{dp}{p^3}\delta_B'(p)$$

converges the integration and the transition to the limit can be interchanged. This yields

$$u'|_{g=0} = -\frac{2}{\pi}k^2\int_0^\infty \frac{dp\,f_B'(p)}{p^2-k^2-i\varepsilon}. \qquad (7')$$

The same also occurs when the regularization procedure is carried out which corresponds to the introduction in $L^{in}(i, f)$ of an additional factor $\nu^*(k_f)\nu(k_i)$ containing a function $\nu(k)$ which falls off sufficiently rapidly for $k \gg \Lambda$, where Λ is the "cut-off" momentum.

Conditions (6) and (7) together uniquely determine the solution of equation (4) for a given C. The value of this quantity is determined by condition (2): the solution of equation (4) which depends on C as a parameter must lead to the quantity $\delta'(k)/k$ which tends to C as $k \to \infty$. For $C \neq 0$ utilizing the WKB method one can easily bring (2) into the form

$$u|_{k\to\infty} = \sqrt{\frac{f_B'(k)}{C}}\exp\left(-ik\int_0^g dg\,C\right), \qquad (8)$$

where the sign of C agrees with the sign of $f_B'(k)$ for $k \to \infty$ (we do not consider the case of oscillating $f_B'(k)$ which is of little interest).

The case C = 0 requires separate consideration. In this case equation (4) has the solution u = A + gB where A and B are functions of k determined by the conditions (6) and (7). The first of these conditions gives A = 1, while the second gives for a renormalizable or a regularized theory (cf. (7′))

$$B(k) = -\frac{2}{\pi}k^2\int_0^\infty \frac{dp\,f_B'(p)}{p^2-k^2-i\varepsilon}.$$

As regards a nonrenormalizable point interaction, in this case the last integral diverges and the corresponding solution turns out to be meaningless.

If the function B(k) is finite then utilizing (4) we arrive at an expression for the scattering amplitude

$$f(k) = f_B(k)\left[1-\frac{2}{\pi}k^2\int_0^\infty dp\frac{f_B(p)}{p^2-k^2-i\varepsilon}\right]^{-1}. \qquad (9)$$

One can verify that this solution is in agreement with condition (2). We note that in the case of a regularized nonrenormalizable theory by letting Λ tend to infinity in (9) we obtain a zero value for the scattering amplitude. This is the well-known difficulty of the vanishing of charge[3].

3. In this section we consider the solutions of equation (4) with $C \neq 0$ in the case when the problem has a single dimensionless parameter $\xi = gk^n$ $(n > 1/2)$. In particular, the problems discussed in[4,6] (cf. also Sec. 5 below) belong to such a self-preserving class.

The Born amplitude in the case under consideration has the form $f_B = Agk^{n-1}$ where A is a constant (for the sake of definiteness we take the sign of g to be positive). Using the notation $C = \alpha g^{1/n-1}$, where α is an as yet unknown numerical factor, and making the transition in (5) to the variable ξ we bring this equation to the form

$$d^2u/d\xi^2 + \alpha^2\xi^{2/n-2}u = 0. \qquad (5')$$

Imposing on its solutions the condition (6) we obtain

$$u(\xi) = \frac{-i\pi}{\Gamma(n/2)}\left(\frac{\sigma}{2}\right)^{n/2}[aH^{(1)}_{n/2}(\sigma) - (1-a)H^{(2)}_{n/2}(\sigma)], \quad (6')$$

where $\sigma = |\alpha| n\xi^{1/n}$, while $H^{(1,2)}_\nu$ are Hankel functions. In principle the arbitrary constant a could be obtained from condition (7). But it is much simpler to utilize directly relation (8) which determines the sign of the phase of u for large k. We recall that in this limit the signs of the phases of the Hankel functions are different and this enables us to make the necessary choice. At the same time we can also find the value of α. We emphasize that condition (8) defines the function u not at one point but on an infinite segment; and this in the last analysis is what gives the possibility of simultaneous determination of α and a. After that it is, of course, necessary to check that condition (7) is satisfied.

Taking into account that according to what was said in section 2, the signs of A and α must be the same and considering the asymptotic behavior of the Hankel functions

$$H^{(1,2)}_{n/2}(\sigma) \sim \sqrt{\frac{2}{\pi\sigma}}\exp\left[\pm i\left(\sigma - \frac{\pi}{4}(n+1)\right)\right]$$

we obtain from (8)

$$a = 0, \quad \alpha = \frac{2}{n}\left(A\Gamma^2\left(\frac{n}{2}\right)\frac{n}{2\pi}\right)^{1/n} \quad (A > 0),$$

$$a = 1, \quad \alpha = -\frac{2}{n}\left(|A|\Gamma^2\left(\frac{n}{2}\right)\frac{n}{2\pi}\right)^{1/n} \quad (A < 0).$$

In accordance with this we have

$$\sigma = 2\left(|A|\Gamma^2\left(\frac{n}{2}\right)\frac{n\xi}{2\pi}\right)^{1/n}. \qquad (10)$$

From the relations given in section 2 one can also easily obtain the scattering amplitude itself

$$f(k) = \int_0^g dg\frac{f_B'(k)}{u^2}.$$

Going over to dimensionless variables and utilizing the integrals

$$\int_0^x \frac{dx}{x}\left(H_\nu^{(1,2)}(x)\right)^{-2} = \pm\frac{\pi i}{4}\left(1+\frac{H_\nu^{(2,1)}(x)}{H_\nu^{(1,2)}(x)}\right),$$

which can be easily evaluated taking into account the expression for the Wronskian

$$H_\nu^{(1)}(x)H_\nu^{(2)\prime}(x) - H_\nu^{(1)\prime}(x)H_\nu^{(2)}(x) = 4/\pi i x.$$

we obtain for $A \gtrless 0$

$$f(k) = -\frac{1}{2ik}(1 + H_{n/2}^{(1,2)}(\sigma)/H_{n/2}^{(2,1)}(\sigma)). \quad (11)$$

From here one can also easily find the scattering phase

$$\operatorname{tg}\delta(k) = \mp J_{n/2}(\sigma)/N_{n/2}(\sigma) \quad (A \gtrless 0). \quad (12)$$

Utilizing this expression one can easily verify that condition (7) is satisfied. Verification shows that for $n < 3$ this condition is satisfied for either sign of A. But for $n \geq 3$ condition (7) turns out to be valid only for $A < 0$. This fact corresponds to the appearance for $A > 0$ of poles of f(k) on the first sheet of the complex energy plane (cf.[7], which gives the distribution of the zeroes of the Hankel functions for arbitrary values of n). At the same time the appearance of such poles is forbidden by the initial equation (1) (cf.[4]). We note in this connection that the appearance of solutions of equation (4) with $A > 0$ which do not satisfy equation (1) is simply associated with the fact that Eq. (4) is obtained by means of an additional differentiation of the original equation (1). Condition (7) is introduced just in order to reject the additional solutions which arise in this process.

In concluding this section we point out that the models investigated in[4,6] are completely contained in the scheme discussed above. Thus, in the renormalizable point interaction[4] model $\xi = gk$, $n = 1$, $H_{1/2}^{(1,2)}(\sigma) = \mp i\sqrt{2/\pi\sigma}\, e^{\pm i\sigma}$, and we obtain from (11) and (12) the solution given in[4]. For the nonrenormalizable point model[6] $\xi = gk^3$, $n = 3$, $H_{3/2}^{(1,2)}(\sigma) = \mp i\sqrt{2/\pi\sigma}(1 \pm i\sigma)e^{\pm i\sigma}$, and the solution is obtained which was found in[6]. The case $n = 4$ corresponds to the four-fermion interaction model which will be discussed below in Sec. 5.

4. As has been emphasized already, Eq. (1), lying at the basis of the calculations described above, corresponds to a more general formulation of the scattering problem than in the dynamic theory. Therefore, among the solutions of (1) there exist both a dynamic and an additional axiomatic solution. The solutions differ in the value of the constant C: the dynamic solution corresponds to $C = 0$, while the axiomatic one corresponds to $C \neq 0$.

One can easily verify the validity of the above statement first of all by means of a direct solution of the dynamic problem and comparison of the results obtained with those given above. The corresponding Schrödinger equation (cf.[4]) indeed does give a solution which coincides with (9) (the latter corresponds just to $C = 0$). More generally it can be easily shown that the solution with $C \neq 0$ cannot be set in correspondence with a unitary scattering "semi-matrix" $S(t, -\infty)$. Indeed, the quantity $d(S^+(t, -\infty)S(t, -\infty))/dg$ can be expressed in terms of an integral similar to I (cf.[4] and Sec. 2), and is therefore proportional to C^2. Of course, in the limit $t \to \infty$ unitarity is reestablished. This is also supported by the evaluation of the effective Hamiltonian appearing in the Schrödinger equation

$$i\frac{dS(t,-\infty)}{dt} = H_{\mathrm{eff}}(t)S(t,-\infty).$$

Utilizing for $S(t, -\infty)$ the expression given in[4] we obtain

$$H_{\mathrm{eff}}(k,k') = \sqrt{\frac{k}{\delta'(k)}}\frac{\sin\delta(k)}{k}\sqrt{\frac{\delta_B'(k')}{k'}}A(g), \quad (13)$$

$$A(g) = -2\pi\left[1 + 4\pi\int\frac{d^3q}{q^3}\sqrt{\frac{\delta_B'(q)}{\delta'(q)}}\sin\delta(q)\right]^{-1}.$$

H_{eff} in the case $C = 0$ reduces to the initial interaction Hamiltonian $-g_0\int d^3x L^{\mathrm{in}}(x)$, where

$$g_0 = -\frac{gA(g)}{2\pi} = g\left[1 + g\frac{2}{\pi}\int_0^\infty\frac{dq}{q}\delta_B'(q)\right]^{-1}$$

is the initial charge. At the same time for $C \neq 0$ expression (13) does not coincide with the initial Hamiltonian and turns out to be explicitly nonhermitian.

In[4] it has been pointed out already that regularization of the interaction which leads to a falling off in the value of $\delta_B'(p)$ with increasing p necessarily leads to the loss of the additional solution. We give the corresponding proof which reduces to the demonstration of the fact that in this case $C = 0$. From Eq. (1), taking into account the identity

$$\int_0^\infty\frac{dp}{p^2-k^2} = 0$$

for $k \neq 0$, we have

$$\frac{\delta'(k)}{k} = \frac{\delta_B'(k)}{k} \times \exp\left[-\frac{4k^2}{\pi}\int_0^g dg\int_0^\infty\frac{dp}{p^2-k^2}\left(\frac{\delta'(p)}{p} - C\right)\right]. \quad (14)$$

If, in particular, $\delta_B'(k) = 0$ for $k > \Lambda$, then (14) directly yields $C = 0$. In the more general case $\delta_B'(k) \to 0$ for $k \to \infty$ we obtain that the integral with respect to p in the index of the exponential converges uniformly with respect to k[5)] and the index itself as $k \to \infty$ turns out to be equal to

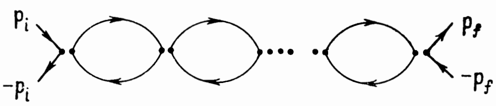
$$\frac{4}{\pi}\int_0^g dg \int_0^\infty dp\left(\frac{\delta'(p)}{p} - C\right).$$

This quantity could diverge only at the lower limit of the integration with respect to g. But for $g \to 0$

$$C = \lim_{k\to\infty}\frac{\delta_B'(k)}{k} = 0, \qquad \int dp\,\frac{\delta'(p)}{p} \to \int dp\,\frac{\delta'_B(p)}{p} < \infty.$$

Therefore, the exponential in (14) is finite and does not grow without bound with increasing k. In view of the falling off of the first factor in (14) we obtain C = 0.

5. The method described above can be used to discuss the simplest model of relativistic four-fermion interaction corresponding to the sum of the following diagrams of the dynamic perturbation theory[6)]:



This model describes the scattering of a particle by an antiparticle and corresponds to the following zero-order interaction Lagrangian:

$$L^{in}(i,f) = -(\bar{u}(\mathbf{p}_i)\Gamma v(-\mathbf{p}_i))(\bar{v}(-\mathbf{p}_f)\Gamma u(\mathbf{p}_f))$$

(the exchange term is not taken into account, $\Gamma = 1, \gamma_5$).

The splitting up of the expression for $L^{in}(i, f)$ taken together with the two-particle nature of the intermediate states enables us to utilize fully the apparatus developed above. Writing the matrix elements of the scattering matrix S and of the Lagrangian L in the form

$$\langle i|S-1|f\rangle = -\delta(E_i - E_f)L^{in}(i,f)\rho^{-1}(E)(e^{2i\delta(E)}-1),$$
$$\langle i|L(0)|f\rangle = L^{in}(i,f)\chi^*(E_f)\chi(E_i),$$

where $\rho(E) = -2\pi p E \Sigma L^{in}(i, i)$ (summation over the spin indices), and substituting the expressions given above into the equations of[4] we obtain the relation between $\chi(E)$ and the scattering "phase":

$$\delta(E) = -\frac{\rho(E)}{8\pi^2}\int_0^g dg|\chi(E)|^2$$

and the equation for the phase:

$$\frac{\delta''(E)}{\delta'(E)} = \frac{2}{\pi}(E-m)\int_m^\infty \frac{d\omega\,\delta'(\omega)}{(\omega-m)(\omega-E)}. \qquad (15)$$

Introducing the new variable $k = \sqrt{E-m}$, one can easily bring (15) into the form of Eq. (1), and this enables us to utilize the results obtained in Sec. 3.

In future we shall restrict ourselves to a discussion of the self-similar limiting case, when $m \ll k^2$ and $gm^2 \ll 1$. Then for $\Gamma = 1, \gamma_5$ we obtain $\rho(E) = 4\pi E^2 = 4\pi k^4$ and $\delta'_B(k) = -k^4/2\pi$. Thus, the problem under consideration belongs to the class of problems discussed in Sec. 3 and corresponds to n = 4.

Utilizing expressions (11) and (12) we obtain

$$\operatorname{tg}\delta(k) = J_2(\sigma)/N_2(\sigma), \qquad (16)$$

$$f(k) = \frac{e^{2i\delta(k)}-1}{2ik} = \frac{-1}{2ik}\left(1 + \frac{H_2^{(2)}(\sigma)}{H_2^{(1)}(\sigma)}\right), \qquad (17)$$

where $\sigma = (2gE^2/\pi^3)^{1/4}$ and $k = \sqrt{E}$[7)]. It is necessary to emphasize the principal characteristic features of the solution obtained:

a) after carrying out charge renormalization it remains finite in contrast to the dynamic solution;

b) it is nonanalytic with respect to the coupling constant and contains terms of the type $g^{1/2}$, $g^{3/2}$ ... and ln g;

c) it increases along certain directions in the first sheet of the complex energy plane as an exponential with an index proportional to $\sqrt{E}$.

The last property means that we indeed do go outside the framework of the class of functions of moderate growth[10–12]. This circumstance is apparently closely connected with the nonrenormalizability of the interaction[13,14] [8)].

In conclusion we emphasize that the aim of this paper consisted only of a comparison of the dynamic and the axiomatic approaches within the framework of the simplest mathematical models of nonrenormalizable interactions; in particular, we did not investigate at all the limits of applicability of the approximation utilized in this section. The problem of the extent to which the conclusions obtained in this paper remain valid in going over to a real four-fermion interaction is under investigation at present and will be the subject of a separate paper.

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5)From the solutions given earlier it follows that $\delta'(p)/p - C$ is of order $1/p^2$ as $p \to \infty$.

6)It is this set of diagrams which was discussed in [8] where it is concluded that the scattering amplitude vanishes in the dynamic method (however, cf. [9]). As will be seen from what follows later, this derivation ceases to be valid in the axiomatic method.

7)We emphasize that the quantity k does not coincide with the momentum of the particle which for m E is simply equal to E.

8)For a renormalizable interaction with n = 1, one can always choose a solution which decays exponentially on the first sheet.

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APPENDIX

We evaluate the quantity I (cf. Sec. 2). Breaking up each integral into two parts $\int_0^N + \int_N^\infty$ where N is a large but finite number we verify that those integrals which contain finite limits make no contribution to I due to their uniform convergence. Consequently,

$$I = \frac{8}{\pi}\left(\int_N^\infty dp \int_N^\infty dq - \int_N^\infty dq \int_N^\infty dp\right) \frac{p\delta'(p)\,\delta'(q)}{q\,(p^2-k^2)\,(q^2-p^2)}.$$

replacing $\delta'(p)/p$ by its limiting value C and carrying out straightforward calculations we obtain after replacing the variables p, q → Np, Nq for N → ∞:

$$I = \frac{8k^2C^2}{\pi}\left(\int_1^\infty dp \int_1^\infty dq - \int_1^\infty dq \int_1^\infty dp\right) \frac{1}{q^2-p^2} = 2C^2k^2.$$

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