CONTRIBUTION TO THE THEORY OF FOURTH SOUND ABSORPTION

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The damping coefficient of fourth sound in a plane-parallel capillary or in a system of plane-parallel capillaries is calculated by means of the two-fluid hydrodynamical equations for helium II. The sound absorption is due to a viscous mechanism of energy dissipation and to heat flow through the capillary walls.

As is well known, two types of vibrations can arise in helium below the λ point: ordinary (first) sound and second sound, which corresponds principally to temperature oscillations.

A few years ago, Pellam[2] and Atkins[2] called attention to the fact that one can materially change the character of sound propagation in helium by retarding the normal motion in it. The sound in a thin film of helium has been called third, and in a sufficiently thin capillary, fourth sound.

The speed of fourth sound (we shall be interested only in fourth sound), according to Atkins, is equal to[1]

$$u_{40} = \left(\frac{\rho_s}{\rho} u_1^2 + \frac{\rho_n}{\rho} u_2^2\right)^{1/2}. \tag{1}$$

It is correctly measured if the experiments are carried out on a complex system of branched capillaries made from powdered rouge. The dimensions of the very curved capillaries here are of the order of the distances between them. Naturally, such a system presents difficulties for theoretical investigations. We consider the propagation of sound in an isolated capillary and in a system of identical parallel capillaries. Such a geometry permits an exact calculation not only of the sound velocity but also of the absorption coefficient.

To retard the normal component in the process of sound propagation, it is sufficient that the length of the viscous wave λ_v or the length of the free path of the elementary excitations l be much greater than the dimensions of the capillary d. If

$$\lambda_v \gg d \gg l, \tag{2}$$

then the absorption coefficient can be expressed in macroscopic terms—by means of the macroscopic dissipation coefficients (the hydrodynamic case) If now

$$l \gg d, \tag{3}$$

then a microscopic consideration is required. In the present communication, we consider only the hydrodynamic case. The hydrodynamic consideration is valid for sufficiently low frequencies ($\omega \ll 2\eta/d^2\rho_n$, where 2d is the width of the capillary, η and ρ_n the viscosity and the density of the normal component of the helium) and comparatively high temperatures.

The energy dissipation in the superfluid helium is described by five viscosity coefficients and the coefficient of thermal conductivity. The absorption of sound is determined by viscous mechanisms (they play the dominant role in the absorption of first sound) and thermal conduction (the latter is responsible for the absorption of second sound), the various dissipation mechanisms entering additively into the absorption coefficient. This allows us to consider the several mechanisms independently. We first calculate the viscous portion of the absorption coefficient.

The complete linearized set of hydrodynamic equations, without account of the normal (dissipative) thermal conductivity, has the following form:

$$\begin{gathered} \dot{\rho} + \operatorname{div}\mathbf{j} = 0 \\ \frac{\partial j_i}{\partial t} + \nabla_i P = \eta\frac{\partial}{\partial x_k}\left(\frac{\partial v_{ni}}{\partial x_k} + \frac{\partial v_{nk}}{\partial x_i} - \frac{2}{3}\delta_{ik}\frac{\partial v_{nl}}{\partial x_l}\right) \\ + \frac{\partial}{\partial x_i}[\zeta_1 \operatorname{div}(\mathbf{j} - \rho\mathbf{v}_n) + \zeta_2 \operatorname{div}\mathbf{v}_n], \\ \dot{\mathbf{v}}_s + \nabla\mu = \nabla[\zeta_3 \operatorname{div}(\mathbf{j} - \rho\mathbf{v}_n) + \zeta_4 \operatorname{div}\mathbf{v}_n], \\ (\sigma\rho)^{\cdot} + \sigma\rho \operatorname{div}\mathbf{v}_n = 0. \end{gathered} \tag{4}$$

[1]Here and below, we shall neglect thermal expansion in this paper.

Here ρ is the density, σ the specific entropy, P the pressure; η, $\zeta_1 = \zeta_4$, ζ_2, ζ_3 are the viscosity coefficients; $\mathbf{v}_n$ and $\mathbf{v}_s$ the velocities of the normal and superfluid components; $\mathbf{j} = \rho_n \mathbf{v}_n + \rho_s \mathbf{v}_s$, $\rho_{n(s)}$ the density of the normal (superfluid) component, and $\nabla\mu = -\sigma\nabla T + \rho^{-1}\nabla P$.

For simplicity, we shall consider a plane capillary, unbounded in two directions, and choose the z axis along the third direction. The set of equations that have been written down, if we add the dissipative component with the corresponding boundary conditions to the last equation, makes it possible to consider the propagation of sound in the capillary for any relation between the width of the channel and the length of the viscous wave. In particular, one can show that first and second sound are propagated in the capillary when $\omega \gg 2\eta/d^2\rho_n$. When the frequency decreases, the velocity of first sound changes and when $\omega \ll 2\eta/d^2\rho_n$, it becomes equal to the velocity of fourth sound (see (1)). Oscillations corresponding to second sound are damped and cease to propagate.

If we consider waves whose lengths is much greater than the dimensions of the capillary, then the set (4) can be greatly simplified, since v_{zn} and v_{zs} can be set equal to zero, while P, ρ and σ can be assumed to be independent of the coordinate z. Moreover, if we direct the x axis along the direction of propagation of the wave, we then have (we omit the vector indices)

$$\begin{gathered}\dot{\rho} + \partial j/\partial x = 0,\\ \frac{\partial j}{\partial t} + \frac{\partial P}{\partial x} = \left(\frac{4}{3}\eta + \zeta_3\right)\frac{\partial^2 v_n}{\partial x^2} + \eta\frac{\partial^2 v_n}{\partial z^2} + \zeta_1\rho_s\frac{\partial^2(v_s - v_n)}{\partial x^2}\\ \dot{v}_s + \frac{\partial\mu}{\partial x} = \rho_s\zeta_3\frac{\partial^2}{\partial x^2}(v_s - v_n) + \zeta_4\frac{\partial^2 v_n}{\partial x^2},\\ (\sigma\rho)^{\cdot} + \sigma\rho\,\partial v_n/\partial x = 0.\end{gathered} \tag{5}$$

We average the set of equations over the z coordinate. As a result, only the second component of the right side of the second equation is changed. It is written in the form $(\eta/d)(\partial v_n/\partial z)_{z=d}$. Since $v_n|_{z=d} = 0$, we get $\partial v_n/dz = -\beta v_n(0)/d$, where β is a quantity of the order of unity, which can be determined in the exact solution of the two-dimensional problem with the corresponding boundary conditions. We denote

$$\beta\eta/d^2 = R \tag{6}$$

(according to the Appendix, for a plane capillary, $\beta = 3$; for a cylindrical capillary, $\beta = 8$). With account of the notation introduced, the set of equations (5) is rewritten in the following form:

$$\begin{gathered}\dot{\rho} + \partial j/\partial x = 0;\\ \frac{\partial j}{dt} + \frac{\partial P}{\partial x} = \left(\frac{4}{3}\eta + \zeta_2\right)\frac{\partial^2 v_n}{\partial x^2} - Rv_n + \zeta_1\rho_s\frac{\partial^2}{\partial x^2}(v_s - v_n),\\ \dot{v}_s + \frac{\partial\mu}{\partial x} = \rho_s\zeta_3\frac{\partial^2}{\partial x^2}(v_s - v_n) + \zeta_4\frac{\partial^2 v_n}{\partial x^2},\\ (\sigma\rho)^{\cdot} + \sigma\rho\frac{\partial v_n}{\partial x} = 0.\end{gathered} \tag{7}$$

Eliminating v_s and v_n from Eqs. (7), assuming that all the variable quantities depend on time and the coordinates according to an exponential law $e^{-i(\omega t - kx)}$, and neglecting the thermal expansion of the helium, we get

$$\begin{gathered}\left(\omega^2 - \frac{\partial P}{\partial\rho}k^2\right)\rho' = -i\omega\left\{\left[\left(\frac{4}{3}\eta + \zeta_2\right)k^2 + R\right]\frac{\rho'}{\rho} + \left[\left(\frac{4}{3}\eta + \zeta_2 - \zeta_1\rho\right)k^2 + R\right]\frac{\sigma'}{\sigma}\right\},\\ \left(\sigma\frac{\partial T}{\partial\sigma}k^2 - \frac{\rho_n}{\rho_s\sigma}\omega^2\right)\sigma' = i\omega\left\{\left[\frac{R}{\rho} + \left(\frac{4}{3}\frac{\eta}{\rho} + \frac{\zeta_2}{\rho} - \zeta_4\right)k^2\right]\frac{\rho'}{\rho} + \left[\frac{R}{\rho} + \left(\frac{4}{3}\frac{\eta}{\rho} + \frac{\zeta_2}{\rho} - \zeta_4 + \rho\zeta_3 - \zeta_1\right)k^2\right]\frac{\sigma'}{\sigma}\right\}.\end{gathered} \tag{8}$$

The primes denote the amplitudes of the variable parts of the specific entropy and the density. Since $\partial P/\partial\rho$ is the square of the velocity of first sound (u_1^2), and $(\sigma^2\rho/\rho_n)\,\partial T/\partial\sigma$ is the square of the velocity of the second sound (u_2^2), it is convenient to introduce the following notation:

$$\begin{gathered}\tilde{u}_1^2 = \frac{\partial P}{\partial\rho} - i\omega\zeta_1, \qquad \tilde{u}_2^2 = \sigma^2\frac{\rho_s}{\rho_n}\frac{\partial T}{\partial\sigma} - i\omega\frac{\rho_s}{\rho_n}(\rho\zeta_3 - \zeta_1),\\ \tilde{R} = R + (^4/_3\eta + \zeta_2 - \zeta_1\rho)k^2 \quad (\zeta_1 = \zeta_4).\end{gathered} \tag{9}$$

Setting the determinant of the system (8) equal to zero, we get for the square of fourth sound speed $u_4^2 = \omega^2/k^2$:

$$u_4^2 - \left(\frac{\rho_n}{\rho}\tilde{u}_2^2 + \frac{\rho_s}{\rho}\tilde{u}_1^2\right) - i\frac{\rho_n k^2}{\omega\tilde{R}}(u_4^2 - \tilde{u}_1^2) \times (u_4^2 - \tilde{u}_2^2) = 0. \tag{10}$$

Taking into account the fact that we are interested in only one solution of the latter equation (which is weakly damped as $R \to \infty$), and also the smallness of the dissipative components, we have

$$u_4^2 = u_{40}^2 - i\omega\rho_s\zeta_3 + \frac{i\rho_n k^2}{\omega R}(u_{40}^2 - u_1^2)(u_{40}^2 - u_2^2), \tag{11}$$

where u_{40} is determined by Eq. (1).

From the last expression, we easily find that in the case under consideration ($l \ll d$), the sound absorption coefficient due to viscous mechanisms is equal to

$$\Gamma_{\text{vis}} = \frac{\zeta_3\rho_s\omega^2}{2u_{40}^2} + \frac{\rho_n\omega^2}{2R}\frac{\rho_s\rho_n}{\rho^2}\frac{(u_1^2 - u_2^2)^2}{u_{40}^4}. \tag{12}$$

The first component describes the dissipation of the sound energy as a result of the mutual friction of the superfluid and normal components. According to Khalatnikov and Chernikova,[3]

$$\zeta_3 = \frac{3}{\tilde{\beta}\rho^2}\left(\frac{\rho}{u_1}\frac{\partial u_1}{\partial \rho} + \frac{S}{C_{He}}\right)^2 u_1^2 \rho_{n.ph}\, \tau_{ph.r}$$

where S and C_{He} are the entropy and the heat capacity of a unit volume of the liquid, $\rho_{n.ph}$ is the phonon part of the normal density. We shall not write out the rather long expressions for the phonon-roton relaxation time $\tau_{ph.r}$ and the coefficient $\tilde{\beta}$. They are contained in [3].

The second component is connected with the slipping of the normal component. We note that one should observe a significant increase in the damping coefficient of the fourth sound near the λ point, since the velocity of the second sound and ρ_s vanish as $T \to T_\lambda$.[4]

We also note that in the absorption of second sound, a modification of which is fourth sound,[5,6] ζ_2 plays the principal role,[7] while here both η and ζ_3 are involved. If we take it into account that $u_{40}^2 \approx u_1^2\rho_s/\rho$, then Eq. (12) is simplified to become

$$\Gamma_{vis} = \frac{\zeta_3\rho\omega^2}{2u_1^2} + \frac{d^2}{6}\frac{\omega^2\rho_n^2}{\rho_s\eta}\left(1 - \frac{u_2^2}{u_1^2}\right)^2. \quad (13)$$

From the latter formula, it is seen that the temperature dependence of the first component is determined principally by the temperature dependence of the coefficient of second viscosity ζ_3 and the second component not only by the viscosity but also by the temperature dependence of the normal component density.

We shall now explain the role of the thermal conductivity in the propagation and absorption of fourth sound. In the consideration of the viscous mechanisms of dissipation, the role of the walls of the capillary is that the velocity of the normal component on them vanishes; in clarifying the role of the thermal conductivity, it is necessary to take it into account that the thermal emission takes place across the walls. Therefore the complete set of equations of hydrodynamics of helium (in the last equation, one must include the dissipative heat flow), must be supplemented by the equation of thermal conduction outside the gap (in the wall):

$$\dot{T}_w = \frac{\varkappa_w}{C_w}\Delta T_w, \quad (14)$$

where k_w, C_w and T_w are the thermal conductivity, heat capacity and temperature of the wall. The latter is measured from the temperature far away from the capillary, that is,

$$T_w = 0 \quad \text{for} \quad |z| \to \infty. \quad (15)$$

On the boundary with helium, the usual boundary conditions for helium are satisfied:

$$-\varkappa_w\,\partial T_w/\partial z = Q_z \quad (z = d),$$
$$-\varkappa_w\,\partial T_w/\partial z = \alpha(T - T_w) \quad (z = d), \quad (16)$$

where

$$Q_z = -\varkappa\,\partial T/\partial z + \rho T\sigma v_{nz} = \text{energy flux density}$$

κ is the thermal conductivity of the helium and α the specific coefficient calculated by Khalatnikov; the inverse quantity α^{-1} is the thermal resistance of the boundary.[8]

The last equation of the set (4), which is supplemented by the dissipative heat flow, takes the form

$$(\sigma\rho)^{\cdot} = \frac{\varkappa}{T}\Delta T. \quad (17)$$

We have omitted the component $\sigma\rho$ div $\mathbf{v}_n$ since, in the approximation in which we are interested (without account of the slipping of the normal component), the velocity of the normal component must be set equal to zero [here the total set of equations consists of the first, third and fourth equations of the system (4)]. Averaging Eq. (17) over the thickness of the shell, neglecting the dependence on the coordinate z of the quantities σ and ρ, we get

$$T(\sigma\rho)^{\cdot} = \frac{\varkappa}{d}\left(\frac{\partial T}{\partial z}\right)_{z=d} + \varkappa\frac{\partial^2 T}{\partial x^2}. \quad (18)$$

By using the solution of Eq. (14) with the boundary conditions (15) and (16), it is possible to express the first component of the right side of Eq. (18) in terms of the temperature of the helium on the boundary. If we now neglect the dependence of the temperature on the coordinate z, which is justified by the small thickness of the capillary, then Eq. (18) takes the form

$$T(\sigma\rho)^{\cdot} = \frac{\partial T}{\partial\sigma}\left(\varkappa\frac{\partial^2\sigma'}{\partial x^2} - \frac{1}{d}\frac{\sigma'}{1/\alpha + 1/\gamma\varkappa_w}\right), \quad (19)$$

where

$$\gamma = (k^2 - i\omega C_w/\varkappa_w)^{1/2}, \qquad \mathrm{Re}\,\gamma > 0. \quad (20)$$

As $d \to 0$, if the thermal resistance of the boundary $1/\alpha$ and of the wall of the capillary $1/\gamma\kappa_w$ are finite, then σ' vanishes, which changes the character of the wave propagation in the capillary. Assuming $\sigma' = 0$ and $\mathbf{v}_n = 0$ in the system (4), we easily find that the square of the velocity of sound propagation is equal to (see [2]):

$$u_\sigma^2 = (\rho_s/\rho)\,\partial P/\partial\rho.$$

Actually, two equations are left from the system (4), as $\sigma' = 0$ and $\mathbf{v}_n = 0$:

$$\dot{\rho}+\rho_s \operatorname{div} \mathbf{v}_s=0, \quad \dot{\mathbf{v}}_s+\frac{1}{\rho}\nabla P=0, \qquad (21)$$

whence, by eliminating $\mathbf{v}_S$, we obtain

$$\ddot{\rho}-u_\sigma^2\Delta\rho=0, \quad u_\sigma^2=\frac{\rho_s}{\rho}\frac{\partial P}{\partial \rho}=\frac{\rho_s}{\rho}u_1^2. \qquad (22)$$

In first equation of the system (2), keeping the component which describes the friction between the superfluid and the normal components (it contains ζ_3), we get

$$\ddot{\rho}-\frac{\rho_s}{\rho}\frac{\partial P}{\partial \rho}\Delta\rho=\zeta_3\rho_s\Delta\dot{\rho}, \qquad (23)$$

whence we easily find the absorption coefficient (more accurately, the part of it that is independent of the thickness of the capillary):

$$\Gamma^{(\sigma)}_{\text{vis}\,(\zeta)}=\zeta_3\rho\omega^2/2u_1^2. \qquad (24)$$

We note that in this case this is an exact formula (cf. (13)). The index (σ) in the sound absorption coefficient means that, because of the walls, the entropy σ is constant.

For the determination of the coefficient of sound absorption due to thermal conductivity, it is convenient to make use of Eq. (19), expressing σ' approximately in terms of ρ:

$$\sigma'\approx-T\sigma\frac{\partial\sigma}{\partial T}\frac{\dot{\rho}}{R_\sigma}, \qquad (25)$$

$$R_\sigma=\frac{1}{d}\frac{1}{1/\alpha+1/\gamma\varkappa_{\text{w}}}. \qquad (26)$$

Substituting the value of σ' in the equation of motion of the superfluid components, and eliminating $\mathbf{v}_S$ by means of the equation of continuity, we obtain

$$\ddot{\rho}-\frac{\rho_s}{\rho}\frac{\partial P}{\partial \rho}\Delta\rho=\frac{T\rho_s\sigma^2}{R_\sigma}\Delta\dot{\rho}, \qquad (27)$$

whence

$$\Gamma^{(\sigma)}_{\text{therm}}=\frac{T\rho\sigma^2}{u_1^2}\omega^2\operatorname{Re}\frac{1}{R_\sigma}. \qquad (28)$$

By making use of Eq. (30), and neglecting k^2 in comparison with $\omega C_W/\kappa_W$, we get

$$\Gamma^{(\sigma)}_{\text{therm}}=\frac{T\rho\sigma^2}{u_1^2}\omega^2\frac{2}{d}\frac{1/\alpha+(1/2\omega C_{\text{w}}\varkappa_{\text{w}})^{1/2}}{1/\alpha^2+[1/\alpha+(2/\omega C_{\text{w}}\varkappa_{\text{w}})^{1/2}]^2}. \qquad (28a)$$

We now make clear how the second component in Eq. (12) changes if the isothermal character of sound propagation in the capillary is taken into account. For this purpose, we set $\sigma' = T' = 0$ in the set (7) (the last equation need not be considered), and obtain from the second and third equations

$$v_n\approx-\frac{\rho_n}{\rho R}\frac{\partial P}{\partial\rho}\frac{\partial\rho}{\partial x}. \qquad (29)$$

Substituting (29) in the equation of continuity and again eliminating v_S, we find

$$\ddot{\rho}-\frac{\rho_s}{\rho}\frac{\partial P}{\partial\rho}\frac{\partial^2\rho}{\partial x^2}=\frac{\rho_n^2\partial P/\partial\rho}{\rho R}\frac{\partial^2\dot{\rho}}{\partial x^2},$$

and, finally,

$$\Gamma^{(\sigma)}_{\text{vis}\,(\eta)}=d^2\rho_n^2\omega^2/6\rho_s\eta. \qquad (30)$$

Thus, the total absorption coefficient, due to viscous mechanisms, is equal in this case to

$$\Gamma^{(\sigma)}_{\text{vis}}=\frac{\zeta_3\rho\omega^2}{2u_1^2}+\frac{d^2}{6}\frac{\rho_n^2\omega^2}{\rho_s\eta}. \qquad (31)$$

This expression differs from Eq. (13) in the absence of the factor $(1-u_2^2/u_1^2)^2$, which is especially important at low temperatures.

We shall not make a comparison of the absorption coefficients (31) and (28), since the latter depends essentially on the properties of the walls of the capillary. It should be noted that the principal role in the dissipation of sound energy is played by surface mechanisms (the second component in Eq. (31) and (28)). We turn our attention to the fact that $\Gamma^{(\sigma)}_{\text{therm}}$ depends linearly on the thickness of the capillary, while the second component in Eq. (21) is proportional to d^2 (see the definition of R and R_σ—Eqs. (6) and (26)).

The different dependence on the thickness of R and R_σ means that the situation can exist in which R should be regarded as a large parameter[2)] ($R \sim 1/d^2$), while R_σ is small ($R_\sigma \sim 1/d$). This should hold in each case for large thermal resistance of the walls of the capillary. In this case, the dissipative losses from thermal conduction in the propagation of ordinary fourth sound (the velocity of sound is determined by Eq. (1)) are small. Neglecting k^2 in comparison with $\omega C_W/\kappa_W$, we get

$$\Gamma_{\text{therm}}=\frac{u_2^2\rho_n}{2u_{40}^2\rho C_{\text{He}}}\left[\varkappa\frac{\omega^2}{u_{40}^2}+\frac{2}{d}\frac{1/\alpha+\sqrt{1/2\omega C_{\text{w}}\varkappa_{\text{w}}}}{1/\alpha^2+(1/\alpha+\sqrt{2/\omega C_{\text{w}}\varkappa_{\text{w}}})^2}\right]. \qquad (32)$$

Let us estimate numerically the coefficients of absorption associated with the slippage of the normal component and with the thermal emission through the walls of the gap (the second component in Eqs. (13) and (32)). For a gap of thickness $d = 10^{-5}$ cm at $T = 1.5°$ K, we have $\Gamma_{\text{vis}(\eta)} \sim 10^{-9}\omega^2$ [sec^{-1}], while $\Gamma_{\text{therm}} \approx 1$ [sec^{-1}]. In the estimate of Γ_{therm}, the heat transfer coefficient of the quartz was taken to be equal to 0.25 watt/cm^2 deg.[9]

The coefficient of absorption due to thermal conductivity has a comparatively complicated frequency dependence. At low frequency,

$$\Gamma_{\text{therm}} \approx \frac{u_2^2 \rho_n \sqrt{2\omega C_w \varkappa_w}}{4 u_{40}^2 \rho C_{\text{He}} d}. \quad (33)$$

From this formula it is seen that the sound of extremely low frequency cannot be propagated in the capillary, since $\Gamma_{\text{therm}}/\omega$ tends to infinity as the frequency approaches zero.

As a condition of applicability of Eq. (33), we use the inequality

$$\frac{\sqrt{2}}{4} \frac{\rho_n}{\rho} \frac{u_2^2}{u_{40}^2} \sqrt{\frac{C_w \varkappa_w}{C_{\text{He}}^2 \omega}} \ll d \ll \sqrt{\frac{6\eta\rho_s}{\rho_n^2 \omega}} \quad (34)$$

(see Eq. (13) and the footnote[2]).

Thus, for fourth sound to be able to propagate in the capillary it is necessary that the thickness of the capillary be much greater than the length of the thermal wave, and much less than the length of the viscous wave.

We now proceed to a system of equally spaced capillaries, the distance between which is 2d′. Equation (19), as can easily be seen, has the form

$$T(\sigma\rho)^{\cdot} = \frac{\partial T}{\partial \sigma}\left(\varkappa \frac{\partial^2 \sigma}{\partial x^2} - \frac{1}{d} \frac{\sigma'}{1/\alpha + \operatorname{cth}(\gamma d')/\gamma \varkappa_w}\right), \quad (35)$$

while the value of γ is determined by Eq. (20).

If d and d′ are of the same order, and small, the following approximate expression follows from (35) and (20):

$$T(\sigma\rho)^{\cdot} = \frac{\partial T}{\partial \sigma}\left[-\left(\varkappa + \frac{d'}{d}\varkappa_w\right)k^2 + i\omega \frac{d'}{d} C_w\right]\sigma',$$

which it is convenient to write in the form

$$C_{\text{eff}} \dot{T} + T\sigma\dot{\rho} = \varkappa_{\text{eff}} \Delta T, \quad (36)$$

by introducing the effective values

$$C_{\text{eff}} = C_{\text{He}} + \frac{d'}{d} C_w, \quad \varkappa_{\text{eff}} = \varkappa + \frac{d'}{d}\varkappa_w.$$

We now use Eq. (36) for obtaining the dispersion law of the sound waves in the set of capillaries. The complete set of equations consists of Eq. (36) and the following equations:

$$\dot{\rho} + \rho_s \operatorname{div} \mathbf{v}_s = 0, \quad \dot{\mathbf{v}}_s + \rho^{-1}\nabla P - \sigma \nabla T = 0. \quad (37)$$

Eliminating v_s, and then, approximately, T′ (the remaining principal terms in the expansion in powers of the frequency ω), we get

$$\ddot{\rho} - \left(\frac{\rho_s}{\rho}\frac{\partial P}{\partial \rho} + \frac{\rho_s \sigma^2 T}{C_{\text{eff}}}\right)\Delta\rho = -\frac{\rho_s \sigma^2 T}{C_{\text{eff}}^2} i\omega \frac{k^2}{\omega^2} \varkappa_{\text{eff}} \Delta\rho. \quad (38)$$

It is then seen that the velocity of sound propagation in the system of capillaries is equal to

$$u_4' = \left[\frac{\rho_s}{\rho} u_1^2 + \frac{C_{\text{He}}}{C_{\text{eff}}} \frac{\rho_n}{\rho} u_2^2\right]^{1/2} \quad (39)$$

and is identical with u_{40} only when the heat capacity of the solid phase can be neglected in comparison with the heat capacity of the helium.

It follows from Eq. (38) that the coefficient of acoustical damping due to the thermal conductivity is equal to

$$\Gamma^{(1)}_{\text{therm}} = \rho_s \sigma^2 T \varkappa_{\text{eff}} \omega^2 / 2C^2_{\text{eff}} u_4'^4. \quad (40)$$

We still note that in consideration of sound in a set of capillaries one must in principle take into account the transition from u_{40} to u_4' in the formulas that describe the viscous part of the absorption coefficient (see Eq. (12)).

APPENDIX

To obtain $\overset{\circ}{\beta}$, it is necessary to solve the set of equations (4) with the boundary conditions

$$v_{nx} = 0, \quad v_{nz} = 0, \quad v_{sz} = 0 \quad \text{for } z = \pm d. \quad (41)$$

The solution for the correction we shall seek in the form

$$\mathbf{v}_n = L_1 \nabla Q_1 + L_2 \nabla Q_2 + \mathbf{u}, \quad \mathbf{v}_s = N_1 \nabla Q_1 + N_2 \nabla Q_2, \quad \sigma = M_1 Q_1 + M_2 Q_2, \quad (42)$$

where Q_i and **u** are functions of the coordinates and time (the time dependence is taken in the form $e^{-i\omega t}$): L_i, N_i, M_i (i = 1, 2) are the amplitudes which do not depend on the coordinates and on the time.

Substitution of (42) in (4) leads to a set of equations for Q_i and **u** and to coupling between the amplitudes:

$$\Delta Q_1 + k_1^2 Q_1 = 0, \quad \Delta Q_2 + k_2^2 Q_2 = 0, \quad \Delta \mathbf{u} + k_3^2 \mathbf{u} = 0, \quad \operatorname{div} \mathbf{u} = 0; \quad (43)$$

$$N_i = P_i L_i, \quad M_i = D_i L_i \quad (i = 1, 2),$$

$$P_i = \frac{1}{\rho_s} \frac{k_i^2[\omega(4/3\eta + \zeta_2 - \zeta_1\rho_s) + i\rho_n u_1^2] - i\omega^2\rho_n}{i\omega^2 - k_i^2(\omega\zeta_1 + iu_1^2)},$$

$$D_i = \frac{k_i^2 \sigma}{(i\omega - k_i^2 \chi)\rho} \frac{k_i^2[\omega(4/3\eta + \zeta_2) + i\rho u_1^2] - i\omega^2\rho}{i\omega^2 - k_i^2(\omega\zeta_1 + iu_1^2)}, \quad (44)$$

where k_1^2 and k_3^2 (with account of the smallness of the dissipation terms) have the form

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[2] R and R_σ have different dimensions; therefore, they do not admit a direct comparison. We shall give the corresponding estimates below.

$$k_1^2 = k_{\perp 1}^2 + k_\parallel^2 = \frac{\omega^2}{u_1^2}\left[1 + i\frac{\omega}{u_1^2\rho}\left(\frac{4}{3}\eta + \zeta_2\right)\right],$$

$$k_2^2 = k_{\perp 2}^2 + k_\parallel^2$$

$$= \frac{\omega^2}{u_2^2}\left[1 + i\frac{\omega}{u_2^2}\left\{\frac{\rho_s}{\rho_n\rho}\left(\frac{4}{3}\eta - 2\zeta_1\rho + \zeta_2 + \zeta_3\rho^2\right) + \chi\right\}\right],$$

$$k_3^2 = k_{\perp 3}^2 + k_\parallel^2 = i\omega\rho_n/\eta. \quad (45)$$

The solution for Q_i and $\mathbf{u}$ which satisfies (43) and (41) can be written in the form

$$Q_i = e^{-i(\omega t - k_\parallel x)}\cos k_{\perp i}z, \qquad u_x = iAe^{-i(\omega t - k_\parallel x)}\cos k_{\perp 3}z. \quad (46)$$

Substituting (46) in (42) and taking (44) into account, we get

$$v_{nx} = ie^{-i(\omega t - k_\parallel x)}[L_1 k_\parallel \cos k_{\perp 1}z + L_2 k_\parallel \cos k_{\perp 2}z + A\cos k_{\perp 3}z],$$

$$v_{nz} = -e^{-i(\omega t - k_\parallel x)}\left[L_1 k_{\perp 1}\sin k_{\perp 1}z + L_2 k_{\perp 2}\sin k_{\perp 2}z - A\frac{k_\parallel}{k_{\perp 3}}\sin k_{\perp 3}z\right],$$

$$v_{sz} = -e^{-i(\omega t - k_\parallel x)}[L_1 P_1 k_{\perp 1}\sin k_{\perp 1}z + L_2 P_2 k_{\perp 2}\sin k_{\perp 2}z]. \quad (47)$$

In accord with (47) and (41), we obtain a set of homogeneous equations whose determinant can be set equal to zero:

$$\begin{vmatrix} k_\parallel \cos k_{\perp 1} d & k_\parallel \cos k_{\perp 2} d & \cos k_{\perp 3} d \\ k_{\perp 1}\sin k_{\perp 1} d & k_{\perp 2}\sin k_{\perp 2} d & -k_\parallel k_{\perp 3}^{-1}\sin k_{\perp 3} d \\ P_1 k_{\perp 1}\sin k_{\perp 1} d & P_2 k_{\perp 2}\sin k_{\perp 2} d & 0 \end{vmatrix} = 0. \quad (48)$$

According to condition (2) $|k_3|d \ll 1$ and since $|k_3| \gg |k_2| > |k_1|$, then $|k_2|d \ll 1$ and $|k_3|d \ll 1$. Then $\cos k_{\perp i}d$ and $\sin k_{\perp i}d$ (i = 1, 2, 3) in (38) can be expanded in series, and only terms of order $d^2k_{\perp i}^2$ retained. (Such an expansion is valid since the solution obtained for $k_{\parallel 1}^2$ is less than k_3^2 in absolute magnitude.)

Solving Eq. (8) with account of the smallness of the dissipation terms, we get, finally,

$$k_{\parallel 1}^2 = \frac{\omega^2}{u_{40}^2}\left[1 + id^2\frac{\omega\rho_n^2\rho_s}{3\eta\rho^2}\frac{(u_1^2 - u_2^2)^2}{u_{40}^4}\right], \quad (49)$$

$$k_{\parallel 2}^2 = i\frac{3\omega\eta u_{40}^2}{d^2u_1^2u_2^2\rho_n} + \frac{\omega^2}{u_2^2}. \quad (50)$$

The first solution is fourth sound. The second solution has the same structure as k_2^2. However, the presence in the first term of (50) of the factor $1/d^2$ leads to the result that the first component in (50) is larger than the second by several orders of magnitude, so that, in the case $|k_3|d \ll 1$ (the so-called "complete" damping of the normal part[6, 11]), the second sound is modified into very rapidly damped waves. The thermal waves (modified second sound in the partial damping of the nor-normal part[6, 11]) were observed at sufficiently large d. In this case, all the calculations given are inapplicable.

From Eq. (49), we find the absorption coefficient of fourth sound, brought about by the slippage of the normal component:

$$\frac{\Delta_{k_y}^2(t)}{k_m^2(t)} \approx \frac{1}{\ln A} \ll 1. \quad (51)$$

Comparing (51) with (12), we see that $\beta = 3$.

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