THE COULOMB FIELD AND THE NONRELATIVISTIC QUANTIZATION OF SPACE

R. M. MIR-KASIMOV

Joint Institute for Nuclear Research

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The possibility of describing the hydrogen atom in terms of a p space geometry with constant curvature is considered. The analogy with the space quantization of Snyder is pointed out. Various ways of introducing a metric in p-space by projection are investigated.

1. INTRODUCTION

IN the present note we present a geometric interpretation of the Coulomb symmetry based on the analogy with the space quantization of Snyder [1-3] and on the Fock theory of the hydrogen atom.[4]

In Snyder's model of space quantization one replaces the usual coordinate operators in Minkowski space $x_\mu = i\hbar\partial/\partial p_\mu$ ($\mu = 0, 1, 2, 3$), which are interpreted geometrically as shifts in momentum space, by new coordinate operators X_μ. A physical interpretation of the operators X_μ can be obtained by considering a momentum space with constant curvature. In the different field theories with quantized space various p space metrics with constant curvature have been used: an elliptic space [2] and de Sitter spaces of the first and second kind.[3] Independently of the specific choice of the metric, the operators X_μ are in each case infinitesimal rotators from the fifth axis to the μ-th axis in the five-dimensional (pseudo-) Euclidean momentum space of projective coordinates. When the radius of curvature tends to infinity these operators go over into infinitesimal translations of the four-dimensional Euclidean p-space, and they may therefore be called "shift" operators in p-space.

As the nonrelativistic analog it is natural to consider the three-dimensional p-space with constant curvature, excluding the time component. The space of projective coordinates is in this case four-dimensional.

On the other hand, the treatment of the higher symmetries of the Schrödinger equation describing the relative motion of particles interacting according to the Coulomb law (the free motion of the center of mass has been separated out) given by Fock [4] makes essential use of a momentum space with constant curvature. The energy levels and the wave functions are determined in terms of representations of the group of motions in a p-space with constant curvature.

2. THE GROUP OF FREE MOTION E_3 AND THE DYNAMICAL GROUP SO(4)

The time independent part of the free Schrödinger equation is invariant under the group of motions of the three-dimensional Euclidean space E_3, which is a semidirect product of the group SO(3) and the commutative group of three-dimensional translations. The wave functions of the free motion $\exp[ipx]$ are representations of this group. The basis of the Lie algebra of E_3 is formed by the generators of rotation M_{ij} ($i, j = 1, 2, 3$) and translation p_i, which satisfy the following commutation relations:

$$[p_i,\ p_j] = 0,\quad [p_i, M_{jk}] = \delta_{ij}p_k - \delta_{ik}p_j,$$
$$[M_{ij}, M_{kl}] = \delta_{il}M_{jk} + \delta_{jk}M_{il} - \delta_{ik}M_{jl} - \delta_{jl}M_{ik}. \qquad (2.1)$$

The invariants of the Euclidean group are $E_0 = p_ip_i$ and $V_0 = \epsilon_{ijk}M_{jk}p_i = M_ip_i$. The quantity V_0 is similar to the Pauli-Lyubanskiĭ-Bargmann vector, with the help of which the covariant spin can be introduced in the Poincaré group. The invariant V_0 can be interpreted as the projection of the angular momentum on the momentum, and E_0 as the energy.

The translational invariance leads to the law of conservation of momentum. Since $E = p^2/2\mu$ in the absence of interaction, the energy is in this case also conserved. The introduction of a field always destroys the translational invariance. The violation of the translational invariance is "governed" by the potential. In the case of the Coulomb potential the translational invariance is actually not destroyed but modified.

Indeed, the basis of the Lie algebra of SO(4) is formed by the six generators $M_{\alpha\beta}$ ($\alpha, \beta = 1, 2, 3, 4$), which satisfy the commutation relations

$$[M_{4i}, M_{4j}] = -M_{ij},\ [M_{4i}, M_{jk}] = \delta_{ij}M_{4k} - \delta_{ik}M_{4j},$$
$$[M_{ij}, M_{kl}] = \delta_{il}M_{jk} + \delta_{jk}M_{il} - \delta_{ik}M_{jl} - \delta_{jl}M_{ik}. \qquad (2.2)$$

In the next section we shall show that the system of commutators (2.2), which differs from (2.1) only in the first row, goes over into the latter when the Coulomb field is turned off ($e \to 0$).

3. COORDINATE OPERATORS IN DIFFERENT PROJECTIONS OF THE FOUR DIMENSIONAL SPHERE

Together with Fock's stereographic projection, let us consider two more projections: the geodesic projection

$$\xi_i = p_i/\sqrt{p_0^2 + p^2}, \quad \xi_4 = p_0/\sqrt{p_0^2 + p^2} \qquad (3.1)$$

and the orthogonal projection

$$\xi_i = p_i/p_0, \quad \xi_4 = \sqrt{p_0^2 - p^2}/p_0. \qquad (3.2)$$

In the geometric sense all projections are completely equivalent.

The infinitesimal "translations" in p-space with a constant curvature have the following form in the different projections:

$$M_{4i}^{(\text{ст})} = \frac{\delta_{ij}(p_0^2 - p^2) + 2p_i p_j}{p_0^2}\frac{\partial}{\partial p_j}, \qquad (3.3)$$

$$M_{4i}^{(\text{г})} = \frac{\delta_{ij}p_0^2 + p_i p_j}{p_0^2}\frac{\partial}{\partial p_j}, \qquad (3.4)$$

$$M_{4i}^{(\text{op})} = \sqrt{1 - \frac{p^2}{p_0^2}}\frac{\partial}{\partial p_j}. \qquad (3.5)$$

These operators are generalizations of the translations in the Euclidean p space $\partial/\partial p_i$, so that the quantities $i\hbar M_{4i}$ are generalizations of the coordinate operator for a free nonrelativistic particle in a Coulomb field.

The commutators between the operators $M_{4i}^{(\alpha)}$ and the generators of three-dimensional rotations

$$M_{ik} = p_i\frac{\partial}{\partial p_k} - p_k\frac{\partial}{\partial p_i} \qquad (3.6)$$

coincide with (2.2) up to a constant multiplier, which can easily be eliminated.

In going over to a flat p-space corresponding to the free motion, the operators (3.3), (3.4), and (3.5) go over into the usual generators of translations $\partial/\partial p_i$. Indeed, in this case $\xi_4 \approx -1$, therefore $\xi_i \approx 0$ (this follows from the equation of the sphere $\xi_4^2 + \xi_i^2 = 1$). This means that in all three projections $p_i/p_0 \approx 0$.

One can also write down integral equations of motion in the momentum representation in the geodesic and orthogonal projections.

4. CONCLUSIONS

1. The possibility of a quantum-mechanical description of the Coulomb interaction by introducing a metric by an arbitrary projection in a p-space of constant curvature allows one to assert that from the geometric point of view the inclusion of the Coulomb field leads to a change in the Euclidean momentum space corresponding to the free motion in a p-space with constant curvature.

2. The additional integrals of the motion in the Coulomb field, having the meaning of infinitesimal translations in p-space with constant curvature, must be interpreted as generalizations of the coordinate operator for the free particle $x_i = i\hbar\partial/\partial p_i$. This interpretation of the additional integrals implies that the Coulomb field can be regarded as the nonrelativistic analog of the space quantization of Snyder, Gol'fand, and Kadyshevskiĭ.[1-3]

3. The violation of translation invariance, which led to difficulties in the construction of a field theory in a p-space with constant curvature, is here interpreted very simply, since the introduction of a curvature in p-space here means the introduction of an interaction, and the latter always destroys the translational invariance.

4. In the papers using the Snyder quantization of space, the "change of the geometry at small distances" did not appear clearly as a property of the interaction and was in fact used only for cutting off the divergent integrals. As our analysis of the Coulomb symmetry shows, the quantization of space is an effect of the dynamics.

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