THEORY OF FINITE-AMPLITUDE SPIN WAVES

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Nonlinear oscillations of the magnetic-moment density in ferromagnets are investigated. Waves of stationary shape, propagated along the axis of easiest magnetization of the crystal, are considered. It is shown that such waves can be of two types: periodic waves and solitary waves. If the perturbation of the magnetic moment is small, then the periodic wave reduces to the well-known small-amplitude spin wave. The solitary wave is an essentially nonlinear development and cannot be obtained within the framework of linear theory. This wave is a region of perturbed magnetic moment moving uniformly along the crystal. In stationary waves of both types, the vector magnetic-moment density executes a rotation about the axis of easiest magnetization. Both the rotation frequency and the angle between the magnetic moment and the axis of easiest magnetization are functions of the coordinates and of time that are periodic (in the case of a periodic wave) or aperiodic and diminishing at infinity (in the case of a solitary wave). It is shown that a solitary wave, in contrast to a periodic wave, can propagate along the crystal with a speed not exceeding a certain critical value, determined by the properties of the crystal and the value of the external magnetic field.

1. INTRODUCTION

As is well known, perturbations of the magnetic-moment density in magnetically ordered crystals can, at low temperatures, be propagated along the crystal in the form of weakly attenuated spin waves. The structure of spin waves is well known for the cases of a small-amplitude wave: in such a wave, the vector magnetic-moment density rotates about its equilibrium value with a frequency ω_k that depends on the wave vector $\mathbf{k}$ of the spin wave.

It is of interest to study motions of the magnetic moment in ferromagnets when the amplitude of the oscillations of the magnetic moment is comparable with the equilibrium value of this quantity—nonlinear spin waves. The present paper investigates one class of such waves: nonlinear waves of stationary shape; that is, those motions in which the magnetic-moment density depends on the coordinates and on the time only in the combination $\mathbf{r} - \mathbf{V}t$, where $\mathbf{V}$ is the constant speed of the wave.

Waves of stationary shape can develop in media with spatial dispersion because of the dependence of the phase velocity of the oscillations not only on the amplitude but also on the sharpness of the wave front. The dependence of the velocity on the amplitude usually leads to an increase in the sharpness of the wave front, and the dependence of the velocity on the sharpness of the front slows down the increase of the sharpness, and as a result a wave of stationary shape develops. This mechanism of formation of waves of stationary shape, well known for waves on the surface of a liquid or for oscillations of a plasma (see, for example,[1]), is present also in the case of spin waves of large amplitude in magnetically ordered crystals.

In the present paper we show that waves of stationary shape, propagated in a ferromagnet along the direction of equilibrium orientation of the magnetic moment, can be of two types: periodic and solitary waves. If the initial perturbation of the magnetic moment is small, then the periodic wave reduces to the well studied small-amplitude spin wave. The solitary wave is an essentially nonlinear development and cannot be obtained within the framework of linear theory. This wave is a region of perturbation of the magnetic moment that moves uniformly along the crystal.

2. EQUATIONS DESCRIBING A WAVE OF STATIONARY SHAPE

In investigating the oscillations of the magnetic-moment density of a ferromagnet, we shall start from the equation of motion of the magnetic moment (see, for example,[2]),

$$\partial \mathbf{M}/\partial t = g[\mathbf{M}\mathbf{H}^e], \qquad (1)^*$$

where M is the magnetic moment of unit volume, $\mathbf{H}^e$ is the effective field

$$\mathbf{H}^e = \mathbf{H} + \alpha\Delta\mathbf{M} - \beta\{\mathbf{M} - \mathbf{n}(\mathbf{M}\mathbf{n})\}, \qquad (2)$$

H is the magnetic field, satisfying the equations of magnetostatics

$$\operatorname{div}(\mathbf{H} + 4\pi\mathbf{M}) = 0, \qquad \operatorname{rot}\mathbf{H} = 0, \qquad (3)$$

g is the gyromagnetic ratio, n is the unit vector in the direction of the axis of easiest magnetization, α is the exchange-interaction constant, and β is a constant that describes the anisotropy of the magnetic properties of the crystal.

We are interested in stationary waves propagated along the axis of easiest magnetization, and we set the external magnetic field perpendicular to this axis equal to zero. Then we can put the relations (1)—(3) into the form

$$-V\,\partial\mathbf{M}/\partial x = g[\mathbf{M}\mathbf{H}^e], \qquad (4)$$

$$\mathbf{H}^e = \alpha\Delta\mathbf{M} + B_0\mathbf{n} - \beta\mathbf{M} - (4\pi - \beta)\mathbf{n}(\mathbf{M}\mathbf{n}), \qquad (5)$$

where V is the velocity of propagation of the wave and $B_0 = (\mathbf{H} + 4\pi\mathbf{M})\mathbf{n}$ is a quantity independent of x and t (the x axis is chosen in the direction of the axis of easiest magnetization).

On multiplying Eq. (4) scalarly by M and by $\mathbf{H}^e$, we obtain two integrals of the motion:

$$\mathbf{M}^2 = M_0^2 = \text{const}, \qquad (6)$$

$$\alpha\{(\mathbf{M}')^2 - (M_0')^2\} + 2B_0\{\mathbf{M}\mathbf{n} - M_0\} - (4\pi - \beta)\{(\mathbf{M}\mathbf{n})^2 - M_0^2\} = 0.$$

Here M_0^2 is the value of the square of the magnetic-moment density, and M_0' is the value of the derivative $\mathbf{M}'$ at a point where the magnetic moment attains its equilibrium value, $\mathbf{M} = M_0\mathbf{n}$ (a prime on a quantity denotes differentiation with respect to x).

On multiplying (4) scalarly by n and introducing the notation

$$M_y + iM_z = M_\perp e^{i\varphi},$$

we get

$$\varphi' = V(g\alpha M_\perp^2)^{-1}(M_0 - \mathbf{M}\mathbf{n}). \qquad (7)$$

We remark that the equation obtained from (4) by vector multiplication by n is a consequence of the relations (6) and (7).

By taking into account that

$$(\mathbf{M}')^2 = (\mathbf{M}'\mathbf{n})^2 M_0^2/M_\perp^2 + M_\perp^2\varphi'^2, \qquad (8)$$

and using (7), we can put the relation (6) into the form

$$\frac{\alpha}{4\pi - \beta} z'^2 + u^2 z^2 = z(2 - z)\{m_0^2 + z(2p + z)\}, \qquad (9)$$

where

$$z = 1 - \frac{\mathbf{M}\mathbf{n}}{M_0}, \quad m_0^2 = \frac{\alpha}{4\pi - \beta} M_0^{-2}(M_0')^2,$$

$$u^2 = \alpha^{-1}(4\pi - \beta)^{-1}(gM_0)^{-2}V^2, \quad p = (4\pi - \beta)M_0^{-1}B_0 - 1. \qquad (10)$$

On integrating Eq. (9), we get

$$x - Vt = \left(\frac{\alpha}{4\pi - \beta}\right)^{1/2} \int dz\, \{z(2 - z)(m_0^2 + 2pz + z^2) - u^2 z^2\}^{-1/2}. \qquad (11)$$

The relation (11), which determines the dependence of the quantity $\mathbf{M}\cdot\mathbf{n}$ on time and the coordinates, together with the relation (7) fully describes the distribution of the magnetic-moment density of a ferromagnet in a wave of stationary shape.

3. SOLITARY SPIN WAVE

We first consider a stationary wave in which the perturbation of the magnetic moment vanishes asymptotically in front of the forward front of the wave and behind its back front—a so-called solitary wave. In such a wave, obviously, $\mathbf{M}' \to 0$ for $x \to \pm\infty$; therefore, on setting $m_0^2 = 0$ in equation (11) and performing the integration, we get

$$1 - \frac{\mathbf{M}\mathbf{n}}{M_0} = \left\{\xi_0 + \xi_1 \operatorname{ch}\frac{x - Vt}{x_1}\right\}^{-1}; \qquad (12)$$

$$\xi_0 = \frac{p - 1}{4p - u^2}, \quad \xi_1 = \frac{[(1 + p)^2 - u^2]^{1/2}}{4p - u^2},$$

$$x_1^2 = \alpha(4\pi - \beta)^{-1}(4p - u^2)^{-1}, \quad 4p - u^2 > 0. \qquad (13)$$

For $x \to \pm\infty$, obviously, the magnetic moment in a solitary wave is directed along the axis of easiest magnetization, $\mathbf{M} = M_0\mathbf{n}$. It is convenient to characterize the amplitude of a solitary wave by the maximum angle of deviation of the vector magnetic-moment density from its equilibrium orientation; that is, by the quantity

$$\theta_{max} = \arccos\left\{1 - \frac{4p - u^2}{p - 1 + \sqrt{(p + 1)^2 - u^2}}\right\}. \qquad (14)$$

It is easy to see that for $p \to \infty$ we have $\theta_{max} = \pi$. On decrease of the parameter p, the amplitude of the solitary wave decreases. Finally, for $p = u^2/4$ the quantity θ_{max} vanishes.

*$[\mathbf{M}\mathbf{H}^e] \equiv \mathbf{M} \times \mathbf{H}^e$.

For $p < u^2/4$ a solitary spin wave is impossible. Thus a solitary spin wave can propagate only with a speed that does not exceed a certain critical value, $V^2 < V_{cr}^2$, where

$$V_{cr} = 2gM_0\alpha^{1/2}\{B_0/M_0 - (4\pi - \beta)\}^{1/2}. \quad (15)$$

We recall that in other known cases the speed of a solitary wave is bounded below (for example, in the case of a two-temperature plasma the speed of a solitary wave must exceed the speed of sound).

To explain the reason for this peculiarity of a solitary spin wave, we note that for $x \to \pm\infty$ the deviation of the magnetic moment in this wave is small, and therefore the change of magnetic moment far ahead of (or behind) the crest of the wave is described by the linear theory. Then

$$1 - \mathbf{Mn}/M_0 = \mathrm{const}\cdot\exp\{-|(x - Vt)\,\mathrm{Im}\,k|\},$$

where k is the root of the equation $\omega_k = kV$ and ω_k is the frequency of a spin wave of small amplitude,

$$\omega_k = gM_0\{B_0/M_0 - (4\pi - \beta) + \alpha k^2\}. \quad (16)$$

It is easy to see that the condition for existence of a solitary wave, that is the inequality $V^2 < V_{cr}^2$, coincides with the condition for presence of an imaginary part in the root of the equation $\omega_k = kV$.

Turning to the study of the structure of a solitary spin wave, we note that if $\theta_{max} < \pi/2$, the component $M_\perp$ of the magnetic moment perpendicular to the axis of easiest magnetization increases from zero at $x = -\infty$ to a maximum value $M_0 \sin\theta_{max}$ at the crest of the wave and then again decreases to zero. If $\theta_{max} > \pi/2$, the value of $M_\perp$ has two maxima, symmetrically located with respect to the crest of the wave, at which $M_\perp = M_0$. In both cases the vector $M_\perp$ performs a rotation about the axis $\mathbf{n}$ with frequency

$$\omega(x - Vt) \equiv V\varphi'$$

$$= \frac{V^2}{2g\alpha M_0}\left\{1 + \left[2\xi_1 \,\mathrm{ch}\,\frac{x - Vt}{x_1} + 2\xi_0 - 1\right]^{-1}\right\}, \quad (17)$$

the frequency of the rotation grows from a value $\omega_{min} = V^2(2g\alpha M_0)^{-1}$ at $x \to \pm\infty$ to a value

$$\omega_{max} = V^2(g\alpha M_0)^{-1}(1 + \cos\theta_{max})^{-1} \quad (18)$$

at the crest of the wave.

We emphasize that the value of the vector magnetic-moment density is the same, both in magnitude and in direction, in front of the front of a solitary wave and behind the wave; in this respect a solitary wave is significantly different from a moving domain wall, separating regions with different orientations of the magnetic moment.

4. PERIODIC SPIN WAVE

We shall now discuss those oscillations of the magnetic moment of a ferromagnet in which the derivative of the magnetic moment and the deviation of the magnetic moment from the equilibrium value do not vanish simultaneously; that is, $\mathbf{M}_0' \neq 0$. In this case the relation (11) determines a periodic function $z(x - Vt)$, which runs through all values between $z = 0$ and $z = z_m$, where z_m is the smallest positive root of the equation

$$P(z) \equiv 2m_0^2 - z(u^2 - 4p + m_0^2) + 2z^2(1 - p) - z^3 = 0. \quad (19)$$

We see that when $\mathbf{M}_0' \neq 0$, there is propagated in the ferromagnet a periodic wave of stationary shape; the amplitude of this wave is

$$\theta_{max} = \arccos(1 - z_m).$$

We note that a periodic wave is possible for any values of the parameters characterizing the ferromagnet and for an arbitrary value of the velocity V, if only $\mathbf{M}_0' \neq 0$. In fact, for existence of such a wave it is sufficient that the inequality $0 < z_m < 2$ be satisfied; it is easy to demonstrate this inequality by taking into account that $P(0) = 2m_0^2 > 0$, whereas $P(2) = -2u_0^2 < 0$.

Without going into a detailed investigation of the periodic wave of finite amplitude, we consider by way of example the structure of a periodic wave propagating with very high velocity,

$$V^2 \gg (gM_0)^2\alpha\cdot\max\{4\pi - \beta;\ B_0/M_0 - (4\pi - \beta)\}.$$

In this case

$$1 - \frac{\mathbf{Mn}}{M_0} = \frac{2(M_0')^2}{(M_0')^2 + (\alpha g)^{-2}V^2}\sin^2\frac{x - Vt}{x_2}, \quad (20)$$
$$x_2 = M_0\{(M_0')^2 + (\alpha g)^{-2}V^2\}^{-1/2}.$$

The wave length is obviously πx_2; the amplitude of the wave has the form

$$\theta_{max} = \arccos\frac{(\alpha g)^{-2}V^2 - (M_0')^2}{(\alpha g)^{-2}V^2 + (M_0')^2}. \quad (21)$$

The transverse component of the magnetic moment rotates about the direction of easiest magnetization with frequency

$$\omega(x - Vt) = \frac{V^2}{2\alpha g M_0}\left\{1 + \frac{(M_0')^2\sin^2\dfrac{x - Vt}{x_2}}{(M_0')^2\cos^2\dfrac{x - Vt}{x_2} + \dfrac{V^2}{(\alpha g)^2}}\right\}. \quad (22)$$

If the quantity $u^2 + m_0^2 - 4p$ is positive and if the condition $u^2 + m_0^2 - 4p \ll m_0^2$ is satisfied, then the amplitude of the periodic wave is small:

$$z_m = 2m_0^2/(u^2 - 4p + m_0^2). \quad (23)$$

In this case the relation (11) determines the usual harmonic spin wave, in which the transverse component of the magnetic moment, equal in absolute value to

$$M_\perp = M_0\sqrt{2z_m}\sin\frac{x - Vt}{x_3}, \quad x_3^2 = 2z_m(M_0')^{-2}M_0^2, \quad (24)$$

rotates about the axis of easiest magnetization with a constant frequency $\omega = V^2(2\alpha g M_0)^{-1}$. This wave can be interpreted as a superposition of two spin waves with wave vectors k_1 and k_2, the two roots of the equation $\omega_k = kV$.

Finally, we shall turn to the case of those oscillations of the vector magnetic moment in which the component of this vector perpendicular to the axis of easiest magnetization never vanishes. For this we note that the expression in curly brackets in the relation (11), a polynomial of fourth degree in the variable $z = 1 - \mathbf{M}\cdot\mathbf{n}/M_0$, can have, along with the roots $z = 0$ and $z = z_m$, two additional positive roots $z_{1,2}$, enclosed in the interval $(z_m, 2)$. In this case the interval (z_1, z_2), as well as the interval $(0, z_m)$, is a region of permissible values of the variable z. Physically this corresponds to oscillations of the magnetic moment for which

$$z_1 < 1 - \mathbf{Mn}/M_0 < z_2.$$

By way of example, we consider a spin wave that occurs when $M' = 0$ and $V^2 > V_{cr}^2$. In this case $z_m = 0$,

$$z_{1,2} = p - 1 \mp \sqrt{(p+1)^2 - u^2}. \quad (25)$$

On setting $m_0 = 0$ and $4p - u^2 < 0$ in the relation (11), we get

$$1 - \frac{\mathbf{Mn}}{M_0} = 2z_1z_2\left\{(z_1 + z_2) + (z_2 - z_1)\cos\frac{x - Vt}{|x_1|}\right\}^{-1}, \quad (26)$$

where the quantity x_1 is determined by formula (13). In this wave, the transverse component of the magnetic moment performs a rotation about the direction of easiest magnetization with frequency

$$\omega(x - Vt) = \frac{V^2}{2\alpha g M_0}\left\{1 + \frac{z_1z_2}{(z_1 + z_2) + (z_2 - z_1)\cos\frac{x - Vt}{|x_1|} - z_1z_2}\right\} \quad (27)$$

We note that if $z_1 = z_2$, then in a spin wave of finite amplitude (similarly to what takes place in the case of a spin wave of small amplitude with a definite value of the wave vector), the transverse component of the magnetic-moment density is constant in value and performs a rotation with constant frequency about the axis of easiest magnetization.

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