

CONTRIBUTION TO THE THEORY OF SELF FOCUSING OF LIGHT IN NONLINEARLY  
POLARIZING MEDIA

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A successive-approximation method is used to consider a self-focusing beam, with cross section of arbitrary form and with transverse dimensions considerably exceeding the wavelength of light. In the zeroth approximation, the method yields for a cylindrical beam results that are already known. In the higher approximations, a longitudinal electric field appears. A new type of self-focusing cylindrical beam is proposed, in which the electric field intensity is proportional to the radius and to the beam axis. An exact solution of Maxwell's equations is presented for such a beam.

THE first to point out the possibility of self focusing (self trapping) of a beam was Askar'yan<sup>[1]</sup>. A quantitative theory of self focusing was developed by Talanov<sup>[2]</sup> for a plasma medium and by Chiao, Garmire, and Townes (CGT) for solids<sup>[3]</sup>. Apparently self focusing was first observed experimentally by Hercher<sup>[4]</sup> and in liquids by Pilipetskiĭ and Rustamov<sup>[5]</sup> and by Lallemand and Blombergen<sup>[6]</sup>. Many papers on this subject, which has attracted many physicists, have been subsequently published.

In this paper we refine the CGT analysis of a cylindrical beam, and propose also an exact solution for a cylindrical beam having a polarization of a different type.

CGT solved for a light beam a wave equation of the type

$$\Delta \mathbf{E} - \frac{\epsilon}{c^2} \frac{\partial^2 \mathbf{E}}{\partial t^2} = 0, \quad \epsilon = \epsilon_0 + \overline{\epsilon_2 \mathbf{E}^2}(t), \quad (1)$$

where  $\epsilon_0$  is the dielectric constant of an isotropic medium in the case of weak electric fields, and the term  $\overline{\epsilon_2 \mathbf{E}^2}$  represents a statistical inertial (incapable of following the frequency of the light) change of the dielectric constant, connected, for example, with the Kerr effect or with electrostriction. Here  $\epsilon_2$  is a constant and  $\overline{\mathbf{E}^2}$  is the time-averaged value of the square of the electric field of the light wave. An exact solution of (1) was obtained in<sup>[3]</sup> in the form of a linearly polarized plane light beam, and a clever mechanical analogy was presented, with the aid of which it is convenient to interpret the solution of the equation. In addition, they presented an exact solution of (1) in the form of a cylindrical circularly polarized beam, with a transverse electric field in the form

$$\begin{aligned} E_x &= E_0(r) \cos(kz - \omega t), & E_y &= E_0(r) \sin(kz - \omega t), \\ E_z &= 0, & r &= \sqrt{x^2 + y^2}. \end{aligned} \quad (2)$$

For such a solution

$$\operatorname{div} \mathbf{D} = \operatorname{div} \epsilon \mathbf{E} = [\epsilon_0 + 3\epsilon_2 E_0^2(r)] \frac{\partial E_0}{\partial r} \cos(kz - \omega t - \varphi). \quad (3)$$

Here  $\varphi$  is the angle in the cylindrical coordinate system.

Thus, the cylindrical solution obtained by CGT does not satisfy Maxwell's equation  $\operatorname{div} \mathbf{D} = 0$ . The reason for the contradiction lies in the fact that the term  $-\nabla(\nabla \mathbf{E})$  was left out from the left side of the wave equation (1). In the case of plane beam this term is equal to zero and therefore the solution obtained by CGT is correct.

It will be shown below that if the transverse dimensions of the beam are greatly larger than the wavelength of the light, it is possible to analyze the electromagnetic wave by an approximate method<sup>[7]</sup>, which yields the CGT result in the zeroth approximation for a cylindrical beam, and corrections to this result in the higher approximations. In particular in the first approximation, a nonzero axial electric field appears.

Let the light wave propagate along the Oz axis. It is assumed that its amplitude is a slow function of x and y. The latter can be taken into account by introducing into the expression for the electric field  $\mathbf{E}$ , as a function of x and y, a small parameter  $\eta$  as a factor of x and y, namely  $\mathbf{E} = \mathbf{E}(\eta x, \eta y) \exp(ikz)$ . When applied to such a function, the operators  $\partial/\partial x$  and  $\partial/\partial y$  are small quantities of the order of  $\eta$ , and  $\partial/\partial z = ik$  is of zeroth order of smallness.

If we introduce a two-dimensional operator  $\tilde{\nabla} = i\partial/\partial x + j\partial/\partial y$  and denote the projection of  $\mathbf{E}$  on the  $xy$  plane by  $\mathbf{E}$ , then the wave equation  $\nabla(\nabla\mathbf{E}) - \Delta\mathbf{E} = \omega^2 c^{-2}\mathbf{E}$  can be written in terms of projections as follows

$$\begin{aligned} \tilde{\nabla}(\tilde{\nabla}\tilde{\mathbf{E}} + ikE_z) - (\tilde{\nabla}^2 - k^2)\tilde{\mathbf{E}} &= \frac{\omega^2}{c^2}(\epsilon_0 + \epsilon_2\overline{\mathbf{E}^2}(t))\tilde{\mathbf{E}}, \\ ik\tilde{\nabla}\tilde{\mathbf{E}} - \tilde{\nabla}^2 E_z &= \frac{\omega^2}{c^2}(\epsilon_0 + \epsilon_2\overline{\mathbf{E}^2}(t))E_z. \end{aligned} \quad (4)$$

When  $\epsilon_2 \rightarrow 0$  the field ceases to depend on  $x$  and  $y$ , i.e.  $\eta \rightarrow 0$ . Consequently, the smallness of  $\eta$  is connected with the smallness of  $\epsilon_2$ . It is physically obvious that when  $\epsilon_2$  is small the difference between the refractive index inside and outside the "waveguide" is small. With this, the total internal reflection of the wave by the "waveguide" surface is possible only at angles of incidence close to  $\pi/2$ , i.e., at small diffraction-divergence angle, as is the case in a broad beam.

The operators  $\tilde{\nabla}$  in (4) will be assumed small quantities of first order with respect to  $\eta$ . The quantity  $\epsilon_2$  is also small (it will be shown below that it is of the order  $\eta^2$ ). Putting  $\mathbf{E} = \mathbf{E}^0 + \eta\mathbf{E}' + \eta^2\mathbf{E}'' + \dots$ ,  $k = k^0 + \eta k' + \eta^2 k'' + \dots$  and gathering in (4) terms of zeroth order in  $\eta$ , we obtain

$$E_z^0 = 0, \quad k^{02} = \frac{\omega^2}{c^2}[\epsilon_0 + \epsilon_2\overline{\mathbf{E}^2}(t)]. \quad (5)$$

From this follow, alternatively, solutions of two types:

1)  $\tilde{\mathbf{E}}^0(\eta x, \eta y) = \text{const.}$  This is an exact trivial solution in the form of a beam of infinite cross section, when  $\eta$  does not depend on the coordinates. It is not corrected further in the higher approximations and will not be discussed below.

2)  $E_z^0 = 0$ ,  $k^{02} = \omega^2 c^{-2}\epsilon_0$ ,  $\mathbf{E}^0$  is an arbitrary function of the coordinates and  $\epsilon_2$  is of order higher than zeroth with respect to  $\eta$ .

Gathering in (4) the terms of first order in  $\eta$ , we get

$$k' = 0, \quad E_z' = \frac{i}{k^0}\tilde{\nabla}\tilde{\mathbf{E}}^0, \quad (6)$$

$\epsilon_2$  is of order higher than first with respect to  $\eta$ .

Allowance for terms of second order with respect to  $\eta$  in (4) yields

$$\begin{aligned} [\tilde{\Delta} - \Gamma^2 + \frac{\omega^2}{c^2}\epsilon_2\overline{\mathbf{E}^2}(t)]\tilde{\mathbf{E}}^0 &= 0, \\ \Gamma^2 \equiv 2k^0 k'' &, \quad E_z'' = \frac{i}{k^0}\tilde{\nabla}\tilde{\mathbf{E}}', \end{aligned} \quad (7)$$

$\epsilon_2$  is of second order in  $\eta$ .

Equation (7) determines the coordinate dependence of  $\mathbf{E}^0$ . Then (6) determines the dependence of  $\mathbf{E}'_z$  on  $x$  and  $y$ .

In the case of a cylindrical beam, Eq. (7) coincides

with the equation considered by CGT. Thus, in [3] they obtained an electric field which is the zeroth approximation of our method.

Owing to the nonlinearity of Eq. (7), we cannot assign an arbitrary amplitude to the field  $\tilde{\mathbf{E}}^0$ ; to each value of  $k''$  or  $\Gamma^2$  there corresponds its own set of solutions with strictly defined amplitudes. For each solution there exists a unique connection between  $k''$  and the light intensity integrated over the beam cross section.

Let us agree to express the real electric field  $\mathbf{E}(t)$  in terms of the complex amplitude  $\mathbf{E}(\eta x, \eta y)$  by means of the formula

$$\mathbf{E}(t) = 1/2[\mathbf{E}e^{i(kz - \omega t)} + \mathbf{E}^*e^{-i(kz - \omega t)}].$$

Then  $\overline{\mathbf{E}^2}(t) = (1/2)(\mathbf{E} \cdot \mathbf{E}^*)$ .

Gathering in (4) third-order terms, we get

$$\begin{aligned} \tilde{\nabla}^2\tilde{\mathbf{E}}' - 2k^0 k''\tilde{\mathbf{E}}' - 2k^0 k''\tilde{\mathbf{E}}^0 + \frac{\omega^2}{c^2}\epsilon_2\{1/2(\tilde{\mathbf{E}}^0, \tilde{\mathbf{E}}^{0*})\tilde{\mathbf{E}}' \\ + [(\tilde{\mathbf{E}}', \tilde{\mathbf{E}}^{0*}) + (\tilde{\mathbf{E}}'^*, \tilde{\mathbf{E}}^0)]\tilde{\mathbf{E}}^0\} &= 0, \\ ik^0\tilde{\nabla}\tilde{\mathbf{E}}'' + ik''\tilde{\nabla}\tilde{\mathbf{E}}^0 - \tilde{\nabla}^2 E_z' - \frac{\omega^2}{c^2}\epsilon_2 E_z'' \\ - \frac{\omega^2}{2c^2}\epsilon_2(\tilde{\mathbf{E}}^0, \tilde{\mathbf{E}}^{0*})E_z &= 0. \end{aligned} \quad (8)$$

The second of these equations, just as the second equation of (7) determines the next higher correction to  $E_z$  without integration, by merely using the differentiation operation. A similar situation holds for all higher approximations.

We note that formula (4) for  $\epsilon$  can be regarded as an expansion of  $\epsilon$  in powers of the electric field in a nonlinearly polarizing medium. Neglect of the higher powers of the field in this expansion is permissible if the expansion converges rapidly. This usually means that  $\epsilon_2\overline{\mathbf{E}^2}(t) \ll \epsilon_0$ , i.e., that  $\epsilon_2$  is a small parameter in the wave equation, that  $\eta$  is small, and, as shown above, the beam is broad. Thus, it may turn out that the region in which this method is applicable for broad beams coincides with the region in which the quadratic expansion (1) of  $\epsilon$  is valid.

Let us consider now a cylindrical beam of a different type, in which the electric-field force lines form concentric circles about the  $z$  axis. In other words, the electric field is

$$\mathbf{E}(r, z, t) = eE(r)\cos(kz - \omega t). \quad (9)$$

Here  $\mathbf{e}$  is a unit vector perpendicular to  $\mathbf{r}$  and  $z$ .

Substituting (9) in the wave equation and introducing the dimensionless radius  $\rho$  and the dimensionless scalar field amplitude  $\mathcal{E}(\rho)$ , using the formulas

$$\rho = \gamma r, \quad \gamma^2 = k^2 - k_0^2, \quad \mathcal{E} = \frac{\omega}{\gamma c} \sqrt{\frac{\epsilon_2}{2}} E_s \quad (10)$$

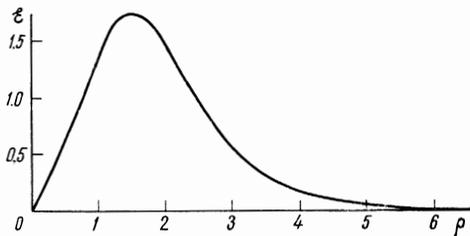
we obtain for the determination of  $\mathcal{E}(\rho)$  the equation

$$\frac{d^2 \mathcal{E}}{d\rho^2} + \frac{1}{\rho} \frac{d\mathcal{E}}{d\rho} - \frac{\mathcal{E}}{\rho^2} - \mathcal{E} + \mathcal{E}^3 = 0. \quad (11)$$

From the condition for the continuity of the electric field it follows that when  $\rho = 0$  we have  $\mathcal{E} = 0$ . As  $\rho \rightarrow \infty$ , no limitations are imposed on  $\mathcal{E}$ . As  $\rho \rightarrow \infty$  Eq. (11) goes over into

$$\frac{d^2 \mathcal{E}}{d\rho^2} + \frac{1}{\rho} \frac{d\mathcal{E}}{d\rho} - \mathcal{E} + \mathcal{E}^3 = 0, \quad (12)$$

which coincides with the equation considered in<sup>[3]</sup> for a cylindrical beam. Equation (12) itself shows three possible asymptotic behaviors of  $\mathcal{E}$  as  $\rho \rightarrow \infty$ <sup>[8]</sup>, namely  $\mathcal{E} \rightarrow 0$ ,  $\mathcal{E} \rightarrow +1$ , and  $\mathcal{E} \rightarrow -1$ . We obtain for (11) a solution of fixed sign under the conditions  $\mathcal{E}(0) = \mathcal{E}(\infty) = 0$ , using the M-20 computer. The results are shown in the figure.



The maximum of the curve is located at  $\rho = 1.5$ . The maximum value of  $\mathcal{E}$  is 1.747.

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