THE JOSEPHSON EFFECT IN HELIUM II

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The phenomenological theory of superfluidity is used for the investigation of the theory of superfluid helium in narrow gaps with dimensions of the order of or smaller than critical.

1. Recently, we proposed [1] to use the averaged wave function of the phenomenological theory of superfluidity [2] for the description of the behavior of liquid helium in narrow pores, and showed the possibility of the existence of an analog to the dc Josephson effect in helium II. [3,4] The derivation of the equation for the wave function, averaged transversely to the channel, [4] was obtained by us without complete rigor. The present research contains a generalization of the method used in [1,3,4], and also a consideration of the case of gaps with dimensions less than critical, which was not considered in these papers.

2. Let a plane gap of width δ (along the y axis) have an infinite depth along the z axis and let its length along the x axis be generally bounded. Then the equilibrium equation of the pehnomenological theory of superfluidity [2] has the form

$$\frac{\partial^2 f}{\partial x^2}+\frac{\partial^2 f}{\partial y^2}+f-v^2 f-f^3=0; \qquad (1)$$

here we have used a dimensionless description, in which the constants are measured in units of $a_0 = 4.3\times10^{-8}(T_\lambda - T)^{-1/2}$ [cm], the velocity v in units of $\hbar/ma_0$ (m is the mass of the helium atom), and the modulus of the wave function f in units of its own equilibrium value for an unbounded motionless liquid (see [1-4]). It is understood that the narrowness of the gap prevents the motion of the normal component ($v_n = 0$, $v \equiv v_s$).

If the width of the gap differs but little from the critical dimension, [2] and if its length is sufficiently great that one can neglect edge effects, then

$$|1-\pi^2/\delta^2|\ll 1, \quad v^2\ll 1, \quad f^2\ll 1, \quad \partial^2 f/\partial x^2\ll 1$$

and the approximate solution of Eq. (1) can be sought in the form

$$f(x,y)=F(x)\cos(\pi y/\delta)+f_1(x,y)+\ldots, \qquad (2)$$

where F(x) and $f_1(x,y)$ are slowly varying functions of x.

The presence of the factor $\cos(\pi y/\delta)$ in the first term of the expansion (2) guarantees the satisfaction of Eq. (1) in the zeroth approximation:

$$\frac{\partial^2 f_0}{\partial y^2}+\frac{\pi^2}{\delta^2}f_0=0$$

and the fulfillment of the boundary condition $f(x, \pm\delta/2) = 0$.

Proceeding to the first approximation, we obtain

$$\frac{\partial^2 f_1}{\partial y^2}+f_1=-\left[\frac{d^2F}{dx^2}+\left(1-\frac{\pi^2}{\delta^2}\right)F-v^2F-\frac{3}{4}F^3\right]\cos\frac{\pi y}{\delta}+\frac{1}{4}F^3\cos\frac{3\pi y}{\delta}.$$

Equating the resonance term to zero (see [5]), we obtain the equation

$$\frac{d^2F}{dx^2}+\left(1-\frac{\pi^2}{\delta^2}\right)F-v^2F-\frac{3}{4}F^3=0. \qquad (3)$$

We note that in the case of an unbounded (in length) gap ($F'' = 0$) and in the absence of flow ($v = 0$), Eq. (3) has a non-vanishing solution only for $\delta \ge \delta_c = \pi$ when $F^2 = \frac{4}{3}(1-\pi^2/\delta^2)$ (see [2]).

We introduce the flux density into consideration:

$$j=f^2v=F^2(x)v(x)\cos^2(\pi y/\delta).$$

The quantity $j_0 \equiv F^2v$ should not change along the gap and it is convenient to put it in Eq. (3) in place of the velocity v(x):

$$\frac{d^2F}{dx^2}+\left(1-\frac{\pi^2}{\delta^2}\right)F-\frac{j_0^2}{F^3}-\frac{3}{4}F^3=0. \qquad (3a)$$

In averaging over the cross section of the gap, we have $(\overline{f^2})^{1/2} = F/\sqrt{2}$ and $\bar{j} = j_0/2$. Therefore, the average equation (which describes, as in [1,3,4], only the change of the wave function along the gap) has the form

$$\frac{d^2}{dx^2}(\overline{f^2})^{1/2}+\left(1-\frac{\pi^2}{\delta^2}\right)(\overline{f^2})^{1/2}-\frac{\bar{j}^2}{(\overline{f^2})^{3/2}}-\frac{3}{2}(\overline{f^2})^{3/2}=0. \qquad (4)$$

Equation (4) coincides with Eq. (1) of [3] and with Eq. (19) of [4], in which one should set $b = 1$, $a^2 = 1 - \pi^2/\delta^2$ ($\delta \geq \pi$) and $c = 3/2$ (for a plane gap). It is also easy to see that if we were to consider not a plane gap but a cylindrical capillary, then the cosine in Eq. (2) would be replaced by the Bessel function J_0 and the coefficient in the last term of Eq. (4) would be a constant $c \approx 2.1$ (see [4]).

3. Inasmuch as Eqs. (4) and (3a) differ only by a numerical factor in the last term, all the solutions of Eq. (4) found by us in [3,4] apply directly to the case of a single capillary. In particular, the confirmation of the possibility of the existence of a superfluid current through the capillary with $\delta = \delta_c$, uniting two large volumes of helium II, is valid (the analog of the dc Josephson current through a normal metal).

We now consider the case $\delta < \delta_c$. Equation (3a) in this case assumes the penetration of superfluidity in the narrow gap adjoining the large volume of helium II. For example, for a semi-infinite gap, extending to infinity in the positive x direction, Eq. (3a) has the solution (for $j_0 = 0$)

$$F = \sqrt{\frac{8}{3}}\left[\lambda \sinh^{-1}\left(\frac{x}{\lambda} + \sinh\frac{\sqrt{8/3}}{\lambda F_0}\right)\right]^{-1}, \qquad (5)$$

where $\lambda^2 \equiv (\pi^2/\delta^2 - 1)^{-1}$, and F_0 is the value of the function F at some point $x = 0$. Neglecting edge effects, we can let this point coincide with the beginning of the gap and set $F_0 = 1$. Equation (5) shows that, far from the origin of the gap ($x \gg \lambda$), the density of the superfluid component (which is proportional to F^2) falls off according to the exponential law $e^{-2x/\lambda}$.

For a gap of finite length ($-d \leq x \leq d$), in which there exists a superfluid flow of liquid from the half-space $x < -d$ to the half-space $x > d$, we can also obtain a solution of Eq. (3a), which we shall not write out. We only note that the critical current falls off with increase in λ^{-1} (for decrease in δ). For small values of λ^{-1} and large d, this decrease is described by Eq. (35) of [4], where a^2 should be replaced by $-\lambda^{-2}$, c by $3/4$, and b set equal to unity.

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