NONLINEAR LONGITUDINAL WAVES IN ELECTRON BEAMS

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Exact solutions are obtained for the description of stationary nonlinear longitudinal waves in nonrelativistic electron beams with external magnetic field.

THE linear theory of wave processes in electron beams is fairly well represented in the physics literature ([1-3] and elsewhere). The main trend in the nonlinear theory is to devise models that are suitable for electronic computer calculations. There is, however, a number of important papers ([4-7] and others) containing a sufficiently detailed study of stationary nonlinear longitudinal waves in electron-ion beams without a magnetic field. Allowance for the magnetic field greatly complicates the problem.

We present below a nonlinear theory of stationary longitudinal waves in nonrelativistic electron beams with an external magnetic field. The exact results obtained are simple and easily visualized. They offer evidence that a pure electron beam can have in the presence of a magnetic field states in which nonlinear longitudinal waves oriented along the magnetic field behave in the same manner as in an electron beam against a stationary ion background in the absence of a magnetic field.

In a coordinate system moving with velocity equal to the phase velocity of the wave, a stationary wave in a nonrelativistic electron beam is represented by a stationary flow, described by the following system of equations

$$\Delta\varphi = 4\pi\rho, \tag{1}$$

$$\operatorname{div} \rho\mathbf{v} = 0, \tag{2}$$

$$(\mathbf{v}\nabla)\mathbf{v} = \eta\nabla\varphi - [\mathbf{v}\boldsymbol{\omega}_H], \qquad (3)^*$$

where φ is the potential of the electric field, ρ the absolute magnitude of the space-charge density, $\mathbf{v}$ the velocity of the electron beam, η the absolute magnitude of the electron specific charge, $\omega_H = \eta c^{-1}\mathbf{H}$, where $\mathbf{H}$ is the intensity of the external magnetic field, and c is the speed of light.

We shall assume that $\partial/\partial x = 0$, and the vector $\mathbf{H}$ is directed along the z axis. Then the condition that the vectors $\rho\mathbf{v}$ and $\boldsymbol{\Omega} = \operatorname{curl}\mathbf{v} - \boldsymbol{\omega}_H$ are solenoidal enables us to introduce the function $\psi = \psi(y, z)$ and $\psi_0 = \psi_0(y, z)$, such that

$$\frac{\partial\psi}{\partial z} = \Omega_y, \quad -\frac{\partial\psi}{\partial y} = \Omega_z; \quad \frac{\partial\psi_0}{\partial z} = \rho v_y, \quad -\frac{\partial\psi_0}{\partial y} = \rho v_z. \tag{4}$$

Since

$$\Omega_y = \frac{\partial}{\partial z}v_x, \quad \Omega_z = -\frac{\partial}{\partial y}v_x - \omega_H, \tag{5}$$

we can assume that

$$\psi = v_x + \omega_H y. \tag{6}$$

Thus,

$$v_x = \psi - \omega_H y; \quad v_y = \frac{1}{\rho}\frac{\partial\psi_0}{\partial z}, \quad v_z = -\frac{1}{\rho}\frac{\partial\psi_0}{\partial y}. \tag{7}$$

The vector lines $\boldsymbol{\Omega}$ and $\rho\mathbf{v}$ lie respectively on the surfaces $\psi(y, z) = \text{const}$ and $\psi_0(y, z) = \text{const}$, i.e., the functions ψ and ψ_0 are the current functions for the electronic fluid.

Taking the scalar products of (3) with $\mathbf{v}$ and $\boldsymbol{\Omega}$, respectively, after first writing (3) in the form

$$\operatorname{grad}\left(\frac{v^2}{2} - \eta\varphi\right) = [\mathbf{v}(\operatorname{rot}\mathbf{v} - \boldsymbol{\omega}_H)], \tag{8}$$

we get

$$(\mathbf{v}\nabla\Phi) = 0, \quad (\boldsymbol{\Omega}\nabla\Phi) = 0, \tag{9}$$

where $\Phi = v^2/2 - \eta\varphi$. The conditions (9) mean that

$$\frac{1}{\rho}\frac{\partial(\psi_0, \Phi)}{\partial(z, y)} = 0, \quad \frac{\partial(\psi, \Phi)}{\partial(z, y)} = 0, \tag{10}$$

i.e.,

$$\psi = \psi(\Phi), \quad \psi_0 = \psi_0(\Phi). \tag{11}$$

We shall assume that $\psi = \psi(\xi)$, $\psi_0 = \psi_0(\xi)$, and $\Phi = \Phi(\xi)$, where $\xi = \xi(y, z)$. The normalization of ξ is arbitrary.

Thus, we can consider the following system as the initial system of equations:

$$\frac{\partial^2\varphi}{\partial y^2} + \frac{\partial^2\varphi}{\partial z^2} = 4\pi\rho,$$

*$[\mathbf{v}\omega_H] \equiv \mathbf{v} \times \omega_H$.

$$\eta\frac{\partial\varphi}{\partial y}=\omega_H(\omega_H y-\psi)+\frac{1}{\rho}\frac{d\psi_0}{d\xi}\left[\frac{\partial\xi}{\partial z}\frac{\partial}{\partial y}\left(\frac{1}{\rho}\frac{d\psi_0}{d\xi}\frac{\partial\xi}{\partial z}\right)-\frac{\partial\xi}{\partial y}\frac{\partial}{\partial z}\left(\frac{1}{\rho}\frac{d\psi_0}{d\xi}\frac{d\xi}{\partial z}\right)\right], \quad (12)$$

$$2\eta\varphi=(\psi-\omega_H y)^2-2\Phi+\left(\frac{1}{\rho}\frac{d\psi_0}{d\xi}\right)^2\left[\left(\frac{\partial\xi}{\partial y}\right)^2+\left(\frac{\partial\xi}{\partial z}\right)^2\right].$$

The independent functions are $\rho(y, z)$, $\varphi(y, z)$, and $\xi(y, z)$. The form of the functions $\psi(\xi)$, $\psi_0(\xi)$, and $\Phi(\xi)$ is governed by the state of the unperturbed beam.

We represent the quantities ρ, φ, and ξ in the form $\rho=\bar\rho+\tilde\rho$, $\varphi=\bar\varphi+\tilde\varphi$, and $\xi=\bar\xi+\tilde\xi$, where the bars denote quantities pertaining to the unperturbed beam and the tildes denote the perturbations. We shall assume that the unperturbed state of the electron beam is such that

$$\frac{\partial}{\partial z}\bar\xi(y, z)=0.$$

This enables us to normalize ξ in such a way that $\bar\xi=y$. Such a state of the electron beam can be realized, for example, under the following conditions:

$$\bar\rho=\text{const},\quad \bar v_z=\text{const},\quad \bar v_y=0,\quad \bar v_x=-\frac{4\pi\bar\rho\eta}{\omega_H}y.$$

Subtracting from (12) the corresponding equations for the unperturbed beam, we obtain

$$\frac{\partial^2\tilde\varphi}{\partial y^2}+\frac{\partial^2\tilde\varphi}{\partial z^2}=4\pi\bar\rho\left(\frac{\rho}{\bar\rho}-1\right),$$

$$\eta\frac{\partial}{\partial y}\tilde\varphi=\omega_H(\bar\psi-\psi)+\frac{1}{\rho}\frac{d\psi_0}{d\xi}\left[\frac{\partial\tilde\xi}{\partial z}\frac{\partial}{\partial y}\left(\frac{1}{\rho}\frac{d\psi_0}{d\xi}\frac{\partial\tilde\xi}{\partial z}\right)-\left(1+\frac{\partial\tilde\xi}{\partial y}\right)\frac{\partial}{\partial z}\left(\frac{1}{\rho}\frac{d\psi_0}{d\xi}\frac{\partial\tilde\xi}{\partial z}\right)\right],$$

$$2\eta\tilde\varphi=2(\bar\Phi-\Phi)+(\psi-\bar\psi)(\psi+\bar\psi-2\omega_H y)+\left(\frac{1}{\rho}\frac{d\psi_0}{d\xi}\right)^2\left[\left(1+\frac{\partial\tilde\xi}{\partial y}\right)^2+\left(\frac{\partial\tilde\xi}{\partial z}\right)^2\right]-\left(\frac{1}{\bar\rho}\frac{d\bar\psi_0}{d\xi}\right)^2. \quad (13)$$

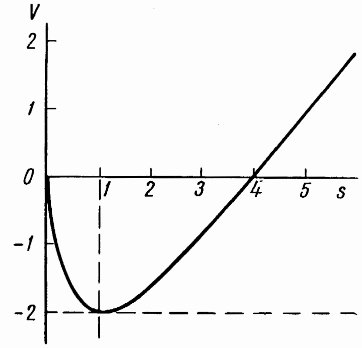
We investigate longitudinal waves, with $\tilde\xi=0$. Since ψ, ψ_0, and Φ depend on ξ in the same manner as $\bar\psi$, $\bar\psi_0$, and $\bar\Phi$ depend on $\bar\xi$, we get when $\tilde\xi=0$:

$$\bar\psi-\psi=0,\quad \bar\Phi-\Phi=0,\quad \bar\psi_0-\psi_0=0.$$

The system (13) then simplifies greatly: $\partial\tilde\varphi/\partial y=0$, i.e., $\varphi=\varphi(z)$,

$$2\eta\tilde\varphi=\left(\frac{d\bar\psi_0}{dy}\right)^2\left(\frac{1}{\rho^2}-\frac{1}{\bar\rho^2}\right), \quad (14)$$

$$\frac{d^2\tilde\varphi}{dz^2}=4\pi\bar\rho\left(\frac{\rho}{\bar\rho}-1\right). \quad (15)$$



Substituting (14) in (15) and using (7), we get

$$d^2s/dl^2=2(1/\sqrt{s}-1), \quad (16)$$

where

$$s=(\bar\rho/\rho)^2,\quad l=z\omega_p/|\bar v_z|,\quad \omega_p{}^2=4\pi\eta\bar\rho.$$

Equation (16) describes one-dimensional finite motion with potential energy $V(s)=2s-4\sqrt{s}$ (see the figure), i.e.,

$$d^2s/dl^2=-dV/ds. \quad (17)$$

Integrating (17), we get

$$(ds/dl)^2+2V(s)=C,\quad -4\leqslant C<\infty. \quad (18)$$

Let us determine the period L of the vibrational motion for Eq. (18). For case $-4\le C<0$ we obtain

$$L=2\int_{s_1}^{s_2}\frac{ds}{(C-2V(s))^{1/2}}, \quad (19)$$

where s_1 and s_2 are the roots of the equation $2V(s)=C$. Carrying out the integration in (19), we get

$$L=2\pi. \quad (20)$$

Thus, the wavelength λ of the steady-state perturbation is determined from the relation

$$L=\lambda\omega_p/|\bar v_z|. \quad (21)$$

Going over to the stationary coordinate frame, we obtain in lieu of (21)

$$L^2=\lambda^2\omega_p{}^2/(v_0-v_{\text{ph}})^2, \quad (22)$$

where v_0 is the velocity of the unperturbed beam in the stationary coordinate system and v_{ph} is the phase velocity of the wave.

From (22), we get, using (20),

$$v_{\text{ph}}=v_0\pm\omega_p\lambda/2\pi. \quad (23)$$

Formula (23) is the dispersion relation for the ordinary fast and slow space-charge waves. It shows that the phase velocity of the wave does not depend on its amplitude.

The possible existence of periodic solutions of Eq. (18) for the case $0 \leq C < \infty$ calls for additional research. In this case it is possible to construct periodic discontinuous solutions. As shown by V. M. Smirnov[7], these are apparently not realized.

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