A CAPACITOR IN THE FIELD OF A GRAVITATIONAL WAVE

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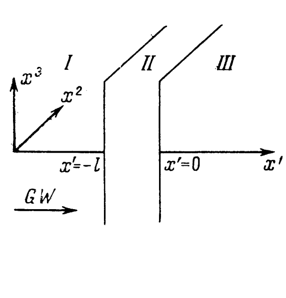
The problem of an electromagnetic field in a charged capacitor located in the field of a gravitational wave is considered. It is shown that the capacitor becomes a source of electromagnetic waves with a frequency equal to that of the gravitational wave and an amplitude $\sim hE$, where h is the perturbation in the metric tensor and E is field strength in the capacitor at h = 0. The possibility of using such a system as a detector (with an efficiency $\sim\sqrt{\epsilon/\mu}$) of gravitational waves is discussed. Some advantages of such a detector at high frequencies are pointed out.

At present there exist several proposals for methods of detecting gravitational waves (cf., for example, the review article [1]). In particular, it has been proposed to register the gravitational radiation by the following of its effects: the mutual displacement of test masses, the displacement of interference fringes, and the excitation of (electro-) mechanical vibrations of extended masses. Below we shall discuss the possibility of using the electromagnetic radiation arising when two charged test bodies are moved in the field of a gravitational wave.

Let us consider the following problem. We have a system of two infinite planes uniformly charged with the densities ρ and $-\rho$ respectively, and separated by the distance l (cf. the figure). A plane gravitational wave is incident normal to this system. We want to determine the electromagnetic field inside and outside such a capacitor. Neglecting the static contribution of the charge and of the masses of the plates to the metric of space-time, the problem reduces to the solution of the Maxwell equations [2]

$$F^{ik}{}_{;k} = \frac{4\pi}{c} j^i, \quad F^{*ik}{}_{;k} = 0 \tag{1}$$

in space-time with a prescribed metric corresponding to the gravitational field. Here F^{ik} is the electromagnetic field tensor, F^{*ik} is the dual



tensor to F^{ik}, and j^i is the current four-vector.

In order to simplify the mathematics, we give the gravitational field some definite form, in particular, we assume that the gravitational wave is generated by a linear quadrupole mass oscillator. [3] When transformed to the coordinate system of the observer, for whom the wave propagates along the x^1 axis, the field of such an oscillator (in the wave zone) close to the observer can be written in terms of the perturbations h_{ik} in the metric tensor:

$$\begin{gathered} h_{11} = {}^1/_2\varphi(2\cos^2\theta - \sin^2\theta),\ h_{33} = -h_{22} = {}^1/_2\varphi\sin^2\theta \\ h_{00} = {}^1/_2\varphi(1+\cos^2\theta), \\ h_{13} = -h_{03} = \varphi\cos\theta\sin\theta, \\ h_{01} = -\varphi\cos^2\theta,\ \ h_{02} = h_{12} = h_{23} = 0, \\ \varphi = A\exp\{i\omega c^{-1}(x^0 - x^1)\}. \end{gathered} \tag{2}$$

Here θ is the angle between the x^1 axis and the direction of the oscillations of the mass oscillator.

Taking account of the symmetry of our problem and the smallness of the perturbations h_{ik}, we find from (1) the following expression for the longitudinal component of the electromagnetic field:

$$F^{01} = E_1^{(0)}(1 + {}^1/_2\varphi\sin^2\theta). \tag{3}$$

Before we determine the other components of the tensor F^{ik}, we turn to the problem of measuring the electromagnetic field in the general theory of relativity. In curved space-time, only local measurements [4] performed by an observer "traveling" on his own world line have a clear and definite meaning. At each point (event) of the world line the observer uses local Cartesian coordinates defined by three space-like vectors orthogonal to one another and to the four-velocity of the observer. This orthonormal tetrad together with the tensor of some physical quantity must give the results of measurements performed by the observer. For example, the intensity of the electric or magnetic fields can be found by the formula

$$E_3 = F_{(30)} = F^{ik}\lambda_{(3)i}\lambda_{(0)k};$$
$$H_2 = -F_{(13)} = -F^{ik}\lambda_{(1)i}\lambda_{(3)k}, \qquad (4)$$

etc., where the numbers in the lower parentheses label the vectors of the tetrad. Consider now an observer located at each moment t of the world time at one and the same point of the charged plate $x^i = c^i$, $c^i = \text{const}$, $i = 1, 2, 3$, such that his local (2)- and (3)-axes are oriented along the surface. This specification of the observer together with the conditions of orthonormality determines uniquely its tetrad; the nonzero components of the tetrad are

$$\lambda^0_{(0)} = 1 + {}^1\!/_4\varphi \sin^2\theta, \quad \lambda^1_{(1)} = 1 - {}^1\!/_4\varphi(2\cos^2\theta - \sin^2\theta),$$
$$\lambda^2_{(2)} = 1 + {}^1\!/_4\varphi \sin^2\theta, \; \lambda^3_{(3)} = 1 - {}^1\!/_4\varphi \sin^2\theta, \; \lambda^0_{(1)} = -\varphi\cos^2\theta,$$
$$\lambda^3_{(1)} = \lambda^0_{(3)} = -\varphi\cos\theta\sin\theta. \qquad (5)$$

It is easy to see[1] that all components of F^{ik} except F^{01} and $F^{23} = 0$ satisfy one and the same wave equation independently of where we consider the field—inside or outside the capacitor. To find a solution we must thus impose boundary conditions at $x^1 = 0$ and $x^1 = -l$.

Let us now assume that the plates of the capacitor are transparent for electromagnetic radiation; then the boundary conditions will be determined by the discontinuity of E and H [written with the help of (4)] and the condition that there be no electromagnetic waves incident from outside. The wave equation for F^{02}, F^{12}, F^{03}, and F^{13} does not explicitly contain terms corresponding to the action of the gravitational field, and thus the curvature of space-time may show up precisely in the boundary conditions. In particular, the fact that the boundary conditions for F^{02} and F^{12} are independent of φ allows us to set them equal to zero. For the components F^{03} and F^{13} we obtain

$$F_{\mathrm{I}}^{13} = -F_{\mathrm{I}}^{03} = {}^1\!/_2 E_1^{(0)} A\cos\theta\sin\theta\,(1 - e^{2ikl})\,e^{ik(x^0+x^1)}, \qquad (6a)$$
$$F_{(13)\mathrm{II}} = -H_{2,\mathrm{II}} = F_{(03)\mathrm{II}} = -E_{3,\mathrm{II}}$$
$$= E_1^{(0)} A\cos\theta\sin\theta e^{i(kx^0+\pi/2)}\sin kx^1, \qquad (6b)$$
$$F_{\mathrm{III}}^{13} = F_{\mathrm{III}}^{03} = 0. \qquad (6c)$$

These solutions show that a charged capacitor in the field of a gravitational wave can serve as a source of electromagnetic radiation. There will be no radiation in the forward direction, i.e., in the direction of propagation of the gravitational wave. This somewhat unexpected result (which holds for arbitrary l) can be interpreted in the following way. The gravitational wave reaching the first plate acts on its charge like some external force. Part of the resultant action of the electromagnetic perturbation propagates together with the gravitational wave and reaches the second oppositely charged plate after the time $\tau = l/c$. Since, however, the variable gravitational field, as an external force, acts in the same way on positive and negative charges, the electromagnetic perturbation induced near the second plate will have the opposite sign of that induced near the first plate. If we now take into account that the only difference between the plates is the sign of the charge, it becomes clear why such a system does not radiate in the forward direction.

There is electromagnetic radiation for all $l \neq n\lambda/2$ ($n = 1, 2, 3, \ldots$) in the direction opposite to the direction of propagation of the gravitational wave, and for $l = \lambda(2n+1)/4$ the amplitude of this radiation is even comparable with the field inside the capacitor. It is interesting that our system ceases to be a radiator when the dimensions of the capacitor are multiples of half the wavelength of the gravitational (and in our case, also of the electromagnetic) wave, i.e., at the resonance frequencies for the free electromagnetic field. However, we do not obtain a resonance in the sense that the field amplitude is increased inside the capacitor. Moreover, the amplitude of the electromagnetic field inside the capacitor with transparent walls is in general independent of the dimensions of the capacitor itself and hence, of whether our system radiates or not. The fact that for all $\lambda/2 \neq l/n$, $n = 1, 2, 3, \ldots$, the charged capacitor with transparent walls in the field of a gravitational wave can emit radiation with a frequency equal to the frequency of the gravitational wave evidently allows us to use it as a source of excitations in resonators. This would increase considerably the effectiveness of the detection of a gravitational wave compared to the direct measurement of the variable field inside the capacitor itself.

Thus, measuring the variable electromagnetic field induced by the gravitational field, we may use the charged capacitor (together with an electric antenna and an amplifier) as a gravitational detector. What now is its sensitivity? The maximal power transmitted by the antenna to the receiver is

$$P_r = \frac{c}{8\pi}E^2\sigma = \frac{c}{8\pi}E_1^{(0)2}\;\sigma A^2\cos^2\theta\sin^2\theta\sin^2 ky; \qquad (7)$$

here σ is the effective area of the antenna equal to $D^2\eta\lambda^2/4\pi = g\lambda^2/4\pi$, where D is the directivity coefficient, η is the efficiency, and g is the gain of the antenna. The quantity y coincides with the coordinate x^1 of the antenna if the latter is located inside the capacitor, and $y = l$ if the antenna is located outside the capacitor in the region I (cf. the figure). Expressing A through the energy flow of the gravitational wave averaged over the time and a sphere, t_{0r} (erg/cm²sec),[3] and substituting the value of σ, we find for the received power

$$P_r = \frac{E_1^{(0)2}}{8\pi} g t_{0r} \frac{60\pi^2 G}{\omega^4} \sin^2 ky. \tag{8a}$$

The received power is thus proportional to the square of intensity of the electromagnetic field [$E_1^{(0)}$ is the field inside the capacitor in Minkowski space-time) inside the capacitor. However, its magnitude cannot be taken arbitrarily large, since for values of the electromagnetic field close to 10^9 V/m = 3.33×10^4 esu the capacitor begins to break down. Applying to the plates of the capacitor a potential difference close to the threshold value, we obtain the following expression for the maximal received power

$$P_r = \frac{175\pi^2}{\omega^4} g t_{0r} \sin^2 ky. \tag{8b}$$

For the two special cases, a) the dimensions (in x^1) of the capacitor are comparable to the wavelength of the electromagnetic wave, so that we can fulfil the condition $ky = \pi/2$, and b) the dimensions of the capacitor are small compared to the wavelength of the electromagnetic wave, we obtain

$$P_r = \frac{175\pi^2}{\omega^4} g t_{0r}, \quad ky = \frac{\pi}{2}; \tag{9a}$$

$$P_r = 4{,}9 \cdot 10^{-20} \frac{l^2}{v^2} g t_{0r}, \quad l \ll \lambda. \tag{9b}$$

The analysis of (9a) and (9b) shows that the use of a charged capacitor as a gravitational antenna is useful only in the region of high frequencies (this remark is especially true in the region of ultralow frequencies, where the efficiency of the antenna is practically proportional to ω^2).

Let us compare the sensitivity of the proposed detector with the sensitivity of the piezodetectors of Weber[3] for high frequencies. However, first we make more precise Weber's expression (Ref. [3], p. 169) for the power absorbed by a crystal interacting with the gravitational wave. We have

$$P_{abs} = \frac{2\pi k^2 G\rho}{c k_s^2} \frac{VQ}{\omega} \frac{1}{(k_s l)^2} t_{0r} \approx 3 \cdot 10^{-26} \frac{VQ}{\omega} \frac{1}{(k_s l)^2} t_{0r},$$
$$l = n\lambda_s/2; \quad n = 1, 2, 3, \ldots, \tag{10}$$

where the quantities with the index s refer to the acoustic wave, and Q is the quality factor. The difference with respect to the corresponding formula of Weber consists in the presence of the factor $1/(k_s l)^2$, which lowers considerably the absorption cross section of the gravitational antenna in the case when several ($n \gg 1$) half-wavelengths of the acoustical wave fit in the length of the crystal ($V \sim n$, $1/(k_s l)^2 \sim 1/n^2$, $P_{abs} \sim 1/n$). The use of piezoelectric crystals in which the incident gravitational wave induces components of the field intensity which do not change sign during each half-period of the acoustical wave, does not (as the corresponding calculations show) present any advantages over the usual crystals, and in particular, does not allow us to avoid the restrictions imposed by the factor $1/(k_s l)^2$. The piezoelectricity is in this sense no more than a convenient means of selecting the energy assembled by the crystal in the space-time of variable curvature.

From (8a) and (10) we obtain the ratio of the absorption cross section of a single crystal $\sigma_{g,cr}$ over the absorption cross section $\sigma_{g,c}$ of a detector using the radiation of a charged condenser in the field of the gravitational wave:

$$\frac{\sigma_{gcr}}{\sigma_{gc}} = \frac{\rho v_s^2/2}{E_1^{(0)2}/8\pi} \frac{Q}{15g} \frac{V_{cr}}{(\lambda/2)^3 (2l/\lambda_s)^2}; \quad ky = \frac{\pi}{2}. \tag{11}$$

It is easy to see that crystal detectors with $l = \lambda_s/2$ are in general more advantageous. However, for high frequencies it is a difficult technical problem to obtain a crystal with a side equal to half the acoustical wavelength. Thus it may turn out that the registration of the gravitational radiation by the method proposed in the present paper is more convenient from this point of view. Moreover, if by some means or other one were able to overcome the barrier of the electric field intensity $E \approx 10^9$ V/m at which the capacitor breaks down, our method may also be preferable at lower frequencies.

Let us now assume that all space (inside as well as outside the capacitor) is filled with matter with the dielectric constant ϵ and the magnetic permeability μ. In the case it is necessary to use the equations of the electrodynamics of moving media in covariant form.[5] Following Landau and Lifshitz,[6] we denote by F^{ik} the tensor which yields the fields E and B after "projection" on the vectors of the orthonormal tetrad. We have

$$F_I^{03} = \psi[(\sqrt{\varepsilon\mu} + 1) e^{i(k_0+3k)l} + (\sqrt{\varepsilon\mu} - 1) e^{i(k_0+k)l} - 2\sqrt{\varepsilon\mu} e^{2ikl}] e^{i(\omega t + kx^1)};$$

$$F_{II}^{03} = \sqrt{\varepsilon\mu}\psi\{[(\sqrt{\varepsilon\mu} - 1) - (\sqrt{\varepsilon\mu} + 1) e^{i(k+k_0)l}] e^{i(\omega t - kx^1)} - [(\sqrt{\varepsilon\mu} + 1) e^{2ikl} - (\sqrt{\varepsilon\mu} - 1) e^{i(k+k_0)l}] e^{i(\omega t + kx^1)}\};$$

$$F_{\mathrm{III}}^{03} = \psi[(\sqrt{\varepsilon\mu} - 1) - 2\sqrt{\varepsilon\mu}e^{i(k+k_0)l} + (\sqrt{\varepsilon\mu} + 1)e^{2ikl}]e^{i(\omega t - kx^1)};$$

$$\psi = \frac{E_1 A \cos\theta \sin\theta}{(\sqrt{\varepsilon\mu} - 1)^2 e^{2ikl} - (\sqrt{\varepsilon\mu} - 1)^2};$$

$$k_0 = \frac{\omega}{c}; \qquad k = \sqrt{\varepsilon\mu}k_0. \tag{12}$$

The analysis of (12) shows that the dielectric increases the effectiveness of the detector. The point is that the induced field E remains the same and the magnetic field intensity becomes $H \sim E\sqrt{\epsilon/\mu}$; thus the introduction of ϵ leads to an increase in the energy flux density of the electromagnetic energy $S = c(E \times H)/4\pi$.[6] A factor $\sqrt{\epsilon/\mu}$ appears in the formulas for the absorption cross section of the detector (9a) and (9b). Thus an increase of the effectiveness of our detector by one or two orders of magnitude can be achieved by introducing a dielectric with high permittivity.

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