SOVIET PHYSICS JETP VOLUME 24, NUMBER 6 JUNE, 1967

NONLINEAR EFFECTS IN COHERENT AMPLIFICATION OF SPIN WAVES BY A CHARGED PARTICLE BEAM

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Submitted to JETP editor July 18, 1966

J. Exptl. Theoret. Phys. (U.S.S.R.) **51**, 1934-1939 (December, 1966)

Nonlinear effects during coherent amplification of spin waves by a charged particle beam are considered. It is assumed that the amplitude of the amplified wave exceeds the background amplitude considerably. The beam density is assumed to be small. The analysis is performed in the hydrodynamic approximation. The maximum amplitudes of the magnetic-moment oscillations of a ferromagnet are estimated. It is shown that in amplification connected with the Cerenkov effect the maximum amplitude is proportional to the square of the increment, whereas in amplification connected with the Doppler effect it is proportional to the increment raised to the 3/2 power.

1. Coherent amplification of spin waves by a beam of charged particles was considered in the linear approximation in a number of papers[1-3]. Equations were derived for the conditions under which the spin waves become amplified, for the minimum beam particle velocities required to satisfy these conditions, and for the growth increments of the spin waves.

In this paper we investigate how spin-wave amplification is influenced by the nonlinearity of the equations describing the interactions of a beam of charged particles with a ferrodielectric. It is assumed here that the initial amplitude of the amplified spin wave (in a frequency interval which is much shorter than the growth increment) greatly exceeds the amplitudes of all the remaining waves. The particle motion is considered in the hydrodynamic approximation. The beam particle density is assumed to be small. The unperturbed particle velocity $\mathbf{v}_0$ is parallel to the magnetic field, which is directed along the easiest-magnetization axis of the ferromagnet.

In this case, amplification takes place when the frequency ω_S of the amplified wave and its wave vector $\mathbf{k}$ are connected with the particle velocity by the relation $\omega_S = \mathbf{k}\cdot\mathbf{v}_0$ (Cerenkov instability), or by the relation $\omega_S = \mathbf{k}\cdot\mathbf{v}_0 - \omega_B$ (the anomalous Doppler effect), where ω_B is the cyclotron frequency of the electron. Nonlinear interactions cause the translational velocity of the particles in the beam, and also the growth increment of the spin, to decrease with increasing oscillation amplitude. We estimate in this paper the maximum values of these amplitudes. It turns out that if the relative growth increment of the spin waves is equal to γ, then when $\omega_S = \mathbf{k}\cdot\mathbf{v}_0$ the maximum amplitude of the magnetic-moment oscillations is $m_{max} \sim \gamma^2 M_0$, and when $\omega_S = \mathbf{k}\cdot\mathbf{v}_0 - \omega_B$ its value is $m_{max} \sim \gamma^{3/2} M_0$ (M_0 is the magnetic moment per unit volume of the ferromagnet).

2. Let the unperturbed particle velocity in the beam, the constant magnetic field $\mathbf{H}_0$, and consequently also the magnetic induction $\mathbf{B}_0$ be parallel to the axis of easiest magnetization, which is chosen in the z direction. We start from Maxwell's equations for the alternating electric and magnetic fields $\mathbf{e}$ and $\mathbf{h}$:

$$\operatorname{curl}\mathbf{e} = -\frac{1}{c}\frac{\partial\mathbf{b}}{\partial t}, \quad \operatorname{curl}\mathbf{h} = \frac{\varepsilon}{c}\frac{\partial\mathbf{e}}{\partial t} + \frac{4\pi}{c}\mathbf{j}, \tag{1}$$

the hydrodynamic equations for the particle velocity $\mathbf{v}$ in the beam and for their density n:

$$\frac{\partial\mathbf{v}}{\partial t} + (\mathbf{v}\nabla)\mathbf{v} = \frac{e}{m}\left\{\mathbf{e} + \frac{1}{c}[\mathbf{v}, \mathbf{B}_0 + \mathbf{b}]\right\},$$

$$\frac{\partial n}{\partial t} + \operatorname{div}(n\mathbf{v}) = 0, \tag{2*}$$

and the Landau and Lifshitz equations for the alternating part of the magnetic moment $\mathbf{m}$ of a uniaxial ferromagnet:

$$\begin{aligned}\dot{m}_x - \hat{\Omega} m_y &= -gM_0 b_y + g(\beta m_y - b_y)m_z + g m_y b_z,\\ \dot{m}_y + \hat{\Omega} m_x &= gM_0 b_x + g(b_x - \beta m_x)m_z - g m_x b_z,\\ \dot{m}_z &= g(m_x b_y - m_y b_x).\end{aligned} \tag{3}$$

The z axis is parallel to the easiest magnetization axis

*$[\mathbf{v}, \mathbf{B}_0 + \mathbf{b}] \equiv \mathbf{v}\times(\mathbf{B}_0 + \mathbf{b})$.

$$\hat{\Omega} = \hat{\Omega}_e + 4\pi g M_0$$
$$= gM_0(-\alpha\nabla^2 + \beta + H_0 M_0^{-1}) + 4\pi g M_0,$$

α and β are constants characterizing exchange and anisotropy, g is the gyromagnetic ratio[4], and m and e are the mass and charge of the electron.

At low beam density, the influence of the nonlinear terms in (3) is small compared with the influence of the nonlinear terms in (2). We shall henceforth disregard the nonlinear terms in Eqs. (3). The main fact that leads to the cessation of the amplification is the nonlinearity of the equations in (2).

3. Let us consider the spin-wave amplification connected with the Cerenkov effect. If we neglect the nonlinear terms in (2) and assume that **e** and **v** vary like $\exp(i\mathbf{k}\cdot\mathbf{r} - i\omega t)$, then

$$v_x^{(1)} = \frac{e}{m\Delta}\left\{-i(\omega - k_z v_0)\left[e_x\left(1 - \frac{k_z v_0}{\omega}\right) + \frac{k_x v_0}{\omega}e_z\right] - \omega_B e_y\left(1 - \frac{k_z v_0}{\omega}\right)\right\},$$

$$v_y^{(1)} = \frac{e}{m\Delta}\left\{\omega_B\left[e_x\left(1 - \frac{k_z v_0}{\omega}\right) + \frac{k_x v_0}{\omega}e_z\right] - i\omega e_y\left(1 - \frac{k_z v_0}{\omega}\right)^2\right\},$$

$$v_z^{(1)} = i\frac{e}{m}\frac{e_z}{\omega - k_z v_0}, \tag{4}$$

where $\Delta = \omega_B^2 - (\omega - k_z v_0)^2$, $\omega_B = |e|B_0/mc$, and the coordinates are chosen such that the wave vector **k** lies in the zOx plane.

The relation between the components of the electric field in the spin wave of frequency $\omega_s(\mathbf{k})$ is

$$e_x = -\frac{k_z}{k_x}e_z, \quad e_y = i\frac{k_z}{k_x}\frac{\Omega_e}{\omega_s}e_z.$$

It is seen therefore that when $\omega \approx \mathbf{k}\cdot\mathbf{v}_0$ the modulus $|v_z|$ is much larger than $|v_y|$ or $|v_x|$, so that v_x and v_y can be disregarded in (1) and (2).

Eliminating from (1) and (3) the magnetic field, the magnetic moments, and the components e_x and e_y of the electric field, we can obtain

$$\hat{D}\frac{\partial}{\partial\varphi}e_z = \frac{4\pi e}{k_z\varepsilon}\left\{\hat{D}\frac{k_z^2}{k^2} - \frac{\varepsilon}{c^2k^2}\frac{\partial^2}{\partial t^2} \times \left[\frac{\partial^2}{\partial t^2} + \hat{\Omega}^2 - 4\pi g M_0\hat{\Omega}\frac{k_z^2}{k^2}\right]\frac{k_x^2}{k^2}\right\}n, \tag{5}$$

where

$$\hat{D} = \left(\frac{\partial^2}{\partial t^2} + \hat{\Omega}_s^2\right)\frac{\partial^2}{\partial\varphi^2} - \frac{\varepsilon}{c^2k^2}\frac{\partial^2}{\partial t^2}\left[2\left(\frac{\partial^2}{\partial t^2} + \hat{\Omega}\hat{\Omega}_e\right) + 4\pi g M_0\hat{\Omega}\frac{k_x^2}{k^2}\right],$$

$$\varphi = \mathbf{kr}, \quad \hat{\Omega}_s^2 = \hat{\Omega}_e(\hat{\Omega}_e + 4\pi g M_0 k_x^2/k^2). \tag{6}$$

In solving the system (2) and (5) we shall use the successive-approximation method developed by Bogolyubov and Mitropol'skiĭ[5]. We put in first approximation

$$e_z = a\cos(\varphi - \psi), \tag{7}$$
$$da/dt = \alpha a, \quad d\psi/dt = \Omega_s + \beta \tag{8}$$

(Ω_s—frequency of the oscillations at a beam density n = 0). From (2) it follows that

$$v_z^{(1)} = eam^{-1}(\alpha^2 + \beta^2)^{-1}[\alpha\cos(\varphi - \psi) - \beta\sin(\varphi - \psi)], \tag{9}$$

$$n^{(1)} = eam^{-1}n_0k_z(\alpha^2 + \beta^2)^{-2}[2\alpha\beta\cos(\varphi - \psi) - (\beta^2 - \alpha^2)\sin(\varphi - \psi)], \tag{10}$$

and Eq. (5), with allowance for (10), yields

$$\xi \equiv (\beta + i\alpha) = \left(-\frac{1}{2} + i\frac{\sqrt{3}}{2}\right)\Omega_s\left(\frac{\omega_p^2}{c^2k^2}\frac{2\pi g M_0}{\Omega_e}\frac{k_x^2}{k^2}\right)^{1/3}, \tag{11}$$

where $\omega_p^2 = 4\pi e^2 n_0/m$.

In the second approximation, oscillations with double frequency appear

$$v_z^{(2)} = -\left(\frac{ea}{m}\right)^2 k_z\frac{\beta^3 - 3\alpha^2\beta}{4(\alpha^2 + \beta^2)^3}\cos 2(\varphi - \psi), \tag{12}$$

$$n^{(2)} = -\frac{3}{4}\left(\frac{ea}{m}\right)^2 n_0k_z^2\frac{\beta^3 - 3\alpha^2\beta}{(\alpha^2 + \beta^2)^4} \times [\beta\cos 2(\varphi - \psi) + \alpha\sin 2(\varphi - \psi)], \tag{13}$$

$$e_z^{(2)} = \frac{3}{8\varepsilon}\frac{ea^2}{m}\omega_p^2\frac{k_z^3}{k^2}\frac{\beta^3 - 3\alpha^2\beta}{(\beta^2 + \alpha^2)^4} \times [\alpha\cos 2(\varphi - \psi) - \beta\sin 2(\varphi - \psi)]. \tag{14}$$

The condition for the applicability of perturbation theory is $|v^{(2)}| \ll |v^{(1)}|$ or $eam^{-1}k_z \ll |\xi|^2$.

From (12)–(14) we see that $|e^{(2)}/e^{(1)}| \ll |v^{(2)}/v^{(1)}| \approx |h^{(2)}/h^{(1)}|$. The electric vector of the oscillations with the double frequency is parallel to the vector **k** ($e_x^{(2)} = e_z^{(2)}k_x/k_z$ and $e_y^{(2)} = 0$). Therefore the magnetic induction in this approximation vanishes ($\mathbf{b} = c\mathbf{k}\times\mathbf{e}/\omega$). Since $\mathbf{m} = \mathbf{b}[1 - \mu^{-1}(\omega, \mathbf{k})]/4\pi$ (where $\hat{\mu}(\omega, \mathbf{k})$ is the magnetic permeability tensor), there are likewise no double-frequency oscillations of the magnetic moment.

To find the corrections to the growth increment let us calculate the third approximation. To this end we assume that

$$da/dt = [\alpha + A(a)]a, \quad d\psi/dt = \Omega_s + \beta + B(a), \tag{15}$$

where A(a) and B(a) are proportional to a^2 are chosen such that (5) contains no resonant terms with $\cos(\varphi - \psi)$ and $\sin(\varphi - \psi)$.

In the third approximation we are interested only in A and B, and therefore we write out only those terms in $v_z^{(3)}$ and $n^{(3)}$ that contain the first harmonic:

$$v_z^{(3)} = -eam^{-1}(\alpha^2+\beta^2)^{-1}(9\alpha^2+\beta^2)^{-1}$$
$$\times\{[-4\alpha\beta A + (3\alpha^2-\beta^2)B + (3\alpha^2+\beta^2)C]\sin(\varphi-\psi)$$
$$+[(3\alpha^2-\beta^2)A + 4\alpha\beta B - 2\alpha\beta C]\cos(\varphi-\psi)\}, \quad (16)$$

$$n^{(3)} = eam^{-1}n_0k_z(\alpha^2+\beta^2)^{-2}(9\alpha^2+\beta^2)^{-2}$$
$$\times\{[-C(69\alpha^4\beta+5\beta^5+42\alpha^2\beta^3)$$
$$+A(40\alpha^3\beta^2+12\alpha\beta^4-36\alpha^5)-B(78\alpha^4\beta+12\alpha^2\beta^3-2\beta^5)]$$
$$\times\sin(\varphi-\psi)+[C(63\alpha^5+15\alpha\beta^4+110\alpha^3\beta^2)$$
$$-A(78\alpha^4\beta+12\alpha^2\beta^3-2\beta^5)-B(40\alpha^3\beta^2+12\alpha\beta^4-36\alpha^5)]$$
$$\times\cos(\varphi-\psi)\}, \quad (17)$$

where

$$C = \frac{1}{8}\left(\frac{ea}{m}\right)^2 k_z^2\frac{\beta^3-3\alpha^2\beta}{(\alpha^2+\beta^2)^3}.$$

Substituting (17) in (5) we obtain the conditions for the absence of resonant terms:

$$72B - 6\sqrt{3}A = -{}^{47}/_2C,\quad 6\sqrt{3}B + 72A = -57\sqrt{3}C/2. \quad (18)$$

Hence the time variation of the phase and the amplitude is determined by the equations

$$\frac{da}{dt} \approx \alpha\left[1-\frac{1}{10}\left(\frac{ea}{m}\frac{k_z}{|\xi|^2}\right)^2\right]a,$$

$$\frac{d\psi}{dt} \approx \Omega_s + \beta\left[1+\frac{1}{10}\left(\frac{ea}{m}\frac{k_z}{|\xi|^2}\right)^2\right]. \quad (19)$$

From (19) we see that the growth increment of the amplitude $(a^{-1}\partial a/\partial t)$ decreases with time, and simultaneously the resonance condition $d\psi/dt - k\cdot v_0 - \beta = 0$ is violated.

Amplification stops apparently at amplitudes $a_{max} \sim (m/e)|\xi|^2/k$. This condition can be written in the form

$$b_{max}/B_0 \sim |\xi|^2/\omega\omega_B \sim \gamma_0^2,$$

where γ_0 is the relative growth increment, or in the form $m_{max}/M_0 \sim \gamma_0^2$.

4. We consider now the amplification of spin waves in the case of the anomalous Doppler effect. We confine ourselves to the case when the wave vector of the amplified wave is parallel to the z axis. We seek a solution of (1)—(3) in the form

$$v_x + iv_y = ca(t)\exp\{i(\Phi+\vartheta(t))\},$$
$$e_x + ie_y = B_0\varepsilon(t)\exp\{i(\Phi+\psi(t))\}, \quad (20)$$

where $\Phi = kz - \omega t$, and $\epsilon(t)$, $\vartheta(t)$, and $\psi(t)$ are slowly varying functions of the time.

Assuming zero beam density, we can find that the frequency of the spin waves is

$$\omega_s = \Omega_e \equiv gM_0(\alpha k^2 + \beta + H_0/M_0).$$

Retaining in (1)—(3) terms of the lowest (nonzero) order in the beam density, we obtain the system

$$\frac{da}{d\tau} = \varepsilon\left(\beta\frac{ck}{\omega}-1\right)\cos(\vartheta-\psi),\quad \frac{d\beta}{d\tau} = -\frac{ck}{\omega}\varepsilon a\cos(\vartheta-\psi),$$
$$\frac{d\varepsilon}{d\tau} = \frac{1}{2}q^2a\cos(\vartheta-\psi),\quad \frac{d\psi}{d\tau} = \frac{1}{2}q^2\frac{a}{\varepsilon}\sin(\vartheta-\psi),$$
$$\frac{d\vartheta}{d\tau} = 1-\frac{\omega}{\omega_B}\left(\beta\frac{ck}{\omega}-1\right)-\frac{\varepsilon}{a}\left(\beta\frac{ck}{\omega}-1\right)\sin(\vartheta-\psi), \quad (21)$$

where

$$q^2 = 2\frac{\Omega-\Omega_e}{\Omega_e}\frac{4\pi nc^2m}{(ck/\omega)^2B_0^2},\quad \tau = \omega_Bt,\quad \beta = \frac{v_z}{c}.$$

If we assume that a and ϵ are small, then we obtain from the system (21) for the growth increment

$$\frac{\alpha}{\omega_B} \equiv \gamma = q\left(\frac{1}{2}\frac{\omega_B}{\omega_s}\right)^{1/2}. \quad (22)$$

The system (21) coincides formally with that obtained by Krasovitskiĭ and Kurilko[6] for the amplification of oscillations by a beam in a nonmagnetic medium with dielectric constant $\epsilon > 1$. It was shown in[6] that the amplitude of the solution (21) first increases, reaches a certain maximum value, and then begins to decrease. Using the results of[6], we can get an expression for the minimal amplitudes of the magnetic moment as functions of their initial value m_0. We have:

a) If $m_0/M_0 \ll \gamma^{3/2} \ll 1$, then

$$m_{max} = \sqrt{6}\gamma^{3/2}B_0/4\pi, \quad (23)$$

b) but if $\gamma^{3/2} \ll m_0/M_0 \ll 1$, then

$$m_{max} = m_0\left[1+4\frac{\gamma^2}{(4\pi m_0/B_0)^{4/3}}\right]^{1/2}. \quad (24)$$

If the initial amplitudes are small, then their relative change is large and the maximum amplitude is attained within a time

$$T \sim \frac{1}{\gamma\omega_B}\ln\frac{m_0}{M_0\gamma^{3/2}}.$$

We see from (11) and (12) that the growth increment is larger in the case $\omega_s \approx \mathbf{k}\cdot v_0$ than in the case $\omega_s = \mathbf{k}\cdot v_0 - \omega_B$, since the growth increment is proportional to $n^{1/3}$ in the first case and to $n^{1/2}$ in the second.

The maximum amplitude in the case of the Cerenkov effect is proportional to $n^{2/3}$, i.e., it is larger than in the case of the anomalous Doppler effect, where is is proportional to $n^{3/4}$.

In conclusion, the author is grateful to A. I. Akhiezer and V. G. Bar'yakhtar for a discussion of the work, and also to V. B. Krasovitskiĭ and V. I. Kurilko for an opportunity to become acquainted with their paper prior to publication.

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Translated by J. G. Adashko
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