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CONTRIBUTION OF THE PERIPHERAL PART OF MANY-PARTICLE STATES TO THE UNITARITY CONDITION FOR $\pi\pi$ SCATTERING

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The unitarity condition for the $\pi\pi$-scattering partial wave is deduced by taking into account that part of many-particle intermediate states which is produced and which goes over to the final state as a result of peripheral interaction. The contribution of this part of the many-particle intermediate states is expressed in terms of physical partial waves. It is proposed to expand the amplitude in a series that converges over the whole z_s plane (z_s is the cosine of the scattering angle in the s channel) and possesses the property that its first N coefficients can be expressed in terms of the first N partial waves only. The expansion is also used to obtain an expression for the partial wave jump $A_l(s)$ on the left cut; this expression converges for all values of s. Application of this expression for the jump $A_l(s)$ on the left cut permits one to remove the divergences which usually appear in the bootstrap problem for exchange, in the crossed channel, of a particle with a spin greater than or equal to unity.

1. In recent years a large number of papers have appeared, dealing with self-consistent calculations of masses and coupling constants (widths) of the ρ meson. It became clear from these papers that many-particle intermediate states play an important role in the unitarity condition for the $\pi\pi$-scattering amplitude. Some of these intermediate states can be approximated by the two-particle state $\pi\omega$ (and also $\pi\varphi$), and the contribution of this state is decisive for obtaining a ρ-meson mass close to the experimental value[1,2]. However, in all of these papers the width of the ρ meson turns out to be double the experimental value. The persistence of this result evidently indicates that this disagreement is not due to the rough methods used in solving the bootstrap problem with the two-particle unitarity condition. It would seem that the contribution to the unitarity condition of the many-particle intermediate states is essential.

In this paper an attempt is made to take into account, in the unitarity condition for the pion-pion scattering, the contribution of many-particle states produced as a result of peripheral interaction of the initial particles, i.e., exchange of one particle (π, ω, φ), and going over into the final state also by means of peripheral interaction. One may hope that allowance for the additional inelasticity will give rise to a narrowing of the ρ resonance so that its width will be closer to its experimental value.

It is a most favorable circumstance that the peripheral part of the many-particle unitarity condition is again expressed in terms of two-particle amplitudes. It turns out to be possible to express it in terms of physical partial waves, and it can be hoped that a good approximation is given by a few low partial waves. In this manner we again obtain a closed problem for a few low partial waves.

The main idea used in this paper was proposed by Ter-Martirosyan already in 1960[3,4]. It consists in representing the double spectral function in the form of two terms, each of which corresponds to a two-particle intermediate state in one of the channels and is determined by the Mandelstam equation[5]. The double spectral functions corresponding to two-particle intermediate states in the t or u channels correspond in the s channel to many-particle intermediate states produced as a result of peripheral interactions and going over to the final state via peripheral interactions. Thus, the unitarity condition derived in this paper for $\pi\pi$ scattering with the peripheral part of the many-particle states taken into account represents in essence the equation obtained by Ter-Martirosyan[3] projected into partial waves.

The Mandelstam equation expresses the spectral function corresponding to the two-particle intermediate states in the t or u channels in terms of the absorptive part of the s channel for unphysical momentum transfers.

It is shown in this paper that the absorptive part (as well as the entire amplitude) can be represented in terms of a double series, which converges in the entire z_s plane, up to the cut (z_s is the cosine of the scattering angle in the s channel); the series is such that its first N terms contain only the first N partial waves. It is true that we cannot specify the number of terms N which is sufficient for a good approximation of the absorptive part in the unphysical region, because such an estimate requires the knowledge of the maximum of the modulus of the absorptive part on the cut, of which nothing is known. One may hope, however, that this number is not large.

The proposed representation of the absorptive part in terms of a double series makes it also possible to obtain an expression, which converges for all s, for the jump of the partial amplitudes on the left cut. The jump over the left cut is expressed in terms of physical partial waves, and if we confine ourselves to a finite number of partial waves, as will certainly be necessary in any practical calculation, then such a cut-off series will, just like the exact jump of the partial wave, also be bounded at infinity by a constant. Therefore the use of the proposed expression for the jump of the partial wave over the left cut makes it possible to eliminate the divergence that arises in the bootstrap problem when exchange of a particle with spin larger or equal to unity occurs in the cross channels.

2. In this section we obtain for the amplitude a series expansion that converges in the entire z-plane. An analogous expansion was proposed, investigated, and applied to the problem of self-consistent calculation of the ρ meson by several authors.[6-8] The coefficients of the expansion, proposed in those papers, are expressed in terms of derivatives of the amplitude with respect to z, which is inconvenient, since the unitarity condition and the bootstrap equations are formulated more naturally in terms of partial waves. Therefore in practical calculations[8] it is necessary to make an incorrect transition from the derivatives of the amplitude to partial waves, which in fact makes the expansion improper. In particular, as the number of omitted terms in the expansion is increased, the accuracy becomes poorer.[8]

The coefficients of the expansion proposed here are expressed in terms of partial waves, with the first N coefficients containing only the first N partial waves.

The $\pi\pi$-scattering amplitude in the state with isospin I can be expressed in terms of an expansion in partial waves

$$A^I(s,t)=\sum_l(2l+1)A_l{}^I(s)P_l(z_s),$$

$$A_l{}^I(s)=\frac{\sqrt{s}}{2q_s}e^{i\delta_l{}^I}\sin\delta_l{}^I, \qquad (1)$$

where the summation is over values of l having the same parity as I. This expansion converges within a Mandelstam ellipse with semi-axis $z_0(s) = 1 + t_0/2q_s^2$. Making use of the integral representation for the Legendre polynomials

$$P_l(z)=\frac{1}{2\pi i}\oint_+\frac{v^l dv}{[1-2zv+v^2]^{1/2}}, \qquad (2)$$

where the integral is taken over a closed contour going in the positive direction (counterclockwise) around the branch points of the denominator, the expansion (Eq. (1)) can be rewritten in the form

$$A^I(s,t)=\frac{1}{2\pi i}\oint_+\frac{dv}{[1-2zv+v^2]^{1/2}}\sum_l(2l+1)A_l{}^I(s)v^l$$

$$=\frac{1}{2\pi i}\oint_+\frac{g^I(s,v)\,dv}{[1-2zv+v^2]^{1/2}}. \qquad (3)$$

The function $g^I(s, v)$, is an analytic function of v with two cuts: from $v_0 = z_0 + (z_0^2-1)^{1/2}$ to ∞ and from $-v_0$ to $-\infty$. Let us map the v-plane onto the interior of a unit circle by means of the conformal mapping:

$$x=v_0/v-\sqrt{(v_0/v)^2-1}, \qquad v=2v_0x/(1+x^2). \qquad (4)$$

The edge of the cut in the v-plane is mapped thereby onto the unit circumference. Then the function $g^I[s, v(x)]$, being analytic inside the circle, can be expanded in a series in powers of x, which converges up to the contour

$$g^I[s,v(x)]=\sum_n B_n{}^I(s)x^n \qquad (5)$$

(the parity of n equals the parity of I).

The coefficients $B_n^I(s)$ can be easily expressed in terms of $A_l^I(s)$ and $v_0(s)$:

$$g^I[s,v(x)]=\sum_l(2l+1)A_l{}^I(s)\left[\left(\frac{2v_0x}{1+x^2}\right)^l=(2v_0)^l x^l.\right.$$

$$\left.\cdot\sum_{k=0}^{\infty}\binom{-l}{k}x^{2k}\right]=\sum_n x^n\sum_l(-1)^{n-l}\frac{(2l+1)\Gamma(^1/_2(n+l))}{\Gamma(l)\Gamma(^1/_2(n-l)+1)}$$

$$\times(2v_0)^l A_l{}^I(s). \qquad (6)$$

It follows hence that

$$B_n{}^I(s)=\sum_l(-1)^{(n-l)/2}\frac{(2l+1)\Gamma(^1/_2(n+l))}{\Gamma(l)\Gamma(^1/_2(n-l)+1)}(2v_0)^l A_l{}^I(s). \qquad (7)$$

The summation in Eqs. (6) and (7) is over all nonnegative n and l having the same parity as I. Let us note that the summation in Eq. (7) stops at $l = n$, owing to the factor $\Gamma(1/2(n-l)+1)$ that appears in the denominator.

The convergence of the series (5) on the contour is determined by the character of the singularity on it. Singularities corresponding to individual many-particle thresholds are weak and do not prevent the series from converging. Therefore the convergence of the series will be determined by the behavior of the function at the point $x = \pm i$ ($v = \infty$), which is related to the behavior of $A^I(s, t)$ at $|t| \to \infty$. If the function $g^I(s, v)$ is bounded at infinity by a power of v, then the series (5) can be somewhat modified in order that it converge on the contour. Let $|g(s, v)| < c|v|^m$. Then we shall expand in a series in x not the function g(s, v), but the function $(1+x^2)^m g(s, v)$ which is bounded at the point $x = \pm i$. We shall obtain a series that converges on the contour:

$$g^I[s, v(x)] = \frac{1}{(1+x^2)^m} \sum_n \bar{B}_n{}^I x^n, \tag{8}$$

where

$$\bar{B}_n{}^I(s) = \sum_l^n (-1)^{(n-l)/2} \frac{(2l+1)\Gamma(^1/_2(n+l)-m)}{\Gamma(l-m)\Gamma(^1/_2(n-l)+1)}$$
$$\times (2v_0)^l A_l{}^I(s). \tag{9}$$

Here, as in Eqs. (6) and (7), n and l have the same parity as I.

If we substitute the expansion (5) in (3) we obtain for $A^I(s, t)$ a series that converges in the entire z-plane up to the cut:

$$A^I(s,t) = \sum_n^\infty B_n{}^I(s) \frac{1}{2\pi i} \oint \frac{[x(v)]^n\, dv}{[1-2zv+v^2]^{1/2}}$$
$$= \sum_n^\infty B_n{}^I(s) F_n(z, z_0). \tag{10}$$

Let us consider the functions used for the expansion

$$F_n(z, z_0) = \frac{1}{2\pi i} \oint \frac{[x(v)]^n\, dv}{[1-2zv+v^2]^{1/2}} \tag{11}$$

For $n = 0$ we have simply $F_0(z, z_0) = P_0(z) = 1$. For $n \neq 0$ the function $F_n(z, z_0)$ is an analytic function of z, with two cuts: from z_0 to ∞ and from $-z_0$ to $-\infty$. Explicitly, the function $F_n(z, z_0)$ is expressible in terms of elliptic integrals (since two square roots of quadratic polynomials appear in the integrand of Eq. (11)).

In the limit as $z_0 \to \infty$ we have

$$F_n(z, z_0) \to v_0{}^{-n} P_n(z),$$

and

$$B_n{}^I(s) \to v_0{}^n (2n+1) A_n{}^I(s),$$

so that the series (10) goes over into conventional expansion in partial waves, Eq. (1), which in this case converges in the entire z-plane.

A comparison of the expansion (10) with the expansion in partial waves, Eq. (1), shows that the first N terms of one expansion differ from the first N terms of the other only by the addition of exponentially decreasing tail of partial waves with $l > N$. In the physical region $-1 \leq z \leq 1$ this tail is small. As an example let us consider one resonant P-wave in the region of the ρ meson. This wave corresponds to the term $B_1^1(s) F_1(z, z_0)$ in the expansion (10) and to $3A_1^1(s) P_1(z)$ in the expansion in partial waves. The term $B_1^1(s) F_1(z, z_0)$ contains, in addition to $3A_1^1(s) P_1(z)$, an F-wave admixture, amounting to 4% of the P-wave, an H-wave admixture amounting to 0.3% of the P-wave, etc. In the physical region the admixture of higher partial waves is small. However, as we go into the region of larger z, the contribution from the tail of higher partial waves increases and becomes comparable with the contribution from the first N partial waves, amounting to an effective cut-off, so that for $z \to \infty$ all functions are bounded by a constant.

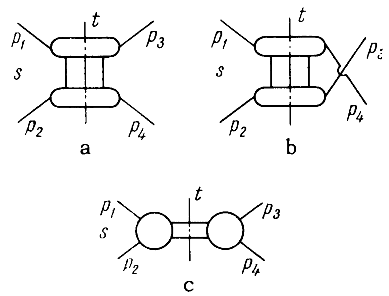
The expansion (10) can be rewritten in the form:

$$A^I(s,t) = \lim_{N\to\infty} \sum_n^N B_n{}^I(s) F_n(z, z_0)$$
$$= \lim_{N\to\infty} \sum_l^N (2l+1) A_l{}^I(s) P_{lN}(z, z_0), \tag{12}$$

where $P_{lN}(z, z_0)$ is obtained if v^l is expanded in the integral representation, Eq. (2), in a power series in x and the first N terms are retained

$$P_{lN} = \frac{1}{2\pi i} \oint_+ \frac{dv}{[1-2zv+v^2]^{1/2}} \sum_{k=0}^{(N-l)/2} (-1)^k \frac{\Gamma(l+k)}{\Gamma(l)\Gamma(k+1)}$$
$$\times (2v_0)^l x^{2k+l}. \tag{13}$$

3. Let us pass now to the calculation of the contribution to the unitarity condition for the $\pi\pi$-scattering partial amplitude of that part of the many-particle states which is produced and goes over into the final state as a result of the peripheral interaction. The unitary graphs corresponding to this process are shown in Fig. 1a and 1b. In order to distinguish the double spectral functions corresponding to graphs 1a and 1b from the functions corresponding to graph 1c, we shall call the double spectral functions corresponding to the graph 1c two-particle spectral functions of the s-channel and denote them by $\rho_{st}(s, t, u)$ and $\rho_{su}(s, t, u)$.



The spectral functions corresponding to the graphs 1a and 1b will be called peripheral spectral functions of the s-channel and denoted by $\rho_{ts}(s, t, u)$ and $\rho_{us}(s, t, u)$.

It is easily seen that if we consider the exchange of the particles π, K, $\overline{K}$, ω, and φ, then as a consequence of conservation laws there are no graphs which have simultaneously two-particle intermediate states in two channels. Therefore, if we neglect contributions from graphs having only many-particle intermediate states in all channels the complete double spectral function $A^I_{st}(s, t, u)$, can be written in the form

$$A_{st}{}^{I}(s,t,u) = \rho_{st}{}^{I}(s,t,u) + \rho_{ts}{}^{I}(s,t,u). \qquad (14)$$

The two-particle spectral function of the s-channel $\rho^I_{st}(s, t, u)$ is determined by the Mandelstam equation in the s-channel, and the peripheral spectral function of the s-channel is determined by the Mandelstam equation in the t-channel.

Since the amplitude $A^I(s, t, u)$ is symmetric (I = 0, 2) or antisymmetric (I = 1) with respect to exchange of t and u, it is sufficient to consider graphs corresponding to two-particle states only, for example, in the t-channel. Equation (14) makes it possible to calculate the jump of the partial wave over the right cut corresponding both to the two-particle intermediate states and the peripheral part of the many-particle intermediate states. For the jump in the partial wave we have the following expression:

$$\operatorname{Im} A_l{}^I(s) = \frac{2}{\pi}\int_{t_0}^{\infty} A_{st}{}^I(s,t)\,Q_l\left(1+\frac{t}{2q_s{}^2}\right)\frac{dt}{2q_s{}^2}. \qquad (15)$$

The partial-wave projection of the two-particle spectral function, determined by the Mandelstam equation in the s-channel, gives rise to the well known result[9]

$$\operatorname{Im} A_l{}^I(s) = \vartheta(s-4\mu^2)\frac{2q_s}{\sqrt{s}}|A_l{}^I(s)|^2 + \delta_{I1}\vartheta[s-(m+\mu)^2]\frac{2q_{\omega s}}{\sqrt{s}}|A_l{}^{\pi\omega}(s)|^2 + \ldots, \qquad (16)$$

where ϑ is the unit step function. Here we have written explicitly the contribution from the two-particle intermediate states $\pi\pi$ and $\pi\omega$; q_s is the momentum of the pion in the center of mass system corresponding to a $\pi\pi$-pair energy equal to s; $q_{\omega s}$ is the same for the $\pi\omega$ pair,

$$q_s{}^2 = s/4-\mu^2,\quad q_{\omega s}{}^2 = {}^1\!/_4 s^{-1}[s-(m-\mu)^2] \times [s-(m+\mu)^2]; \qquad (17)$$

μ is the pion mass, m is the ω-meson mass; $A^{\pi\omega}_l(s)$ is the amplitude for the transition $\pi\pi \to \pi\omega$, corresponding to the total angular momentum l and parity $(-1)^l$.

Let us express now the peripheral spectral function of the s-channel in terms of the absorptive part $A^I_s(s, t)$. To this end we first make use of the crossing symmetry

$$A^I(s,t,u) = \beta_{Ij}A^j(t,s,u),\quad \beta_{Ij} = \begin{pmatrix} {}^1\!/_3 & 1 & {}^5\!/_3 \\ {}^1\!/_3 & {}^1\!/_2 & -{}^5\!/_6 \\ {}^1\!/_3 & -{}^1\!/_2 & {}^1\!/_6 \end{pmatrix}, \qquad (18)$$

from which it follows that

$$\rho_{ts}{}^I(s,t,u) = \beta_{Ij}\rho_{ts}{}^j(t,s,u). \qquad (19)$$

The quantity $\rho^j_{ts}(t, s, u)$ is the two-particle spectral function of the t-channel corresponding to an isospin j in the t-channel. It is determined by the Mandelstam equation[5]

$$\rho_{ts}{}^j(t,s,u)_{\pi\pi} = \frac{2q_t}{\pi\sqrt{t}}\int_{-\infty}^{\infty} dz_1\,dz_2[A_s{}^j(t,z_1)A_s{}^{j*}(t,z_2) + A_u{}^j(t,z_1)A_u{}^{j*}(t,z_2)]\frac{\vartheta(-K)}{\sqrt{-K(z_t,z_1,z_2)}}, \qquad (20)$$

where

$$-K(z_t,z_1,z_2) = z_t{}^2+z_1{}^2+z_2{}^2-2z_tz_1z_2-1 = \frac{2s}{(2q_t{}^2)^3}[(t-4\mu^2)p^2-s_1s_2],$$

$$p^2 = {}^1\!/_4 s^{-1}[s-(\sqrt{s_1}-\sqrt{s_2})^2][s-(\sqrt{s_1}+\sqrt{s_2})^2]. \qquad (21)$$

The subscript $\pi\pi$ indicates the state in the t-channel to which this ρ corresponds. Since $A^j_u(t, -z) = (-1)^j A^j_s(t, z)$, the contribution of the second term in Eq. (20) is equal to the contribution of the first. If we make use of the crossing symmetry, Eq. (18),

$$A_s{}^j(t,s_1) = \beta_{jI_1}A_s{}^{I_1}(s_1,t) \qquad (22)$$

and go over to integration over s_1 and s_2, then we obtain for the peripheral double spectral function of the s-channel the following expression:

$$\rho_{ts}{}^I(s,t,u)_{\pi\pi} = C^I_{I_1I_2}\frac{2}{\pi\sqrt{st}}\int_{4\mu^2}^{\infty} A_s{}^{I_1}(s_1,t)A_s{}^{I_2*}(s_2,t)\cdot \times\cdot\frac{\vartheta(-K)\,ds_1\,ds_2}{[(t-4\mu^2)p^2-s_1s_2]^{1/2}}. \qquad (23)$$

where

$$C^I_{I_1I_2} = \sum_j \beta_{Ij}\beta_{Ij_1}\beta_{jI_2}, \qquad (24)$$

or explicitly

$$C^0_{I_1I_2} = \begin{pmatrix} 1/3 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 5/3 \end{pmatrix}, \quad C^1_{I_1I_2} = \begin{pmatrix} 0 & 1/3 & 0 \\ 1/3 & 1/3 & 5/12 \\ 0 & 5/12 & 5/4 \end{pmatrix},$$

$$C^2_{I_1I_2} = \begin{pmatrix} 0 & 0 & 1/3 \\ 0 & 1/4 & 3/4 \\ 1/3 & 3/4 & 7/12 \end{pmatrix}.$$

The expression for the peripheral spectral function contains the value of the absorptive part of the s-channel for unphysical values of momentum transfer ($t > 4\mu^2$). We can express the absorptive part for these values of t in terms of physical partial waves of the s-channel by making use of the expansion obtained in the second part of the paper (Eq. (12)). Upon substitution of Eq. (12) into Eq. (23) and projection into partial waves we obtain

$$\operatorname{Im} A_l^I(s)_{\pi\pi} = \vartheta(s - 16\mu^2) C^I_{I_1I_2} \times \lim_{N\to\infty} \sum_{l_1l_2}^{N} \int_{4\mu^2}^{\sqrt{s_1}+\sqrt{s_2}=\sqrt{s}} ds_1\, ds_2 \Gamma^l_{l_1l_2}(s, s_1, s_2, N) \times \operatorname{Im} A_{l_1}^{I_1}(s_1) \operatorname{Im} A_{l_2}^{I_2}(s_2). \qquad (25)$$

Here we have introduced the notation

$$\Gamma^l_{l_1l_2}(s, s_1, s_2, N) = \frac{\sqrt{2}(2l_1+1)(2l_2+1)}{\pi^2\sqrt{s}\, q_s^3} \operatorname{Re} \times \int_{4\mu^2+s_1s_2/p^2}^{\infty} P_{l_1N}\left[1 + \frac{t}{2q_{s_1}^2},\ z_0(s_1)\right] \times P_{l_2N}\left[1 + \frac{t}{2q_{s_2}^2},\ z_0(s_2)\right] \frac{Q_l(1 + t/2q_s^2)\, dt}{\sqrt{t}\,[(t-4\mu^2)p^2 - s_1s_2]^{1/2}}. \qquad (26)$$

The summation in Eq. (25) is over values I_1 and I_2 equal to 0, 1, and 2 and also over physical values of partial waves having the same parity as the isospin. In the integrand there appear values of imaginary parts of partial waves at energies $\sqrt{s_1}$ and $\sqrt{s_2}$ at least by 2μ smaller than $\sqrt{s}$.

4. In an analogous fashion one can calculate the peripheral spectral function corresponding to the exchange of π and ω mesons in the t-channel. The imaginary part of the $\pi\pi$-scattering amplitude in the t-channel corresponding to the two-particle intermediate state $\pi\omega$, is equal to

$$\operatorname{Im} A^1(t, s, u)_{\pi\omega} = \frac{q_{\omega t}}{2\pi\sqrt{t}} \sum_{\varepsilon_i} \int d\Omega A^{\pi\omega}(t, s_1, \varepsilon_i) A^{\pi\omega *}(t, s_2, \varepsilon_i), \qquad (27)$$

where ϵ_i is the polarization vector of the ω meson.

From symmetry properties it follows that the amplitude for the transition $\pi\pi \to \pi\omega$ may be written in the form

$$A^{\pi\omega}(t, s, \varepsilon) = \varepsilon^{\alpha\beta\gamma\delta} p_{1\alpha} p_{3\beta} q_{\omega\gamma} \varepsilon_\delta F(t, s, u), \qquad (28)$$

where F(t, s, u) is a function that is symmetric under permutation of all three variables. The presence of just one invariant function corresponds to the fact that one has only one independent helicity amplitude or that only one direction of the polarization vector is possible, perpendicular to $\mathbf{p}_1$ and $\mathbf{q}_\omega$ in the center-of-mass system ($\mathbf{p}_1 + \mathbf{p}_3 = 0$).

Substituting Eq. (28) into the unitarity condition (27) and summing over polarizations we obtain

$$\operatorname{Im} A^1(t, s, u)_{\pi\omega} = \frac{q_{\omega t}^3 q_t^2 \sqrt{t}}{2\pi} \int d\Omega (z_t - z_1z_2) F(t, z_1) F^*(t, z_2), \qquad (29)$$

where z_t, z_1, and z_2 are the cosines of scattering angles in the t-channel

$$z_t = \cos\vartheta_{\mathbf{p}_1\mathbf{p}_2}, \quad z_1 = \cos\vartheta_{\mathbf{p}_1\mathbf{q}}, \quad z_2 = \cos\vartheta_{\mathbf{p}_2\mathbf{q}},$$

$$z_t = 1 + \frac{s}{2q_t^2}, \quad z_{1,2} = \frac{t - m^2 - 3\mu^2}{4q_{\omega t}q_t} + \frac{s_{1,2}}{2q_{\omega t}q_t}. \qquad (30)$$

By analytic continuation of the unitarity condition (29) one obtains the following expression for the double spectral function:

$$\rho_{ts}^1(t, s, u)_{\pi\omega} = \frac{4}{\pi} q_{\omega t}^3 q_t^2 \sqrt{t} \int_{1+4\mu^2/2q_t^2}^{\infty} (z_t - z_1z_2) F_s(t, s_1) F_s^*(t_1, s_2) \times \frac{\vartheta(-K)\, dz_1\, dz_2}{\sqrt{-K(z_t, z_1, z_2)}} = \frac{q_{\omega t}\sqrt{t}}{\pi} \int_{4\mu^2}^{\infty} F_s(s_1, t) F_s^*(s_2, t) \times \frac{(z_t - z_1z_2)\vartheta(-K)}{\sqrt{-K(z_t, z_1, z_2)}} ds_1\, ds_2. \qquad (31)$$

In the derivation of Eq. (31), use was made of the symmetry of the function F(s, t, u).

Now, as in the case of $\pi\pi$ scattering, we express $F_s(s, t)$ in terms of physical partial waves of the transition $\pi\pi \to \pi\omega$. The expansion of the $\pi\pi \to \pi\omega$ in a partial-wave series has the form

$$A^{\pi\omega}(s, t) = \sum_{l=1}^{\infty} \frac{(2l+1)}{\sqrt{l(l+1)}} A_l^{\pi\omega}(s) P_l^1(z_s), \qquad (32)$$

where $A_l^{\pi\omega}(s)$ is the partial wave for the production of the $\pi\omega$ pair with total and orbital angular momentum l, $P_l^1(z) = -\sin\vartheta dP_l(z)/dz$ is the associated Legendre polynomial of first order. The summation is over odd l. Correspondingly we have for $F_s(s, t)$

$$F_s(s,t) = -\frac{1}{q_{\omega s}\, q_s \sqrt{s}} \sum_{l=1}^{\infty} \frac{2l+1}{\sqrt{l(l+1)}} \operatorname{Im} A_l^{\pi\omega}(s) P_l'(z_s). \quad (33)$$

In analogy with the case of the $\pi\pi$-scattering amplitude, this series can be transformed into a series that converges in the entire plane with the cut z_s:

$$F_s(s,t) = -\frac{1}{q_{\omega s}\, q_s \sqrt{s}} \lim_{N\to\infty} \sum_{l=1}^{\infty} \frac{2l+1}{\sqrt{l(l+1)}} \operatorname{Im} A_l^{\pi\omega}(s) P_{lN}'(z_s, z_0), \quad (34)$$

where

$$P'_{lN}(z_s, z_0) = dP_{lN}(z_s, z_0) / dz_s.$$

Substituting this expansion into Eq. (31) and projecting into partial waves we obtain

$$\operatorname{Im} A_l^I(s)_{\pi\omega} = \vartheta(s - 16\mu^2)\beta_{I1} \times \lim_{N\to\infty} \sum_{l_1,l_2=1}^{N} \int_{4\mu^2}^{\sqrt{s_1}+\sqrt{s_2}=\sqrt{s}} ds_1\, ds_2 \overline{\Gamma}^l_{l_1 l_2}(s, s_1, s_2, N) \operatorname{Im} A_{l_1}^{\pi\omega}(s_1) \times \operatorname{Im} A_{l_2}^{\pi\omega}(s_2), \quad (35)$$

where

$$\overline{\Gamma}^l_{l_1 l_2}(s, s_1, s_2, N) = \frac{(2l_1+1)(2l_2+1)}{\pi^2 q_{\omega s}^2 q_s^4 s [l_1(l_1+1) l_2(l_2+1)]^{1/2}} \times \int^{\infty} P'_{l_1 N}[z_{s_1}(t, s_1), z_0(s_1)] P'_{l_2 N}[z_{s_2}(t, s_2), z_0(s_2)] \times \frac{q_{\omega t}\sqrt{t}(z_t - z_1 z_2)\vartheta(-K)}{\sqrt{-K(z_t, z_1, z_2)}} Q_l\left(1 + \frac{t}{2q_s^2}\right) dt. \quad (36)$$

The lower limit of integration is determined by the function $\vartheta(-K)$. Summation in Eq. (35) is over odd l_1 and l_2.

5. Let us write out the resultant unitarity condition for the scattering amplitude with two-particle intermediate states $\pi\pi$ and $\pi\omega$ taken into account, as well as the peripheral parts of the many-particle states corresponding to the exchange of π and ω mesons in the t- and u-channels:

$$\operatorname{Im} A_l^I(s) = \vartheta(s - 4\mu^2)\frac{2q_s}{\sqrt{s}} |A_l^I(s)|^2 + \vartheta[s - (m+\mu)^2]\delta_{I1} \times \frac{2q_{\omega s}}{\sqrt{s}} |A_l^{\pi\omega}(s)|^2 + \vartheta(s - 16\mu^2) C^I_{I_1 I_2} \times \lim_{N\to\infty} \sum_{l_1 l_2}^{N} \int_{4\mu^2}^{\sqrt{s_1}+\sqrt{s_2}=\sqrt{s}} ds_1\, ds_2 \cdot \Gamma^l_{l_1 l_2}(s, s_1, s_2, N) \operatorname{Im} A_{l_1}^{I_1}(s_1) \times \operatorname{Im} A_{l_2}^{I_2}(s_2) + \vartheta(s - 16\mu^2)\beta_{I1} \times \lim_{N\to\infty} \sum_{l_1 l_2=1}^{\infty} \int_{4\mu^2}^{\sqrt{s_1}+\sqrt{s_1}=\sqrt{s}} ds_1 ds_2 \overline{\Gamma}^l_{l_1 l_2}(s, s_1, s_2, N) \operatorname{Im} A_{l_1}^{\pi\omega}(s_1) \times \operatorname{Im} A_{l_2}^{\pi\omega}(s_2). \quad (37)$$

The unitarity condition (37) has exact crossing symmetry, although, of course, in practical calculations we retain in the sums only a finite number of terms and the crossing symmetry is then only approximate.

6. Two comments are necessary with respect to Eq. (37) and the method of its derivation. In the first place, in the derivation of the expression for the peripheral spectral function $\rho^I_{ts}(s, t, u)$ we have introduced an expansion for $A^I_s(s, t)$ in the form of Eq. (12). If $A^I_s(s, t)$ increases as $|t| \to \infty$, then such an expansion on the cut itself, in the conventional sense, does not converge. However, integration with the smooth function $Q_l(z)$ improves the convergence of the series, so that if one takes l sufficiently large (to the right of singularities in the l-plane), then the integrated series will converge.

In the second place, the integral with Q_l (Eq. (15)) exists only for values of l which lie in the complex l-plane to the right of all singularities. The partial waves with l lying to the left of the singularity farthest to the right are determined by analytic continuation in l from the region in which the integral converges.

If the partial wave $A^I_l(s)$ has a Regge pole $\alpha(s)$, then even that part of the many-particle intermediate states, which is taken into account by Eq. (37), gives rise to moving branch points in the l-plane, $l_n = n[\alpha(s/n^2) - 1] + 1$. However, the jumps over these cuts, obtained when the two-particle spectral function in the t-channel is taken into account, do not correspond to jumps of the real partial waves[10]. This is related to the fact that the jump in the real partial wave is determined by the behavior of $A^I_{st}(s, t)$ as $t \to \infty$, where one has many contributions from many-particles states in the t-channel and where $\rho_{st} + \rho_{ts}$ differs substantially from A_{st}. Therefore allowance for the singularities of the partial wave in the l-plane is outside the framework of the approximation considered here.

One may, however, hope that the jumps in the exact amplitudes over the cuts in the l-plane are small so that they may be neglected. Then if we retain a finite number N of terms in the expansion (37) we obtain a good approximation for the lower partial waves.

7. The expansion considered in the second part of this work for the scattering amplitude, Eq. (12),

together with crossing symmetry, Eq. (18), makes it possible to obtain an expression for the jump in the partial $\pi\pi$-scattering wave over the left cut, which converges for all values of s:

$$\operatorname{Im} A_l^I(s) = \beta_{Ij} \int\limits_{4\mu^2}^{4\mu^2 - s} \frac{dt}{2q_s^2} P_l\left(1 + \frac{t}{2q_s^2}\right) \operatorname{Re} A_l^j(t, s, u)$$

$$= \beta_{Ij} \lim_{N\to\infty} \sum_{l_1}^{N} (2l_1 + 1) \int\limits_{4\mu^2}^{4\mu^2 - s} \frac{dt}{2q_s^2} \operatorname{Im} A_{l_1}^j(t)$$

$$\times \operatorname{Re} P_{l_1 N}\left[1 + \frac{s}{2q_t^2},\ z_0(t)\right] P_l\left(1 + \frac{t}{2q_s^2}\right). \qquad (38)$$

For the absorptive part in the t-channel we have used an expansion of the type (12), which does not converge on the cut if the absorptive part increases at infinity. Thus we have followed a formally nonrigorous procedure by introducing this expansion under the integral sign. However, the resultant expression for the jump in the partial wave over the left cut converges regardless of whether the absorptive part increases at infinity or not. The point is that the partial wave is bounded as one goes to infinity along the left cut. This can be shown by making use of the unitarity condition on the right cut and assuming that the partial wave does not have too strong a singularity at infinity.[11,12] Consequently the effective growth of the absorptive part at infinity is cut off upon integration with the Legendre polynomial, and therefore the series does not diverge.

More formally this can be shown as follows. In the complex s-planes the expansion (38) for the partial wave converges up to the cut, since the corresponding expansion (12) converges in that region. On the cut the partial wave is a smooth function at all finite points, and at infinity it is bounded. It follows from this that the series will also converge on the cut.

Let us note that expansion (38) automatically gives for the jump in the partial wave over the left cut an expression that is bounded by a constant, even if we retain in it only a finite number of partial waves. Thus, the use of Eq. (38) eliminates the divergence of the bootstrap problem, due to the exchange of a particle with spin larger than or equal to unity in the t- or u-channel.

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