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PRYCE'S THEOREM AND THE NEUTRINO THEORY OF PHOTONS

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Recently possibilities have been found and widely discussed by which it would seem that one avoids the difficulty in the neutrino theory of photons which was pointed out by Pryce. In the present paper it is proved in general form that for free fields and arbitrary N-particle states the requirements of invariance under space rotations, of statistics, and of genuine neutrality are incompatible for a compound photon (Pryce's theorem). The proof presented here takes in all of the constructions that have been suggested recently.

1. INTRODUCTION

THE neutrino theory of photons had only a short existence (six years) and then was refuted by Pryce's theorem. The idea was first put forward by de Broglie in 1932.[1] He proposed a simple model of the photon constructed from a neutrino and an antineutrino with equal energies and momenta. If these particles are assumed to be free, then they must move in the same direction. In fact, let us denote the four-momentum of the photon by p_μ and those of the neutrino and antineutrino by k_μ and k'_μ. Then

$$p_\mu = k_\mu + k_\mu'. \tag{1}$$

Taking the square of (1) and summing over μ, we get $\cos\theta = 1$, where θ is the angle between **k** and **k'**.

It was soon noted that the de Broglie photon does not obey Bose statistics. In fact, the existence of two photons in the same state for example, with momentum **p**) would mean the existence of two neutrinos in the same state (with momentum **p**/2), which is impossible if neutrinos are fermions.

In 1935 Jordan[2] proposed a model of the photon constructed from two neutrinos each of which is in a superposition of states with different momenta. With an appropriate choice of the superposition coefficients such a model would assure the correct statistics. Thereafter there were a great many papers[3-8] in which attempts were made to construct the photon from neutrino fields. It appeared that complete success in this direction had been achieved by Kronig[7] in 1936. In 1938 the development of the neutrino theory of photons was brought to a halt for a long time, actually until 1963, by Pryce's theorem.[9]

Pryce noted that the fixing of the relative phases of neutrino states with different helicities means the choice of a vector **a** perpendicular to the momentum of the neutrino. Rotation of this vector around the axis lying along the momentum of the neutrino changes the relative phases of states with different helicities. When a photon state is constructed from neutrino states with fixed phases, the photon state depends on this arbitrarily chosen transverse vector **a**. At the same time there is nothing in the nature of the neutrino that would fix this choice, and therefore the construction should be invariant under a rotation of **a**. It turned out that such an invariance is in contradiction with the condition of statistical compatibility. Accordingly, what Pryce showed is essentially the incompatibility of the statistical properties with invariance relative to a certain group of axial rotations.

A different interpretation of this difficulty in the neutrino theory of photons was given in 1963 by Barbour, Bietti, and Touschek.[10] They assert that it is impossible to construct, from the components of the wave functions of two massless fermions obeying the Dirac equation with m = 0 and having collinear momenta, a transverse four-vector—or, in the words of the authors, in the neutrino theory the photon is always longitudinal. The proof they give is as follows.

One considers a fermion which obeys the Dirac equation and whose mass is λm while its momentum is λk,

$$(i\gamma_\mu \lambda k_\mu + \lambda m)\psi(\lambda k) = 0. \tag{2}$$

The neutrino is the limiting state of this particle for $m \to 0$. By means of Eq. (2) we get the relation

$$2m\bar{\psi}(\tau k)\gamma_\mu\psi(\lambda k) = -2ik_\mu\bar{\psi}(\tau k)\psi(\lambda k), \qquad (3)$$

where λ and τ are numbers. The left member of this relation contains the only four-vector that can be constructed from the components of $\psi(\tau k)$ and $\psi(\lambda k)$, and the right member is a longitudinal four-vector. With a gauge transformation the amplitude of a photon state of the type (3) can always be reduced to zero.

It can be shown, however, that Eq. (2) and the relation (3) are not invariant under transformations which change the absolute value of k. And indeed when we go to the limit $m \to 0$ the relation (3) becomes the trivial identity 0 = 0. In fact, for m = 0 we have $\gamma_\mu k_\mu \psi(\lambda k) = \bar{\psi}(\tau k)\gamma_\mu k_\mu = 0$, from which it follows that

$$0 = {}^1\!/_2 k_\nu \bar{\psi}(\tau k)(\gamma_\nu\gamma_\mu + \gamma_\mu\gamma_\nu)\psi(\lambda k) = k_\nu\bar{\psi}(\tau k)\delta_{\mu\nu}\psi(\lambda k)$$

In the four-component theory there is no difficulty in practically constructing all transverse four-vectors from the wave functions of neutrinos with collinear momenta. In the representation in which

$$\alpha = \begin{pmatrix} \sigma & 0 \\ 0 & \sigma \end{pmatrix}, \quad \gamma_4 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \quad \gamma_j = -i\gamma_4\alpha_j, \quad \Sigma = \begin{pmatrix} \sigma & 0 \\ 0 & \sigma \end{pmatrix},$$

let us write out the four solutions u_ρ^δ of the Dirac equation with m = 0 that correspond to the two values of the sign of the energy $S_H = \omega/k$ and the two values of the helicity $\Gamma = \Sigma\mathbf{k}/k$:

$$\begin{array}{lccccc} S_H & = & +1 & -1 & +1 & -1 \\ \Gamma & = & +1 & -1 & -1 & +1 \end{array}$$

$$\sqrt{\frac{2}{1+n_3}}\, u_\rho^\delta(n) = \begin{pmatrix} 1 \\ \frac{n_1+in_2}{1+n_3} \\ 0 \\ 0 \end{pmatrix} \begin{pmatrix} -\frac{n_1-in_2}{1+n_3} \\ 1 \\ 0 \\ 0 \end{pmatrix} \begin{pmatrix} 0 \\ 0 \\ -\frac{n_1-in_2}{1+n_3} \\ 1 \end{pmatrix} \begin{pmatrix} 0 \\ 0 \\ 1 \\ \frac{n_1+in_2}{1+n_3} \end{pmatrix}$$

$(\delta,\ \rho = 1,\ -1;\ \mathbf{n} = \mathbf{k}/k)$

From the Dirac equation for the neutrino one gets the following condition for the vector $\bar{u}(\mathbf{n})\gamma_j u(\mathbf{n})$ to be orthogonal to the direction of propagation $\mathbf{n}$:

$$u^+(\mathbf{n})u(\mathbf{n}) = 0. \qquad (4)$$

Using (4), we find that there are always four linearly independent transverse four-vectors:

$$\bar{u}_{+1}^{+1}(\mathbf{n})\gamma_\mu u_{-1}^{-1}(\mathbf{n}); \quad \bar{u}_{-1}^{+1}(\mathbf{n})\gamma_\mu u_{+1}^{-1}(\mathbf{n});$$

$$\bar{u}_{-1}^{-1}(\mathbf{n})\gamma_\mu u_{+1}^{+1}(\mathbf{n});$$

$$\bar{u}_{+1}^{-1}(\mathbf{n})\gamma_\mu u_{-1}^{+1}(\mathbf{n}). \qquad (5)$$

Of course the condition (4) is not relativistically invariant: for any two four-vectors there is always a Lorentz frame in which their space components are not orthogonal. This corresponds to the situation with the potential A_μ of the free electromagnetic field, whose transversality in all Lorentz frames can be secured by means of a supplementary gauge transformation.

Summarizing, we can say that this particular difficulty does not exist in the neutrino theory of photons. The only actual difficulty is that the construction of a transverse four-vector is incompatible with the requirements of the statistics.

2. PRYCE'S THEOREM

We shall assume that all of the states of an elementary particle are described by a single irreducible representation of the proper inhomogeneous Lorentz group. For a particle with zero mass the helicity is an invariant of this group, and therefore such a particle is characterized by a single value of the helicity. The antiparticle has the opposite helicity. In fact, for the pseudoeuclidean space-time the operations of time displacement and of space inversion commute. Therefore the total Hamiltonian must commute[11] with the operator of space inversion I_s, which changes the sign of the helicity. According to the hypothesis of the conservation of combined parity, I_s = CP, where C is the charge-conjugation operator and P is the spatial parity; i.e., the operator I_s, which changes the sign of the helicity, converts particle into antiparticle.

This conception means in particular that the photon is a particle with a definite helicity (for example, a right-circularly polarized photon); the particle with the opposite helicity (the left-circularly polarized photon) is the antiphoton. The genuine neutrality of the photon is a sufficient condition for the physical realization of superpositions of photon and antiphoton states (a photon with linear polarization).

Suppose we have several different massless fields with spin ½. In the Hilbert space of their state vectors we define operators for creation of neutrino and antineutrino with momentum **k**: $c_i^+(\mathbf{k})$, $d_i^+(\mathbf{k})$, i = 1, 2, . . . , M. In what follows it will be convenient for us to classify these fields according to the value of the helicity; therefore we denote by $a_i(\mathbf{k})$ the operator of the i-th neutrino (or antineutrino) field corresponding to helicity $+\frac{1}{2}$, and by $b_i(\mathbf{k})$ that for helicity $-\frac{1}{2}$.

The problem of the neutrino theory of photons is that in the Hilbert space of the state vectors of neutrinos (or of several types of neutrinos) we are able to construct from the operators $a_i^+(\mathbf{k})$ and $b_i^+(\mathbf{k})$ certain operators $\kappa^+(\mathbf{p})$, $\omega^+(\mathbf{p})$, $\eta^+(\mathbf{p})$, $\xi^+(\mathbf{p})$. These are to be creation operators for states of the neutrino field which we identify with photon states (with circular and linear polarizations). If from the commutation and transformation properties of the neutrino operators a_i and b_i there then follow the correct analogous properties for the photon operators κ, ω, η, ξ, then we can without any contradictions construct from the latter operators the Heisenberg operators for the field strengths $\mathbf{E}(x)$ and $\mathbf{H}(x)$ of the free electromagnetic field, obeying Maxwell's equations and the usual commutation relations.

We shall make the following assumptions about the neutrino and photon fields:

1) Let U_θ be an operator which in the space of neutrino state vectors describes a rotation by angle θ around the direction of the momentum $\mathbf{k}$ as an axis. Then

$$U_\theta a_i(\mathbf{k}) U_\theta^{-1} = e^{i\theta/2} a_i(\mathbf{k}), \quad U_\theta b_i(\mathbf{k}) U_\theta^{-1} = e^{-i\theta/2} b_i(\mathbf{k}). \tag{6}$$

2) If the neutrinos are fermions, then the operators $a_i(\mathbf{k})$ and $b_i(\mathbf{k})$ obey the following commutation relations:

$$[a_i^+(\mathbf{k}), a_j(\mathbf{k}')]_+ = [b_i^+(\mathbf{k}), b_j(\mathbf{k}')]_+ = \delta_{ij}\delta(\mathbf{k}-\mathbf{k}'). \tag{7}$$

All other anticommutators are equal to zero.

If, on the other hand, the neutrinos are parafermions, then we can write different commutation relations for the operators $a_i(\mathbf{k})$ and $b_i(\mathbf{k})$, depending on the value of the maximum occupation number.[12]

3) There exist photons with right and left circular polarization, whose annihilation operators $\kappa(\mathbf{p})$ and $\omega(\mathbf{p})$ have the following transformation properties:

$$U_\theta \varkappa(\mathbf{p}) U_\theta^{-1} = e^{i\theta}\varkappa(\mathbf{p}), \quad U_\theta \omega(\mathbf{p}) U_\theta^{-1} = e^{-i\theta}\omega(\mathbf{p}). \tag{8}$$

4) Simultaneously there exist photons with linear polarization, whose annihilation operators $\xi(\mathbf{p})$ and $\eta(\mathbf{p})$ are linear combinations of $\kappa(\mathbf{p})$ and $\omega(\mathbf{p})$ and satisfy the transformation relations

$$U_\theta \xi(\mathbf{p}) U_\theta^{-1} = \xi(\mathbf{p})\cos\theta + \eta(\mathbf{p})\sin\theta,$$
$$U_\theta \eta(\mathbf{p}) U_\theta^{-1} = -\xi(\mathbf{p})\sin\theta + \eta(\mathbf{p})\cos\theta. \tag{9}$$

5) It is also assumed that the photon operators satisfy the following commutation relations:

$$[\xi(\mathbf{p}), \eta^+(\mathbf{p})]_- = 0, \tag{10}$$

$$[\varkappa(\mathbf{p}), \omega^+(\mathbf{p})]_- = 0. \tag{11}$$

Equations (10) and (11) are the conditions for the photon to be genuinely neutral. No assumptions are made about the equations for the photons or the neutrinos.

We identify with a photon state an N-particle state of the neutrino field(s) of one or more types. The creation operators (κ^+, ω^+, ξ^+, η^+) for photon states are expressed in terms of a_i^+ and b_i^+. Pryce's theorem, which we shall prove, is that the only quantities that satisfy the conditions (6)—(11) are $\xi = \eta = \kappa = \omega = 0$, i.e., no photon states with the enumerated properties exist in the Hilbert space of neutrino state vectors. All of the present papers on the neutrino theory of photons are based on the assumptions made above, and therefore come under the theorem of Pryce as formulated here.

At first we assume that the neutrinos are fermions, i.e., that the operators $a_i(\mathbf{k})$ and $b_i(\mathbf{k})$ satisfy the commutation relations (7). We shall first of all give the proof of the theorem in the simple case in which there is a single neutrino field (with antineutrinos) and the photon states are two-particle neutrino states. We denote by $|\mu_1, k_1, \mu_2, k_2\rangle$ the state vector of a two-particle state in which one neutrino (or antineutrino) has four-momentum k_1 and helicity μ_1 and the other has k_2 and μ_2. Using the completeness of the system of vectors $|\mu_1, k_1, \mu_2, k_2\rangle$ for two-particle states, we can show that the most general expression for the vector $|\Phi_p\rangle$ of a two-particle state which is an eigenvector of $p_\mu = k_{1\mu} + k_{2\mu}$, with $p_4 = ip$, is

$$|\Phi_p\rangle = \sum_{\mu_1\mu_2}\int_0^1 d\lambda C_{\mu_1\mu_2}(\lambda,\tau)\,|\mu_1, \lambda p, \mu_2, \tau p\rangle \tag{12}$$

with $\tau = 1-\lambda$. The relation (12) means that a photon state with momentum $\mathbf{p}$ is a superposition of two-particle neutrino states with momenta $\mathbf{k}_1$ and $\mathbf{k}_2$ parallel to $\mathbf{p}$, exactly as was the case in the simple model of de Broglie (cf. Sec. 1).

It is then not hard to show that in the state $|\Phi_p\rangle$ the projection of the total angular momentum on the direction of the momentum $\mathbf{p}$ is equal to the sum of the helicities of the neutrinos, i.e., that the orbital angular momentum of the neutrinos does not contribute to the helicity of the photon. And finally, using an argument about the helicity of the photon, we get from (12) the most general expression (on the assumption that photon states are two-particle states of non-interacting neutrinos) for the operators for photons with definite momentum and with right ($\varkappa$) and left (ω) circular polarizations:

$$\varkappa(\mathbf{p}) = \int_0^1 d\lambda A(\lambda, \tau) a(\lambda \mathbf{p}) a(\tau \mathbf{p}),$$

$$\omega(\mathbf{p}) = \int_0^1 d\lambda B(\lambda, \tau) b(\lambda \mathbf{p}) b(\tau \mathbf{p}). \quad (13)$$

Owing to the identity of the two neutrinos and the anticommutation of the two operators a(p) and b(p) the superposition coefficients $A(\lambda, \tau)$ and $B(\lambda, \tau)$ have the following symmetry properties:

$$A(\lambda, \tau) = -A(\tau, \lambda), \quad B(\lambda, \tau) = -B(\tau, \lambda),$$
$$A(\lambda, \lambda) = B(\lambda, \lambda) = 0. \quad (14)$$

According to assumption 4), there must exist operators $\xi(\mathbf{p})$ and $\eta(\mathbf{p})$ for photons with linear polarizations, which are linear combinations of $\varkappa(\mathbf{p})$ and $\omega(\mathbf{p})$:

$$\xi(\mathbf{p}) = \alpha\varkappa(\mathbf{p}) + \beta\omega(\mathbf{p}), \quad \eta(\mathbf{p}) = \gamma\varkappa(\mathbf{p}) + \delta\omega(\mathbf{p}). \quad (15)$$

From (8) and (9) we find

$$\gamma = -i\alpha, \quad \delta = i\beta. \quad (16)$$

It is not hard to verify that $\varkappa(\mathbf{p})$ and $\omega(\mathbf{p})$ as given by (13) satisfy the commutation relations (11), but there is still the condition (10). Substituting (15) in (10) and using (16), we get

$$|\alpha|^2[\varkappa(\mathbf{p}), \varkappa^+(\mathbf{p})]_- - |\beta|^2[\omega(\mathbf{p}), \omega^+(\mathbf{p})]_- = 0. \quad (17)$$

Equation (17) is the last requirement for compatibility of all the conditions 1)—5). We shall show that the relation (17) is possible only for $A(\lambda, \tau) = B(\lambda, \tau) = 0$. In fact, by simple manipulations using (14) and (7) we can get the result

$$|\alpha|^2[\varkappa(\mathbf{p}), \varkappa^+(\mathbf{p})]_- - |\beta|^2[\omega(\mathbf{p}), \omega^+(\mathbf{p})]_-$$
$$= 2\delta(0) \int_0^1 d\lambda \{|\alpha A(\lambda, \tau)|^2$$
$$- |\beta B(\lambda, \tau)|^2\} + 4|\alpha|^2 \int_0^1 d\lambda |A(\lambda, \tau)|^2 a^+(\lambda \mathbf{p}) a(\lambda \mathbf{p})$$
$$- 4|\beta|^2 \int_0^1 d\lambda |B(\lambda, \tau)|^2 b^+(\lambda \mathbf{p}) b(\lambda \mathbf{p}). \quad (18)$$

The operator (18) must give zero when applied to any state vector, and this requires that the coefficients of $n(\lambda\mathbf{p}) = a^+(\lambda\mathbf{p})a(\lambda\mathbf{p})$ and of $\bar{n}(\lambda\mathbf{p}) = b^+(\lambda\mathbf{p})b(\lambda\mathbf{p})$ vanish separately. We cannot take $|\alpha|^2 = |\beta|^2 = 0$, since this would contradict the assumption that linearly polarized photons exist, and consequently we arrive at the conclusion that $A(\lambda, \tau) = B(\lambda, \tau) = 0$.

We now go on to the general case, in which there are M different neutrino fields and a photon state is an N-particle neutrino state. First, it again turns out here that the momenta of all the neutrinos making up a photon are parallel, and the helicity of the photon is equal to the sum of the helicities of the neutrinos. This leads to the most general expression for the operators for right and left circularly polarized photons:

$$\varkappa(\mathbf{p}) = \int_s d^{N-1}\lambda A_\sigma(\{\lambda\}) a_i(\lambda_1\mathbf{p}) \ldots$$
$$\ldots a_k(\lambda_c\mathbf{p}) b_j(\lambda_{c+1}\mathbf{p}) \ldots b_l(\lambda_N\mathbf{p}), \quad (19)$$

$$\varkappa(\mathbf{p}) = \int_s d^{N-1}\lambda B_\sigma(\{\lambda\}) b_i(\lambda_1\mathbf{p})$$
$$\ldots b_k(\lambda_c\mathbf{p}) a_j(\lambda_{c+1}\mathbf{p}) \ldots a_l(\lambda_N\mathbf{p}) \quad (20)$$

Here the integration is over a bounded hypersurface S: $\sum_{i=1}^{N} \lambda_i = 1$, $0 \le \lambda_i \le 1$; σ is a set of indices i, ..., l numbering the neutrino fields; $\{\lambda\}$ is the set of variables of integration; and the superposition coefficients $A_\sigma(\{\lambda\})$ and $B_\sigma(\{\lambda\})$ are antisymmetric under simultaneous odd permutations of the indices of the fields and of the variables of integration. The expression (19) contains as factors n operators a and m operators b, and vice versa for (20): n + m = N, n − m = 2. In the case m = 0 the photon is a two-particle neutrino state. We then have $[\varkappa(\mathbf{p}), \omega^+(\mathbf{p})]_- = 0$, but $[\xi(\mathbf{p}), \eta^+(\mathbf{p})]_- \neq 0$. The proof is essentially the same as in the case we have treated, that of a single neutrino field. For $m \neq 0$ we even have $[\varkappa(\mathbf{p}), \omega^+(\mathbf{p})]_- \neq 0$.

We now go on to the case in which the neutrinos are parafermions. Here we shall confine ourselves to the treatment of two-particle neutrino states only. First let us assume that the maximum occupation number is $n_{max} = 2$ and the operators a(k) and b(k) in (13) satisfy the following commutation relations[12]:

$$c(\mathbf{k})c(\mathbf{k}')c(\mathbf{k}'') + c(\mathbf{k}'')c(\mathbf{k}')c(\mathbf{k}) = 0,$$
$$c^+(\mathbf{k})c(\mathbf{k}')c(\mathbf{k}'') + c(\mathbf{k}'')c(\mathbf{k}')c^+(\mathbf{k}) = \delta(\mathbf{k}-\mathbf{k}')c(\mathbf{k}''),$$

$$c(\mathbf{k})c^+(\mathbf{k}')c(\mathbf{k}'') + c(\mathbf{k}'')c^+(\mathbf{k}')c(\mathbf{k}) = \delta(\mathbf{k}-\mathbf{k}')c(\mathbf{k}'') + \delta(\mathbf{k}'-\mathbf{k}'')c(\mathbf{k}). \quad (21)$$

Here $c(\mathbf{k})$ means $a(\mathbf{k})$ or $b(\mathbf{k})$. It is not hard to see that

$$[\varkappa(\mathbf{p}), \varkappa(\mathbf{p}')]_- = [\omega(\mathbf{p}), \omega(\mathbf{p}')]_- = [\varkappa(\mathbf{p}), \omega(\mathbf{p}')]_- = 0.$$

But the condition for the existence of linearly polarized photons imposes another requirement: the commutators $[\kappa(\mathbf{p}), \kappa^+(\mathbf{p})]_-$ and $[\omega(\mathbf{p}), \omega^+(\mathbf{p})]_-$ must be c-numbers (this requirement is identical with the supplementary assumption with which the commutation relations of the usual statistics can be derived from the equations of motion[13]). We shall show that this requirement contradicts the commutation conditions (21). Using (21), we get

$$[\varkappa(\mathbf{p}), \varkappa^+(\mathbf{p})]_- = \int_0^1 d\lambda |A(\lambda,\tau)|^2 \{a(\lambda\mathbf{p})a^+(\lambda\mathbf{p}) - a^+(\tau\mathbf{p})a(\tau\mathbf{p})\}.$$

For $|A(\lambda, \tau)|^2 \neq 0$ the expression on the right cannot be a c-number, since for $n_{max} = 2$ an interchange in the product of two operators gives an expression which is linearly independent of the original expression, and therefore the entire expression in the right member is a sum of linearly independent operators.

For $n_{max} = 3$ the proof is analogous to that given above, and for $n_{max} > 3$

$$[\varkappa(\mathbf{p}), \varkappa^+(\mathbf{p})]_- = \int_0^1 d\lambda \int_0^1 d\lambda' A(\lambda,\tau)A^*(\lambda',\tau') \times [a(\lambda\mathbf{p})a(\tau\mathbf{p}), a^+(\tau'\mathbf{p})a^+(\lambda'\mathbf{p})]_-$$

cannot be a c-number, since an interchange in the product of four operators gives an expression linearly independent of the original one. The proof as given can be easily extended to the case in which there are two independent neutrino fields.

In the course of the proof of Pryce's theorem we have explicitly assumed that a photon state is a two-particle (or an N-particle) neutrino state. There also exists, however, a different assumption, which, following Pryce,[9] we can explain in the following way.

The photon is not a particle at all. All electromagnetic processes are in reality transitions of neutrinos from one state to another. For example, the process of absorption of a photon with momentum $\mathbf{p}$ consists in a transition of a neutrino from a state with momentum $\mathbf{k} + \mathbf{p}$ to a state with momentum $\mathbf{k}$. In such a model, however, there are difficulties with the vacuum. In fact, instead of (13) we would then have to write the operator for creation of a "photon" in a form such as

$$\varkappa^+(\mathbf{p}) = \int_0^1 d\lambda A(\lambda,\tau) b^+(\lambda\mathbf{p}) a(\tau\mathbf{p}).$$

With such a definition there cannot occur, for example, any emission of a photon into vacuum, since $\kappa^+(\mathbf{p})$ acting on the neutrino vacuum gives zero. Physically this means that if there are no neutrinos capable of accepting the energy and momentum of an excited atom emission of radiation is impossible. On the other hand ejection of neutrinos from states of negative energy means creation of neutrinos and antineutrinos—that is, in this case the photon is again identified with a two-particle neutrino state. To explain the emission of a photon by an atom in vacuum it is of course possible to assume that there is always some number of free neutrinos. But then they must solidly fill the lower levels (otherwise under any conditions there would always be observed photons, for example of visible light, in an absolutely light-tight room), and then the transfer of energy by a neutrino would again mean the production of a free neutrino and a "hole." In any case in such a model the probability of emission of radiation depends on the state of the field of free neutrinos.

3. PRYCE'S THEOREM AND PRESENT-DAY MODELS OF THE NEUTRINO THEORY OF PHOTONS

Discussions of the neutrino theory of photons have been given in a number of recent papers.[14-19]

Let us examine how Pryce's theorem, as proved here, extends to the constructions of Perkins[14] and of Ferretti and Venturi.[15]

Perkins considers essentially the ordinary four-component Dirac equation with m = 0, and from two neutrino operators in the configuration space he constructs in a special way the tensor of the electromagnetic field. For him the photon operators with right and left circular polarizations are then of the following form:

$$\varkappa(p) = \frac{i}{\sqrt{p}} \int_0^p c_2(k) a_1(p-k)\, dk, \quad (22)$$

$$\omega(p) = \frac{i}{\sqrt{p}} \int_0^p c_1(k) a_2(p-k)\, dk. \quad (23)$$

Here a_1 and c_2 are annihilation operators of the neutrino ν_1 and the antineutrino $\bar{\nu}_2$; a_2 and c_1 are the annihilation operators of the neutrino ν_2 and the antineutrino $\bar{\nu}_1$. The neutrinos ν_1 and ν_2 have different helicities. Accordingly, in our terminology, the Perkins model is a two-particle model in the case of two different neutrino fields, and (22) and (23) are essentially a special case of (13) with $A(\lambda, \tau) = B(\lambda, \tau) = ip^{-1/2}$.

The difficulty presented by Pryce's theorem in this model is that it is impossible to construct linearly polarized photons with the commutation relations (10) for their operators. The impossibility of constructing a linearly polarized photon could be interpreted as a loss of genuine neutrality of the photon, since in this case a superposition of right and left circularly polarized photon states can fail to have physical meaning, and then it is impossible to require the commutation relation (10). We shall, however, show at once that the additional assumption with which Perkins derives the Planck distribution for composite photons leads to relativistically noninvariant commutation relations.

Perkins assumes that the only momenta the neutrinos can have are multiples of a small quantity $\epsilon/2$: $k = \epsilon/2, 3\epsilon/2, 5\epsilon/2, \ldots, (2n + 1)\epsilon/2$. Then the operator for absorption of a photon with momentum ϵ can be written uniquely up to a coefficient [this can be understood directly, or can be seen from Eq. (22), where the integral becomes a sum, which in this case is a single term]: $\kappa(\epsilon) = c_2(\epsilon/2)a_1(\epsilon/2)$. It is easy to see that

$$\varkappa(\varepsilon)\varkappa(\varepsilon) = 0. \tag{24}$$

For the operator for a photon with momentum 2ϵ we have

$$\varkappa(2\varepsilon) = c_2(\varepsilon/2)a_1(3\varepsilon/2) + c_2(3\varepsilon/2)a_1(\varepsilon/2), \tag{25}$$

$$\varkappa(2\varepsilon)\varkappa(2\varepsilon)\varkappa(2\varepsilon) = 0, \quad \varkappa(2\varepsilon)\varkappa(2\varepsilon) \neq 0. \tag{26}$$

When we apply to (24) the operation of the Lorentz transformation corresponding to the change $\epsilon \rightarrow 2\epsilon$, we get $\kappa(2\epsilon)\kappa(2\epsilon) = 0$, which contradicts (26).

The physical meaning of the proof we have given is obvious. According to (24), (25), and (26), the maximum number of photons that can be in a state with momentum ϵ is one, and for momentum 2ϵ the number is two. Such a condition is not relativistically invariant, since we can always find a Lorentz reference system in which the former state is characterized by the momentum 2ϵ, and consequently can be occupied by two photons, not only one.

To overcome the difficulty connected with Pryce's theorem, Ferretti and Venturi[15] propose considering a system of two noninteracting neutrinos with total angular momentum $m > 1$. They write out the equation for the noninteracting neutrinos and antineutrinos in configuration space (the two-component theory is considered):

$$\delta_{ki}\delta_{\dot{k}'l'}\frac{\partial}{\partial x_0}\varphi^{\dot{i}l'} = \delta_{\dot{k}'l'}\sigma_{(r)ki}\frac{\partial}{\partial x_r}\varphi^{\dot{i}l'} + \delta_{ki}\sigma^{+}_{(r)\dot{k}'l'}\frac{\partial}{\partial x_r'}\varphi^{\dot{i}l'}. \tag{27}$$

Here $\varphi^{\dot{l}l'}(x, x')$ is the wave function of the neutrino-antineutrino system; primed quantities relate to the neutrino and dotted ones to the antineutrino; $\sigma_{(r)}$ are the Pauli matrices; $l, l' = 1, 2$; $r = 1, 2, 3$. The authors of[15] find the solution of Eq. (27) with definite total momentum p and with the value m for the projection of the angular momentum along the direction of the momentum:

$$\varphi^{\dot{i}l'}_{p,m}(x, x') = f^{\dot{i}l'}(x_1 - x_1', x_2 - x_2')$$
$$\times \exp\left\{-i\left[\left(\frac{p}{2}+k\right)x_0 + \left(\frac{p}{2}-k\right)x_0'\right] + i\left[\left(\frac{p}{2}+k\right)x_3 + \left(\frac{p}{2}-k\right)x_3'\right]\right\};$$

$$f^{\dot{1}1'} = 0, \quad f^{\dot{1}2'} = 0, \quad f^{\dot{2}1'} = -C\frac{2im}{p-2k}(x_- - x_-')^{m-1},$$
$$f^{\dot{2}2'} = C(x_- - x_-')^m \quad (m > 0);$$
$$f^{\dot{1}1'} = 0, \quad f^{\dot{1}2'} = C'\frac{2i|m|}{p+k}(x_+ - x_+')^{|m|-1},$$
$$f^{\dot{2}1'} = 0, \quad f^{\dot{2}2'} = C'(x_+ - x_+')^{|m|} \quad (m < 0), \tag{28}$$

where $x_- = x_1 - ix_2$, $x_+ = x_1 + ix_2$, and C and C′ are constants.

It must be noted, however, that $\varphi^{\dot{l}l'}(x, x')$ is a solution of (27) only for $f^{\dot{1}2'} = f^{\dot{2}1'} = 0$. This is physically understandable and is verified by direct substitution of (28) in (27).

Let us see whether in the problem as stated there can exist a physically meaningful solution $\varphi^{\dot{l}l'}_{p,m}(x, x')$ with a value $m > 1$ for the projection of the total angular momentum along the momentum. In this case the connection between the helicity of the composite photon and the helicities of the neutrinos, which was used in the proof of Pryce's theorem, corresponds to the following easily proved statement.

If a function $\varphi(x, x')$ (a scalar for simplicity) has the following properties

$$-i\left(\frac{\partial}{\partial x_\mu} + \frac{\partial}{\partial x_\mu'}\right)\varphi(x, x') = p_\mu\varphi(x, x');$$

$$p_\mu^2 = 0, \quad \mu = 1\ 2, 3, 4, \tag{29}$$

$$\frac{\partial^2}{\partial x_\mu^2}\varphi(x, x') = \frac{\partial^2}{\partial x_\mu'^2}\varphi(x, x') = 0 \tag{30}$$

and $\varphi(x, x')$ can be expressed as a Fourier integral,

$$\varphi(x, x') = \int d^4k \int d^4k' f(k, k') e^{i(kx+k'x')}, \qquad (31)$$

then $p_j L_j \varphi(x, x') = 0$, where $L_j = -ie_{jlm}(x_l \partial/\partial x_m + x'_l \partial/\partial x'_m)$; e_{jlm} is the unit antisymmetric tensor; j, l, m = 1, 2, 3. The Ferretti-Venturi function

$$\varphi_m(x, x') = C(x_- - x_-')^m \times \exp\left\{-i\left[\left(\frac{p}{2}+k\right)x_0 + \left(\frac{p}{2}-k\right)x_0'\right] + i\left[\left(\frac{p}{2}+k\right)x_3 + \left(\frac{p}{2}-k\right)x_3'\right]\right\}$$

satisfies the requirements (29) and (30), but nevertheless for it we have

$$p_j L_j \varphi(x, x') = m\varphi(x, x').$$

The point is that $\varphi_m(x, x')$ does not obey the condition (31). To be expressible as a Fourier integral it should obey the condition of absolute convergence

$$\int d^2(x - x') \, |\varphi_m(x, x')| < \infty. \qquad (32)$$

It is easy to see that the $\varphi_m(x, x')$ of Eq. (28) does not satisfy (32) for any m (positive or negative) other than m = 0.

Accordingly, the case considered in[15] is one in which the particle is described by a wave function in configuration space but no wave function in momentum space exists. Besides this, the space of the solutions that are found is not a Hilbert space. It must be noted that Ferretti and Venturi[15] did encounter this difficulty in going over to momentum space in a nonexplicit form, and owing to this the relation (8) in their paper is incorrect. But even if we forget this objection for the time being, it unfortunately is still difficult to accept the proof of the commutation relations in[15] as convincing. In fact, the state vector which describes m_1 right circularly polarized photons and m_1' left circularly polarized photons with momentum $\mathbf{q}_1$, m_2 right circularly polarized and m_2' left circularly polarized photons with momentum $\mathbf{q}_2$, and so on, is defined by

$$|m_1, m_1', m_2, m_2' \ldots m_n, m_n'\rangle = B(m_1, m_1', \mathbf{q}_1) B(m_2, m_2', \mathbf{q}_2) \ldots \ldots B(m_n, m_n', \mathbf{q}_n) |0\rangle,$$

where $B(m, m', \mathbf{q}) = A(m, \mathbf{q}) A(-m', \mathbf{q})$; $A(m, \mathbf{q})$ is the operator for creation of the two-particle state (neutrino and antineutrino) in which the total momentum of the two particles is $\mathbf{q}$ and the projection of the total angular momentum along the momentum is m. This two-particle neutrino state is taken for the m-particle photon state describing m right circularly polarized photons. Therefore the vector $|m_1, m_1', \ldots, m_n, m_n'\rangle$ describes a 4n-particle neutrino state and the vector $A(1, \mathbf{q})|m_1, m_1', \ldots, m_n, m_n'\rangle$ describes a (4n + 2)-particle neutrino state. Accordingly the matrix element

$$\langle \bar{m}_1, \bar{m}_1', \ldots, \bar{m}_n, \bar{m}_n' | A(1, \mathbf{q}_n) | m_1, m_1', \ldots, m_n, m_n' \rangle$$

is always equal to zero, a result different from Eq. (10) of[15].

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