FORM FACTORS AND MULTIPOLES IN ELECTROMAGNETIC INTERACTIONS

V. I. DUBOVIK and A. A. CHESHKOV[1)]

Joint Institute for Nuclear Research

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The electric and magnetic multipoles of the first and second kind are determined for arbitrary spin particles. The electromagnetic current operator is parametrized in terms of the particle multipoles and their mean 2n-degree distribution radii. The properties of the current operators under conditions when P and T invariance is imposed are discussed.

1. INTRODUCTION

IN quantum field theory, the electrodynamic properties of a particle are fully determined by its electromagnetic form factors, which are introduced in the parametrization of the matrix element of the current operator. On the basis of the general method of parametrization of matrix elements of local operators, we discuss in this paper in detail the consequences of the conditions imposed on the current operator by the requirements of hermiticity and gauge invariance. The question is clarified of what form factors enter into the electromagnetic current in the case of P- and T-invariance, and what form factors are excluded by these conditions. The form factors of one-particle matrix elements of the current operator are directly expressed in terms of multipole electromagnetic moments of the particle and their mean 2n-degree distribution radii.

A definition is given of a new family of electromagnetic multipole moments (completely forbidden under P- and T-invariance of the current)—magnetic multipoles of the second kind.[1] The first moment in this family is that introduced by Zel'dovich[2] and called anapole $\mathbf{M}^{\mathrm{II}}$—the magnetic dipole of the second kind:

$$\mathbf{M}^{\mathrm{II}} = \int [\mathbf{I}\mathbf{r}^2 - \mathbf{r}(\mathbf{I}\mathbf{r})]\, d^3x.$$

A thoroidal current is an example of a classical system which possesses such a magnetic moment. In the quantum case magnetic multipoles of the second kind of odd order (dipole, octupole, etc.) are forbidden by the P-invariance of the current, and those of even order (quadrupole, etc.) are forbidden by T-invariance of the current. Thus if T-invariance[3] is violated in electromagnetic interactions while P-invariance is preserved, then particles with spin $j \geq 1$ may possess magnetic moments of the second kind of even order.

2. PARAMETRIZATION OF THE CURRENT

Making use of the general method for the parametrization of matrix elements of local operators,[4] we write the matrix element of the electromagnetic current, taken in the Breit system (b.s.) with respect to states with masses κ, κ' and spins j, j′ (m, m′—spin projections on the z axis) in the form

$$\langle \mathbf{q}/2, \varkappa', j', m' | I_\mu(x) | -\mathbf{q}/2, \varkappa, j, m\rangle = \frac{1}{(2\pi)^3} e^{-iqx} \langle \mathbf{q}/2, \varkappa', j', m' | I_\mu(0) | -\mathbf{q}/2, \varkappa, j, m\rangle, \quad (1)$$

$$\langle \mathbf{q}/2, \varkappa', j', m' | I_0(0) | -\mathbf{q}/2, \varkappa, j, m\rangle = e\sqrt{4\pi} \sum_{L,M} (-i)^L \frac{[L]^{1/2}}{[L]!!} q^L Y_{LM}^*(\mathbf{n}_q) \langle jmLM | j'm'\rangle \varphi_{jj'}^{0L}(q^2, \varkappa, \varkappa'), \quad (2)$$

$$\langle \mathbf{q}/2, \varkappa', j', m' | I_i(0) | -\mathbf{q}/2, \varkappa, j, m\rangle = e\sqrt{4\pi} \sum_{J,\Lambda,L,M,\mu} \alpha_{i\mu} (-i)^{J+1} \frac{[L]^{1/2}}{[L]!!} \langle 1\mu LM | J\Lambda\rangle \times \langle jmJ\Lambda | j'm'\rangle q^L Y_{LM}^*(\mathbf{n}_q) \varphi_{jj'}^{1LJ}(q^2, \varkappa, \varkappa'), \quad (3)$$

where $\alpha_{i\mu}$ is the transformation matrix from the orthogonal basis to the canonical, $[L] \equiv 2L + 1$.

In our case, when the matrix element (1) is not diagonal in the masses and spins, the requirement of hermiticity of the current operator leads to the following conditions for each form factor:

$$\varphi_{jj'}(q^2, \varkappa, \varkappa') = (-1)^{j-j'} ([j]/[j'])^{1/2} \varphi^*_{j'j}(q^2, \varkappa', \varkappa). \quad (4)$$

The law of current conservation gives the relation

*Moscow Aviation Institute.

$$(-1)^L(^8/_{3\pi}L[L-1])^{1/2}$$

$$\times\left\{\varphi_{jj'}^{1,L-1,L}-q^2\varphi_{jj'}^{1,L+1,L}\frac{1}{[L]}\left(\frac{(L+1)/L}{[L+1][L-1]}\right)^{1/2}\right\}$$

$$=-iq_0\varphi_{jj'}^{0L}, \qquad (5)$$

$$q_0=(\mathbf{q}^2/4+\varkappa'^2)^{1/2}-(\mathbf{q}^2/4+\varkappa^2)^{1/2}.$$

With Eq. (5) taken into account, the vector part of the current may be written in the form

$$\langle \mathbf{q}/2,\varkappa',j',m'|I_\mu^{(1)}(0)|-\mathbf{q}/2,\varkappa,j,m\rangle$$

$$=\sqrt{4\pi}\sum_{J,\Lambda,M}\frac{(-i)^{J+1}}{[J]!!}\langle jmJ\Lambda|j'm'\rangle$$

$$\times\{\langle 1\mu JM|J\Lambda\rangle[J]^{1/2}q^JY_{JM}^*(\mathbf{n}_q)\varphi_{jj'}^{1JJ}$$

$$+q^{J+1}[J+1]^{-1/2}[\langle 1\mu J-1M|\Lambda\rangle Y_{J+1,M}^*(\mathbf{n}_q)$$

$$+\sqrt{(J+1)/J}\langle 1\mu J-1M|J\Lambda\rangle Y_{J-1,M}(\mathbf{n}_q)]\varphi_{jj'}^{1;J+1,J}$$

$$-iq_0q^{J-1}(-1)^J[J](3/8\pi J)^{1/2}$$

$$\times\langle 1\mu J-1M|J\Lambda\rangle Y_{J-1,M}^*(\mathbf{n}_q)\varphi_{jj'}^{0J}\} \qquad (5a)$$

P-invariance of the current operator imposes additional conditions on the form factors:

$$\varphi_{jj'}^{0L}(q^2,\varkappa,\varkappa')=0 \quad \text{for } L=2n+1,$$

$$\varphi_{jj'}^{1,L\pm1,L}(q^2,\varkappa,\varkappa')=0 \quad \text{for } L=2n+1,$$

$$\varphi_{jj'}^{1LL}(q^2,\varkappa,\varkappa')=0 \quad \text{for } L=2n. \qquad (6)$$

The requirement of T-invariance leads to the following inequalities:

$$\varphi_{jj'}^{0L}(q^2,\varkappa,\varkappa')=(-1)^{L+j-j'}([j]/[j'])^{1/2}\varphi_{j'j}^{0L}(q^2,\varkappa',\varkappa),$$

$$\varphi_{jj'}^{1;L\pm1,L}(q^2,\varkappa,\varkappa')=(-1)^{1+L+j-j'}$$

$$\times([j]/[j'])^{1/2}\varphi_{j'j}^{1,L\pm1,L}(q^2,\varkappa',\varkappa),$$

$$\varphi_{jj'}^{1LL}(q^2,\varkappa,\varkappa')=(-1)^{1+L+j-j'}([j]/[j'])^{1/2}\varphi_{j'j}^{1LL}(q^2,\varkappa',\varkappa) \qquad (7)$$

It is clear from the above formula that if T-invariance is left out the form factors of the nondiagonal matrix element of the current operator are complex, in other words the number of form factors is doubled. Thus j = j′ = 0 for scalar particles, but $\kappa\neq\kappa'$, the form factor for the "mixed charge"[3] acquires an imaginary part, and for spinor particles an additional part appears also for the form factor of the "mixed dipole magnetic moment." These additions will be responsible for decays with violation of T-invariance of the corresponding particles (see, for example, [3]).

3. FORM FACTORS AND MULTIPOLES

The form factors of the one-particle matrix element of the current operator are related in a natural way to the multipole moments of the particle. To this end we change the normalization of the form factors in Eq. (2) and (3):

$$\varphi_{jj}{}^{0L}(q^2,\varkappa,\varkappa)=f^{0L}(q^2)/\langle jjL0|jj\rangle,$$

$$\varphi_{jj}^{1LJ}(q^2,\varkappa,\varkappa)=f^{1LJ}(q^2)/\langle jjJ0|jj\rangle \qquad (8)$$

The requirement of hermiticity for the current operator now makes the form factors real. The current conservation condition gives rise to the relation between form factors

$$f^{1,L-1,L}=q^2f^{1,L+1,L}\frac{1}{[L]}\left(\frac{(L+1)/L}{[L+1][L-1]}\right)^{1/2}. \qquad (9)$$

Therefore

$$\langle \mathbf{q}/2,\varkappa,j,m'|I_\mu^{(1)}(0)|-\mathbf{q}/2,\varkappa,j,m\rangle$$

$$=e\sqrt{4\pi}\sum_{J=1}^{\infty}\frac{(-i)^{J+1}\langle jmJ\Lambda|jm'\rangle}{[J]!!\quad\langle jjJ0|jj\rangle}$$

$$\times\Big\{\langle 1\mu JM|J\Lambda\rangle[J]^{1/2}q^JY_{JM}^*(\mathbf{n}_q)f^{1JJ}(q^2)$$

$$+\frac{q^{J+1}}{[J+1]^{1/2}}\Big[\langle 1\mu J+1M|J\Lambda\rangle Y_{J+1,M}^*(\mathbf{n}_q)$$

$$+\sqrt{\frac{J+1}{J}}\langle 1\mu J-1M|J\Lambda\rangle Y_{J-1,M}^*(\mathbf{n}_q)\Big]f^{1,J+1,J}(q^2)\Big\} \qquad (10)$$

At $q^2=0$ the form factors $f^{0L}(q^2)$ and $f^{1LL}(q^2)$ determine respectively the L-th order electric moments Q_L and the conventional magnetic moments $\mathscr{M}_L^I$:

$$\langle \mathbf{q}/2,\varkappa,j,m'|\hat{Q}_{LM}|-\mathbf{q}/2,\varkappa,j,m\rangle$$

$$=\sqrt{\frac{4\pi}{[L]}}\int d^3x x^LY_{LM}(\mathbf{n}_x)\langle|I_0(x)|\rangle=\frac{\langle jmLM|jm'\rangle}{\langle jjL0|jj\rangle}Q_L, \qquad (11)$$

$$Q_L=ef^{0L}(0); \qquad (11a)$$

$$\langle \mathbf{q}/2,\varkappa,j,m'|\hat{\mathscr{M}}^I_{LM}|-\mathbf{q}/2,\varkappa,j,m\rangle$$

$$=i\sqrt{\frac{4\pi}{[L]}}\sum_{M',\mu}\int d^3x\cdot x^LY_{LM'}(\mathbf{n}_x)\langle|I_\mu^{(1)}(x)|\rangle\langle 1\mu LM'|LM\rangle$$

$$=\frac{\langle jmLM|jm'\rangle}{\langle jjL0|jj\rangle}\mathscr{M}_L{}^I, \qquad (12)$$

$$\mathscr{M}_L^{\mathrm{I}} = ef^{0LL}(0). \qquad (12a)$$

It follows from Eq. (5a) and (10) that the electric and magnetic 2L-th order form factors do not completely determine the current operator.

In analogy with $f^{1LL}(q^2)$, the form factors $f^{1,\,L+1,\,L}(q^2)$ may be called 2^L-th order magnetic form factors of the second kind, unlike the conventional 2^L-th order magnetic form factors (which in the following will be referred to as magnetic form factors of the first kind). The values of $f^{1,\,L+1,\,L}(q^2)$ at $q^2 = 0$ will be called the L-th order magnetic moments of the second kind $\mathscr{M}_L^{\mathrm{II}}$:

$$\langle \mathbf{q}/2, \varkappa, j, m' | \hat{\mathscr{M}}_{LM}^{\mathrm{II}} | -\mathbf{q}/2, \varkappa, j, m\rangle = \left(\frac{4\pi}{[L+1]}\right)^{1/2} \sum \int d^3x x^{L+1} Y_{L+1,\,M'}$$

$$\times(\mathbf{n}_x)\langle | I_\mu^{(1)}(x) | \rangle \langle 1_\mu L+1M' | LM\rangle = \frac{\langle jmLM | jm'\rangle}{\langle jjL0 | jj\rangle} \mathscr{M}_L^{\mathrm{II}}, \qquad (13)$$

$$\mathscr{M}_L^{\mathrm{II}} = ef^{1,\,L+1,\,L}(0). \qquad (13a)$$

The average values of the 2n-degree distribution radii of the electric multipole will be written in the form

$$\langle \mathbf{q}/2, \varkappa, j, m' | \hat{r}_{LM}^{2n} | -\mathbf{q}/2, \varkappa, j, m\rangle = \left(\frac{4\pi}{[L]}\right)^{1/2} \int d^3x \cdot x^{2n+L} Y_{LM}(\mathbf{n}_x) \langle | I_0(x) | \rangle = \frac{\langle jmLM | jm'\rangle}{\langle jjL0 | jj\rangle} [n]! \overline{r_L{}^{2n}}, \qquad (14)$$

$$\overline{r_L{}^{2n}} = \frac{e}{n!} [f^{0L}(q^2)]_{q=0}^{(n)}. \qquad (14a)$$

The corresponding formulae for the magnetic multipoles are identical to the above

$$\langle | \hat{\rho}_{\mathrm{I}LM}^{2n} | \rangle = e \frac{\langle jmLM | jm'\rangle}{\langle jjL0 | jj\rangle} \frac{[n]!}{n!} [f^{1LL}(q^2)]_{q=0}^{(n)}, \qquad (14b)$$

$$\langle | \hat{\rho}_{\mathrm{II}LM}^{2n} | \rangle = e \frac{\langle jmLM | jm'\rangle}{\langle jjL0 | jj\rangle} \frac{[n]!}{n!} [f^{1,\,L+1,\,L}(q^2)]_{q=0}^{(n)}. \qquad (14c)$$

Thus the electromagnetic form factors are directly expressible in terms of multipoles and their mean 2n-degree distribution radii. Thus the electric form factor has the form

$$ef^{0L}(q^2) = Q_L + \sum_{n=1}^{\infty} (-q^2)^n \overline{r_L{}^{2n}}. \qquad (15)$$

In the case of the one-particle matrix element of the current operator, the requirement of P-invariance gives the same conditions (6). The requirement of T-invariance gives now rise to the conditions

$$f^{0L}(q^2) = 0 \quad \text{for } L = 2n+1,$$

$$f^{1,\,L\pm1,\,L}(q^2) = 0 \quad \text{for } L = 2n,$$

$$f^{1LL}(q^2) = 0 \quad \text{for } L = 2n. \qquad (16)$$

4. ELECTROMAGNETIC INTERACTION OF THE MAGNETIC MULTIPOLES OF THE SECOND KIND

It is seen from conditions (6) and (16) that only the magnetic multipoles of the second kind are T-noninvariant characteristics of the particle (under conditions of P-invariance). Particles with spin $j \geq 1$ may possess such multipoles. Thus if the hypothesis on the violation of T-invariance in electromagnetic interactions of hadrons is correct, they will contribute to the cross section for elastic scattering of electrons by particles with spin $j \geq 1$.[5,6] Since magnetic multipoles of the second kind have the parity of electrical multipoles of the same order, they will contribute to radiation of the electric type. At that the intensity of the radiation of the 2^L-th order magnetic moment of the second kind is by two orders of v/c smaller than the 2^L-th order electric moment. However, the static multipole moments of the second kind do not give rise to a magnetic field and do not interact with it,[1] as is indicated by the additional power of q in the form factor of the 2^L-th order magnetic moment of the second kind in the decomposition of the current in multipole moments, Eq. (10).

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