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NONLINEAR MOTION OF A PLASMA WITH AN ARBITRARY ELECTRON VELOCITY DISTRIBUTION

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One-dimensional finite-amplitude motion is investigated in a nonequilibrium plasma consisting of hot electrons and cold ions. We consider stationary waves and simple nonstationary multistream flows (Riemann or simple waves). It is shown that solitary waves of two kinds are possible, depending on the electron velocity distribution: one is a solitary compression wave in which the particle density and electrostatic potential increase, and the other is a solitary rarefaction wave in which these quantities are reduced. In the case of electron distributions that lead to solitary rarefaction waves, the reflection of electrons from the potential barrier that is formed can lead to the formation of a rarefaction quasi-shock wave (in contrast with the compressional quasi-shock wave in a plasma with a Maxwellian distribution). The sign of the variation in the characteristic plasma quantities in a multistream simple wave is determined. It is shown that two cases are possible, depending on the initial electron velocity distribution: one is the normal case in which the discontinuities (or additional ion streams) are formed in the compression regions; the other is an anomalous case in which the discontinuities appear in the rarefaction regions.

INTRODUCTION

It is well known that a collisionless plasma consisting of hot electrons and cold ions can support the propagation of stationary solitary waves, which are uniformly moving symmetric potential hills or wells. Some of the ions in front of the crest of such a wave can be reflected from the potential barrier, leading to the formation of a so-called quasi-shock wave with an oscillatory structure.[1] Quasi-shock waves and solitary waves in a plasma with no magnetic field have been investigated in detail by Moiseev and Sagdeev.[2] In that work, as in all other work known to the present authors, it was assumed that the electron velocity distribution was Maxwellian.

In the present work we investigate solitary waves and quasi-shock waves in a plasma in the general case in which the electron velocity distribution is arbitrary. It will be shown that, depending on the nature of the electron distribution function, the solitary wave can be either a uniformly moving potential hill or a uniformly moving potential well. In the case in which the electron distributions result in a solitary wave in the form of a depression, the rarefaction quasi-shock wave arises by virtue of the reflection of electrons from the potential barrier (rather than ions, as in the usual case). A wave of this kind might be called a rarefaction quasi-shock wave (in contrast from the compressional quasi-shock wave which arises in the case of a Maxwellian electron distribution).

The question of quasi-shock waves and solitary waves is closely related to the question of multistream plasma flows. In particular, in the case of a compressional quasi-shock wave the ions in front of the shock can form three streams. In the case of a solitary rarefaction wave it is possible to have a triple stream flow within the perturbed region, this flow being produced by trapped ions.

In this connection it is also of interest to investigate nonstationary multistream flows in a two-temperature plasma. We shall investigate simple waves in a plasma of this kind because, as is well known, we can trace the evolution of an initial perturbation; in particular, it is possible to establish the conditions under which discontinuities arise. The investigation of simple waves is of interest in its own right since only a region of simple waves (in the absence of discontinuities) can be contiguous to the unperturbed plasma.

A distinguishing feature of a plasma with several ion streams is the possibility of unstable ion-acoustic waves. We shall show that in this case there are two possibilities for the simple waves. Either the wave moves away from the spatial boundary of the instability region in the course of time, or a discontinuity develops at this boundary.

1. SIMPLE WAVES

We consider first the nonlinear multistream flow of a collisionless plasma consisting of cold ions and hot electrons. The system of equations that describe this plasma is

$$\left\{\frac{\partial}{\partial t}+\mathbf{v}\frac{\partial}{\partial \mathbf{r}}+\frac{e}{m}\frac{\partial \varphi}{\partial \mathbf{r}}\frac{\partial}{\partial \mathbf{v}}\right\}F=0,$$

$$\left(\frac{\partial}{\partial t}+\mathbf{u}_j\frac{\partial}{\partial \mathbf{r}}\right)\mathbf{u}_j+\frac{ez_j}{M_j}\frac{\partial \varphi}{\partial \mathbf{r}}=0, \quad \frac{\partial n_j}{\partial t}+\operatorname{div}(n_j\mathbf{u}_j)=0,$$

$$\Delta\varphi=4\pi e\left(\int F d\mathbf{v}-\sum_j z_j n_j\right), \tag{1}$$

where $F(\mathbf{v})$ is the electron distribution function, n_j and u_j are the density and hydrodynamic velocity of ions of species j, and φ is the electrostatic potential (m, M_j, $-e$, and ez_j are the mass and charge of the electron and ion respectively).

We shall be interested in low-frequency waves whose phase velocity is small compared with the mean electron thermal velocity, and limit our analysis to one-dimensional plasma motion; in this case (1) reduces to

$$\left(\frac{\partial}{\partial x}+\frac{e}{m}\frac{\partial \varphi}{\partial x}\frac{\partial}{v_x\partial v_x}\right)F=0, \quad D=\int F d\mathbf{v}; \tag{2}$$

$$\left(\frac{\partial}{\partial t}+u_j\frac{\partial}{\partial x}\right)u_j+\frac{ez_j}{M_j}\frac{\partial \varphi}{\partial x}=0, \quad \frac{\partial n_j}{\partial t}+\frac{\partial}{\partial x}(n_j u_j)=0,$$

$$\frac{\partial^2\varphi}{\partial x^2}-4\pi e\left(D-\sum_j z_j n_j\right)=0 \tag{3}$$

(the x axis is taken in the direction of wave propagation; here and below we omit the subscript x on the velocity component u_x). Integrating (2) and introducing the notation $F(v_x^2; \mathbf{v}_t) = F(\mathbf{v})$ we have

$$F(v_x^2; \mathbf{v}_t; x, t)=F\left(v_x^2-\frac{2e}{m}\varphi(x,t); \mathbf{v}_t\right),$$

$$D(\varphi)=\int F d\mathbf{v}. \tag{4}$$

The relation in (3) together with the equation of state $D = D(\varphi)$, which connects the electron density D with the potential φ, comprise a complete system of equations for describing low-frequency one-dimensional oscillations of a plasma. These equations hold for arbitrary electron velocity distributions so long as the mean energy of the electrons is appreciably greater than the mean ion energy; depending on the form of the electron distribution function the equation of state can assume different forms.

In cases of large-scale motion, in which the characteristic scale for changes in plasma parameters is large compared with the Debye radius, we can neglect the first term in the last equation in (3). Under these conditions (3) allows solutions in the form of simple waves which correspond to motion of the plasma in which perturbations of all quantities propagate with the same velocity; in other words, in this case each of the functions X ($X \equiv \varphi$, n_j, u_j, D, $F(\mathbf{v})$) satisfies an equation of the form

$$\left\{\frac{\partial}{\partial t}+V(x,t)\frac{\partial}{\partial x}\right\}X=0.$$

It is well known that for simple waves all the quantities X can be written in terms of functions of any one of them (for example φ) which, in turn, is a function of x and t. Under these conditions (3) becomes a system of ordinary differential equations for the functions $n_j(\varphi)$ and $u_j(\varphi)$:

$$\frac{dn_j}{d\varphi}=\frac{ez_j}{M_j}\frac{n_j}{(V-u_j)^2}, \quad \frac{du_j}{d\varphi}=\frac{ez_j}{M_j(V-u_j)},$$

$$D=\sum_j z_j n_j, \tag{5}$$

in which the phase velocity $V(\varphi)$ is determined from the compatibility condition for solving this system, that is to say, from the equation

$$\Phi(V)=1, \tag{6}$$

where

$$\Phi(V)=\sum_j S_j^2(V-u_j)^{-2}, \quad S_j^2=\frac{ez_j^2 n_j}{M_j D_1} \tag{7}$$

(we use the notation $D_n \equiv d^nD/d\varphi^n$).

The dispersion equation (6) is an algebraic equation of degree 2N (N is the number of ion streams) and has 2N roots of which two, the largest and the smallest, are always real, while the other $2(N-1)$ can be real or complex. (A plot of the function $\Phi(V)$ is shown in Fig. 1.)

In the range of values of the characteristic plasma quantities in which all 2N roots of the dispersion equation are real, the ion acoustic waves are stable. Under these conditions the plasma can support the propagation of simple waves of 2N kinds, characterized by various phase velocities $V_{\pm j}$ ($j = 1, \ldots, N$). (We use the notation $V_{\pm j}$ to denote roots of the dispersion equation close to

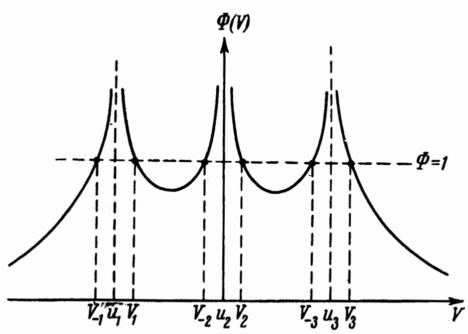

FIG. 1.

u_j, $V_{-j} < u_j < V_j$.) In the compressional regions for any of the simple waves the potential φ increases, as do the electron density and the ion density assigned to each of the N streams; in the rarefaction regions the quantities φ, D, and n_j are reduced.

We note that if the phase velocity of any of the waves is close to the velocity u_j, the ions in the j-th stream, as is evident from (5), will experience a resonance interaction with this wave.

Let us now determine how the shape of the acoustic wave changes as it propagates in the plasma. Using (5) and (6) we find

$$\frac{dV}{d\varphi} = \frac{D_1}{D}\frac{3G - DD_2D_1^{-2}}{(-\partial\Phi/\partial V)}, \qquad (8)$$

$$G = D\sum_j S_j^4 (z_j n_j)^{-1}(V - u_j)^{-4}. \qquad (8')$$

The denominator in (8) is always positive (negative) for a wave with phase velocity V_j (V_{-j}). However, the numerator can be either positive or negative, depending on the nature of the electron distribution function, and consequently on the form of the equation of state $D = D(\varphi)$.

It is not difficult to show that $G \geq 1$; hence, for electron distributions in which $DD_2D_1^{-2} < 3$ the numerator in (8) is always positive. This is the case, in particular, for a Maxwellian electron velocity distribution.[1] In order to analyze the way in which the profile changes for a wave moving with velocity V_j or V_{-j} it will be found convenient to transform to a coordinate system in which ions of the j-th species are at rest. In this reference system points with high density move in both waves with a higher absolute velocity. Hence, as in ordinary hydrodynamics, the discontinuities arise in the compression regions;[2] the self-similar waves are the rarefaction waves (normal case).

There are also, however, classes of distribution functions for which $DD_2D_1^{-2} > 3$ so that the numerator in (8) can be negative. In these cases the discontinuities do not arise in the compression regions, as in the normal case, but in the rarefaction regions; the compression waves are then the self-similar waves (anomalous case). This is the situation, in particular, for distributions characterized by basic velocity regions of the form $(v^2 + v_0^2)^{-\alpha}$ where $3/2 < \alpha < 2$.

We emphasize that in the normal (anomalous) case the discontinuities arise in those regions of the wave $V_{\pm j}$ which are regions of compression (rarefaction) from the point of view of an observer moving with the j-th ion stream. Obviously the concept of regions of compression (rarefaction) is not invariant with respect to transformation to a moving reference system. For example, an observer moving with a velocity greater than V_j (in particular, moving with the (j + 1)-th ion stream) will see the regions of compression as regions of rarefaction and vice versa.

If the dispersion equation (6) has complex roots, the ion acoustic waves become unstable. It is evident that as the boundary of the stability region is approached the phase velocities for any two types of waves approach each other $V_{j-1} \to V_{-j} \to V_c$; the boundary of the stability region and the critical phase velocity are determined from the equations

$$\Phi(V_c) = 1, \qquad \Phi'(V_c) = 0. \qquad (9)$$

(The primes on the function Φ denote differentiation with respect to V.)

In particular, in a plasma containing two ion streams it can be shown from (9) that the stability condition is

$$(u_1 - u_2)^2 \geqslant u_c^2, \quad u_c = (S_1^{2/3} + S_2^{2/3})^{3/2}, \qquad (10)$$

where, near the stability boundary,

$$V_{1,-2} = V_c \mp \frac{1}{\sqrt{3}}(u_c S_1 S_2)^{1/3}\left\{\left(\frac{u_1 - u_2}{u_c}\right)^2 - 1\right\}^{1/2},$$

$$V_c = \frac{u_1 S_2^{2/3} + u_2 S_1^{2/3}}{u_c^{2/3}}. \qquad (11)$$

[1]It has been pointed out to the authors that simple waves in a plasma with several ion streams have been considered for the case of a Maxwellian electron distribution in [3].

[2]We shall use the term "discontinuity" to mean a narrow region in which the gradients of the characteristic plasma quantities become so large that the original equations (5) no longer apply and in which it becomes necessary to take account of the first term in (3). If the values of the gradients increase still further in these regions it is possible for additional ion streams ([4]) or shock waves to appear.

In order to investigate the evolution of a wave close to the boundary of the stability region, we note that the quantity $w = \pm\Phi'(V_{\pm j})$, which is positive in the stability region and vanishes at the boundary, is a function of $x - Vt$ as are all the quantities in the simple wave. Hence, an instability will not arise in a simple wave if the plasma is stable over all space at the initial time; the spatial boundary of the instability region $x = x_c(t)$ is determined from the condition $w(x_c(t), t) = 0$ which moves along the characteristic.

Assuming that

$$\frac{dw}{d\varphi} = \pm\Phi''(V_{\pm j})\frac{dV}{d\varphi}, \quad \Phi'' > 0,$$

we find that the quantity $dw/d\varphi$ is positive (negative) in the normal (anomalous) case. Thus, if in the normal (anomalous) case the spatial boundary of the stability region is in a compression (rarefaction) region at the initial time, a discontinuity will develop at this boundary. However, if the boundary of the stability region is in a rarefaction (compression) region the distance between the crest of the simple wave and the point $x_c(t)$ will increase with time.

We note that at the boundary of the stability region the quantity $dV/d\varphi$, which characterizes the rate of change of the profile of the simple wave (in particular, the formation of discontinuities) becomes infinite, in accordance with (8) and (9).

Let us now consider the case of a plasma penetrated by a low-density ion beam ($n_1 \gg n_2$). Four simple waves can propagate in a plasma of this kind. Two of them are characterized by phase velocities close to the ion acoustic velocity in a plasma with one ion species[3)] $V_{\pm 1} = u_1 \pm S_1$. Using (8) we find that for these waves

$$\frac{dV_{\pm 1}}{d\varphi} = \pm\frac{D_1 S_1}{2D}\left\{3 - \frac{DD_2}{D_1^2}\right\}, \tag{12}$$

so that depending on the form of the electron distribution function the discontinuities can arise either in the compression region (normal case) or in the rarefaction region (anomalous case).

The other two waves have phase velocities approximately equal to

$$V_{\pm 2} = u_2 \pm S_2\{1 - S_1^2(u_2 - u_1)^{-2}\}^{-1/2}. \tag{13}$$

The relations in (5) and (8) then assume the form

$$\frac{dn_1}{d\varphi} = \frac{ez_1 n_1}{M_1(u_2 - u_1)^2}, \quad \frac{dn_2}{d\varphi} = \frac{ez_2 n_2}{M_2 S_2^2}\{1 - S_1^2(u_2 - u_1)^{-2}\},$$

$$\frac{du_1}{d\varphi} = \frac{ez_1}{M_1(u_2 - u_1)},$$

$$\frac{du_2}{d\varphi} = \pm\frac{ez_2}{M_2 S_2}\{1 - S_1^2(u_2 - u_1)^{-2}\}^{1/2},$$

$$\frac{dV_{\pm 2}}{d\varphi} = \pm\frac{3D_1 S_2}{2z_2 n_2}\{1 - S_1^2(u_2 - u_1)^{-2}\}^{1/2}. \tag{14}$$

The quantity $dV/d\varphi$, which characterizes the rate of change of wave shape, is large in waves of this kind ($\sim n_2^{-1/2}$); regardless of the form of electron distribution function it is possible to have only the usual direction of change in the wave profile (normal case).

We note that an ion stream with a low velocity interacts very intensely with these waves; in particular, $dn_2/dn_1 \gg n_2/n_1$.

2. STATIONARY WAVES

As we have just seen, an ion acoustic wave that propagates in a plasma with linear dispersion will exhibit a wave profile that changes with time. (As indicated in[5] the only exception is the case of a plasma consisting of electrons characterized by a distribution of the form $(v^2 + v_0^2)^{-2}$ and one ion species; in a plasma of this kind the acoustic wave propagates without distortion.)

In regions of the wave in which the characteristic scale size of the inhomogeneity is comparable with the Debye radius it is necessary to consider the acoustic dispersion described by the first term in the last equation in (3). When dispersion is taken into account, as is well known, it is possible to obtain waves with a stationary profile: these are the solitary waves, the periodic waves, and the quasi-shock waves.[1,2,4]

We shall first consider solitary waves in a plasma with a single ion stream. Transforming to a coordinate system moving with the wave and using (3) we have

$$un = u_0 n_0, \quad {}^1\!/_2 Mu^2 + ze\varphi = {}^1\!/_2 Mu_0^2,$$
$$d^2\varphi / dx^2 = 4\pi e\{D - zn\}, \tag{15}$$

where n_0 and u_0 are the values of the ion density and velocity in the unperturbed regions ($x \to \pm\infty$; the potential φ in the unperturbed region is taken equal to zero). Solving these equations we have

$$\left(\frac{d\varphi}{dx}\right)^2 = 8\pi e\int_0^\varphi d\varphi\{D - zn\}, \quad n = n_0\left\{1 - \frac{2ze\varphi}{Mu_0^2}\right\}^{-1/2}. \tag{16}$$

3)Simple waves in a plasma consisting of hot electrons and cold ions of one kind have been studied in [5].

When there are no trapped or reflected particles the relation in (16) and the equation of state $D = D(\varphi)$ together with the condition for charge neutrality in the unperturbed region $D(0) = zn_0$ determine completely the distribution of the characteristic plasma quantities in the symmetric solitary wave.

In the asymptotic region $|x| \to \infty$, (16) can be written conveniently in the form

$$(d\varphi / dx)^2 = 8\pi e D_1 \{1 - S_0^2 / u_0^2\}, \qquad (17)$$

where S_0 is the value of the ion acoustic velocity for $|x| = \infty$, as given by (17). We see that in the laboratory coordinate system the solitary wave moves with supersonic velocity ($u_0 > S_0$), in which case the perturbation falls off at infinity in accordance with the exponential relation

$$\varphi \sim \exp\{-\varkappa |x|\}, \qquad \varkappa = (8\pi e D_1)^{1/2} (1 - S_0^2 / u_0^2)^{1/2}.$$

The amplitude of the solitary wave φ_m is determined from the equation

$$\int_0^{\varphi_m} d\varphi \{D - zn\} = 0. \qquad (18)$$

For the case of a Maxwellian electron distribution this equation has a solution for which $\varphi_m > 0$. It is evident that the ion and electron densities both increase with the potential in this wave (solitary compression wave).

However, it is possible to have electron velocities distributions for which (18) admits of a solution with negative φ_m rather than positive. In this case the ion and electron densities are reduced in the solitary wave (solitary rarefaction wave). This case is realized, in particular, for distributions of the form $(v^2 + v_0^2)^{-\alpha}$, where $3/2 < \alpha < 2$ in the primary velocity region.

We shall now show that the solitary compression wave arises for the case of a distribution function which leads to an increase in the gradients in compression regions (normal case) and to a solitary rarefaction wave in the case of a distribution function which increases the gradients in the rarefaction region (anomalous case). We note that the integrand in (18) vanishes when $\varphi = 0$ and when $\varphi = \varphi_1$ where φ_1 lies in the interval $(0, \varphi_m)$. Consequently, the quantity

$$A = \frac{d}{d\varphi}\left\{\frac{1}{D^2} - \frac{1}{z^2 n_0^2}\left(1 - \frac{2ze\varphi}{M u_0^2}\right)\right\}$$

vanishes for some value $\varphi = \varphi_2$ in the range $(0, \varphi_1)$. Assuming that by virtue of the condition $u_0 > S_0$ the quantity A is negative for $\varphi = 0$, we have

$$\operatorname{sign} \varphi_m \cdot \frac{d^2}{d\varphi^2}(D^{-2}) > 0. \qquad (19)$$

On the other hand, the profile of the simple wave is distorted in the normal (anomalous) direction if the expression in the curly brackets in (12) is positive (negative); According to (19) this expression has the same sign as the quantity φ_m.[4)]

It is well known that a solitary wave is a degenerate case of a periodic wave with a stationary profile. The latter is described by the same equations (16) as a solitary wave (u_0 and n_0 are the values of u and n at the point at which the electric field vanishes) which degenerate into a solitary wave if the charge density $D(0) = zn_0$ vanishes simultaneously with the electric field.

If, for any reason, the symmetry of the distribution of the physical quantities in the plasma is disturbed, a wave arises for which the leading edge is of the same shape as the trailing edge of the solitary wave but for which there is no exponential decay beyond the wave crest, but oscillatory behavior (quasi-shock wave[1,2]). For the case of a solitary compression wave a mechanism which disturbs the symmetry of the spatial distribution of the quantities and which can lead to the formation of a compressional shock wave is ion reflection.[1,2] In the case of a solitary rarefaction wave the reflection of electrons from the potential barrier plays the same role and a rarefaction shock wave can be produced.

The profile of a rarefaction shock wave is shown schematically in Fig. 2. The potential distribution in the leading edge of the wave (region I) is described, as before, by (16) in which the potential minimum φ_m is determined from (18). In region II we have

$$\left(\frac{d\varphi}{dx}\right)^2 = 8\pi e \int_{\varphi_m}^{\varphi} d\varphi \{D^+ - zn\}, \qquad (20)$$

where $(D - D^+)$ is the density of reflected electrons. Whence, we find that the maximum value of the potential in the oscillation region φ_r and the wavelength λ are given by

$$\int_{\varphi_m}^{\varphi_r} d\varphi \{D^+ - zn\} = 0, \qquad \lambda = \int_{\varphi_m}^{\varphi_r} d\varphi \left\{2\pi e \int_0^{\varphi} d\varphi' (D^+ - zn)\right\}^{-1/2}. \qquad (21)$$

It is evident that the oscillations in the quasi-shock rarefaction shock wave (as in the compression wave) will be damped slowly as the result of dissipative processes.

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4)The function $d(D^{-2})/d\varphi$ is assumed to be monotonic; in cases of nonmonotonic functions the gradients in the simple wave can, depending on the magnitude of the perturbation, increase in compression regions or rarefaction regions.

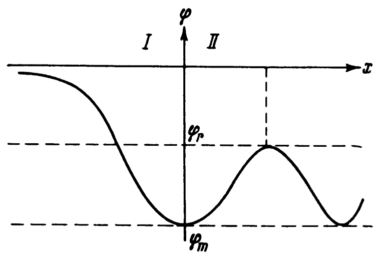

FIG. 2.

As we have noted above, a solitary (quasi-shock) wave moves with supersonic velocity $u_0 > S_0$ with respect to the unperturbed region. However with respect to the plasma behind the shock front the quasi-shock wave moves with a velocity slower than the velocity of sound $u(\varphi) < S(\varphi)$. The point here is that the expression on the right side of (20) is positive and vanishes when $\varphi = \varphi_m$ and $\varphi = \varphi_r$ so that there must be a negative second derivative; on the other hand this derivative is $8\pi e D_1(1 - S^2/u^2)$.

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146