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THE INSTABILITY OF WAVES IN NONLINEAR DISPERSIVE MEDIA

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It is shown that stationary nonlinear waves with a non-decay type of dispersion law can be unstable with respect to slower decay instabilities. In particular, ion-sound and gravitational waves on the surface of a liquid may be unstable.

1. INTRODUCTION

It is well known[1-3] that stationary nonperiodic waves of small (but finite) amplitude in a plasma can be unstable relative to the simultaneous excitation of pairs of waves. Up to the present, the strongest of such instabilities have been considered, for which $\mathbf{k}$, the wave vector of the initial wave, and $\mathbf{k}_1$ and $\mathbf{k}_2$, the wave vectors of the excited waves, are connected by the relations

$$\omega(\mathbf{k}) = \omega_1(\mathbf{k}_1) + \omega_2(\mathbf{k}_2), \quad \mathbf{k} = \mathbf{k}_1 + \mathbf{k}_2, \qquad (1)$$

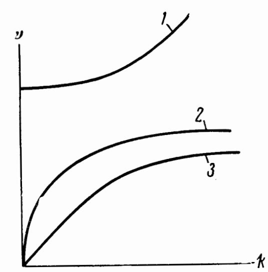
$\omega(\mathbf{k})$, $\omega(\mathbf{k}_1)$, $\omega(\mathbf{k}_2)$ are the dispersion laws of the waves. This instability can be interpreted as a coherent decay of a large number of quanta from a state with wave vector $\mathbf{k}$; Eq. (1) gives the conservation laws for the decay. The instability increment is proportional to the amplitude of the wave. Instabilities of such a type can be possessed by waves not only in a plasma, but also in other nonlinear media with dispersion, for example, capillary waves on the surface of a liquid.

In the present research we consider the instability of waves for which the decay processes (1) are forbidden. We shall show that certain types of such waves, particularly ion-sound waves in a plasma and gravitational waves on the surface of a liquid, can be unstable relative to the excitation of pairs of waves, which correspond to the conservation laws

$$2\omega(\mathbf{k}) = \omega(\mathbf{k}_1) + \omega(\mathbf{k}_2), \qquad 2\mathbf{k} = \mathbf{k}_1 + \mathbf{k}_2. \qquad (2)$$

Oraevskiĭ[2] has already demonstrated the possibility of such instabilities. This instability can be interpreted as coherent scattering of pairs of quanta that are the same state.

Let us see under what dispersion laws the process (2) is possible. Let $\omega(\mathbf{k})$ be a growing wave, convex upward ($d^2\omega/d|\mathbf{k}|^2 < 0$) (curves 2, 3 in the drawing). We direct the vectors $\mathbf{k}$, $\mathbf{k}_1$ and $\mathbf{k}_2$ along



a single straight line. As a consequence of the convexity of the function $\omega(\mathbf{k})$, the following inequality holds:

$$\omega\left(\frac{\mathbf{k}_1 + \mathbf{k}_2}{2}\right) > \frac{\omega(\mathbf{k}_1) + \omega(\mathbf{k}_2)}{2}.$$

By adding to the vectors $\mathbf{k}_1$ and $\mathbf{k}_2$ components that are perpendicular to the vector $\mathbf{k}$, we can convert this inequality into Eq. (2). On the other hand, if $\omega(\mathbf{k})$ is a growing function convex downward, Eq. (2) can be satisfied only for $\mathbf{k} = \mathbf{k}_1 = \mathbf{k}_2$ (curve 1 in the drawing).

2. GENERAL CASE

We consider the problem of the instability of nonlinear waves in the general form. We shall describe the waves by means of the normal variables a(k)—the complex amplitudes of traveling waves. We so normalize the a(k) so that the quadratic part of the Hamiltonian for the waves has the form

$$H_0 = \int \omega(\mathbf{k}) a(\mathbf{k}) a^*(\mathbf{k}) d\mathbf{k}.$$

The total Hamiltonian is

$$H_0 = H + H_{int},$$

Where H_{int} is the interaction Hamiltonian, in which we must take into account four-wave terms corresponding to the process (2), and also three-wave terms, since they also lead to four-wave processes in second-order perturbation theory. The general form of such an interaction Hamiltonian is

$$H_{int} = \int V(\mathbf{k}, \mathbf{k}_1, \mathbf{k}_2)[a^*(\mathbf{k})a(\mathbf{k}_1)a(\mathbf{k}_2) + a(\mathbf{k})a^*(\mathbf{k}_1)a^*(\mathbf{k}_2)] \times \delta(\mathbf{k}-\mathbf{k}_1-\mathbf{k}_2)\,dk\,dk_1\,dk_2 + \frac{1}{3}\int U(\mathbf{k}, \mathbf{k}_1, \mathbf{k}_2) \times [a(\mathbf{k})a(\mathbf{k}_1)a(\mathbf{k}_2) + a^*(\mathbf{k})a^*(\mathbf{k}_1)a^*(\mathbf{k}_2)]\,\delta(\mathbf{k}+\mathbf{k}_1+\mathbf{k}_2)\,dk\,dk_1\,dk_2 + \frac{1}{2}\int W(\mathbf{k}, \mathbf{k}_1, \mathbf{k}_2, \mathbf{k}_3)a^*(\mathbf{k})a^*(\mathbf{k}_1)a(\mathbf{k}_2)a(\mathbf{k}_3) \times \delta(\mathbf{k}+\mathbf{k}_1-\mathbf{k}_2-\mathbf{k}_3)\,dk\,dk_1\,dk_2\,dk_3. \quad (3)$$

Here V, U, and W are functions describing the interaction. They obey the obvious symmetry conditions:

$$V(\mathbf{k}, \mathbf{k}_1, \mathbf{k}_2) = V(\mathbf{k}, \mathbf{k}_2, \mathbf{k}_1), \quad U(\mathbf{k}, \mathbf{k}_1, \mathbf{k}_2) = U(\mathbf{k}, \mathbf{k}_2, \mathbf{k}_1) = U(\mathbf{k}_1, \mathbf{k}, \mathbf{k}_2), \quad W(\mathbf{k}, \mathbf{k}_1, \mathbf{k}_2, \mathbf{k}_3) = W(\mathbf{k}_1, \mathbf{k}, \mathbf{k}_2, \mathbf{k}_3) = W(\mathbf{k}, \mathbf{k}_1, \mathbf{k}_3, \mathbf{k}_2) = W(\mathbf{k}_2, \mathbf{k}_3, \mathbf{k}, \mathbf{k}_1). \quad (4)$$

The equations of motion for the variables a(**k**) have the form

$$\partial a(\mathbf{k}) / \partial t = -i\delta H / \delta a^*(\mathbf{k}), \quad (5)$$

which gives

$$\frac{\partial a(\mathbf{k})}{\partial t} + i\omega(\mathbf{k})a(\mathbf{k}) = -i\int V(\mathbf{k}, \mathbf{k}_1, \mathbf{k}_2)[a(\mathbf{k}_1)a(\mathbf{k}_2)\delta(\mathbf{k}-\mathbf{k}_1-\mathbf{k}_2) + 2a^*(\mathbf{k}_1)a(\mathbf{k}_2)\delta(\mathbf{k}+\mathbf{k}_1-\mathbf{k}_2)]\,dk_1\,dk_2 - i\int U(\mathbf{k}, \mathbf{k}_1, \mathbf{k}_2)a^*(\mathbf{k}_1)a^*(\mathbf{k}_2)\delta(\mathbf{k}+\mathbf{k}_1+\mathbf{k}_2)\,dk_1\,dk_2 - i\int W(\mathbf{k}, \mathbf{k}_1, \mathbf{k}_2, \mathbf{k}_3)a^*(\mathbf{k}_1)a(\mathbf{k}_2)a(\mathbf{k}_3) \times \delta(\mathbf{k}+\mathbf{k}_1-\mathbf{k}_2-\mathbf{k}_3)\,dk_1\,dk_2\,dk_3.$$

Equation (5) can be simplified by using the fact that the decay processes (1) are forbidden. We represent a(**k**) in the form

$$a(\mathbf{k}) = [A(\mathbf{k}, t) + f(\mathbf{k}, t)]\exp[-i\omega(\mathbf{k})t].$$

Here A(**k**) varies slowly in comparison with f(**k**), f(**k**) ≪ A(**k**). In the equation for f, we take into account only those terms which are quadratic in A. Assuming A to be a constant during the time of change of f, we integrate the equation for f with respect to time. We obtain

$$f(\mathbf{k}, t) = -\int V(\mathbf{k}, \mathbf{k}_1, \mathbf{k}_2) \times \left\{\frac{\exp\{it[\omega(\mathbf{k})-\omega(\mathbf{k}_1)-\omega(\mathbf{k}_2)]\}}{\omega(\mathbf{k})-\omega(\mathbf{k}_1)-\omega(\mathbf{k}_2)}A(\mathbf{k}_1)A(\mathbf{k}_2) \times \delta(\mathbf{k}-\mathbf{k}_1-\mathbf{k}_2) + 2\frac{\exp\{it[\omega(\mathbf{k})+\omega(\mathbf{k}_1)-\omega(\mathbf{k}_2)]\}}{\omega(\mathbf{k})+\omega(\mathbf{k}_1)-\omega(\mathbf{k}_2)} \times A^*(\mathbf{k}_1)A(\mathbf{k}_2)\cdot\delta(\mathbf{k}+\mathbf{k}_1-\mathbf{k}_2)\right\}dk_1\,dk_2 + \int\frac{\exp\{it[\omega(\mathbf{k})+\omega(\mathbf{k}_1)+\omega(\mathbf{k}_2)]\}}{\omega(\mathbf{k})+\omega(\mathbf{k}_1)+\omega(\mathbf{k}_2)} \times U(\mathbf{k}, \mathbf{k}_1, \mathbf{k}_2)A^*(\mathbf{k}_1)A^*(k_2)\delta(\mathbf{k}+\mathbf{k}_1+\mathbf{k}_2)\,dk_1\,dk_2.$$

The denominators in the integrals cannot vanish, so that the quadratic terms in A do not make a contribution to the equation for A. We consider in the equation for A those of the terms proportional to Af which contain the slowest exponents. It is obvious that these terms are proportional to products of the type A*AA. Selecting all such terms, we get

$$\frac{\partial A(\mathbf{k})}{\partial t} = -i\int T(\mathbf{k}, \mathbf{k}_1, \mathbf{k}_2, \mathbf{k}_3)\exp\{it[\omega(\mathbf{k}) + \omega(\mathbf{k}_1)-\omega(\mathbf{k}_2)-\omega(\mathbf{k}_3)]\} \times \delta(\mathbf{k}+\mathbf{k}_1-\mathbf{k}_2-\mathbf{k}_3)A^*(\mathbf{k}_1)A(\mathbf{k}_2)A(\mathbf{k}_3)\,dk_1\,dk_2\,dk_3, \quad (6)$$

where

$$T(\mathbf{k}, \mathbf{k}_1, \mathbf{k}_2, \mathbf{k}_3) = -2\frac{V(\mathbf{k}+\mathbf{k}_1, \mathbf{k}, \mathbf{k}_1)V(\mathbf{k}_2+\mathbf{k}_3, \mathbf{k}_2, \mathbf{k}_3)}{\omega(\mathbf{k}_2+\mathbf{k}_3)-\omega(\mathbf{k}_2)-\omega(\mathbf{k}_3)} - 2\frac{U(-\mathbf{k}-\mathbf{k}_1, \mathbf{k}, \mathbf{k}_1)U(-\mathbf{k}_2-\mathbf{k}_3, \mathbf{k}_2, \mathbf{k}_3)}{\omega(\mathbf{k}_2+\mathbf{k}_3)+\omega(\mathbf{k}_2)+\omega(\mathbf{k}_3)} - 2\frac{V(\mathbf{k}, \mathbf{k}_2, \mathbf{k}-\mathbf{k}_2)V(\mathbf{k}_3, \mathbf{k}_3-\mathbf{k}_1, \mathbf{k}_1)}{\omega(\mathbf{k}_3-\mathbf{k}_1)+\omega(\mathbf{k}_1)-\omega(\mathbf{k}_3)} - 2\frac{V(\mathbf{k}, \mathbf{k}_3, \mathbf{k}-\mathbf{k}_3)V(\mathbf{k}_2, \mathbf{k}_2-\mathbf{k}_1, \mathbf{k})}{\omega(\mathbf{k}_2-\mathbf{k}_1)+\omega(\mathbf{k}_1)-\omega(\mathbf{k}_2)} - 2\frac{V(\mathbf{k}_2, \mathbf{k}, \mathbf{k}_2-\mathbf{k})}{\omega(\mathbf{k}_1-\mathbf{k}_3)+\omega(\mathbf{k}_3)-\omega(\mathbf{k}_1)}V(\mathbf{k}_1, \mathbf{k}_2-\mathbf{k}, \mathbf{k}_3) - 2\frac{V(\mathbf{k}_3, \mathbf{k}, \mathbf{k}_3-\mathbf{k})V(\mathbf{k}_1, \mathbf{k}_3-\mathbf{k}, \mathbf{k}_2)}{\omega(\mathbf{k}_1-\mathbf{k}_2)+\omega(\mathbf{k}_2)-\omega(\mathbf{k}_1)} \quad (7)$$

Equation (6) can be obtained more rigorously by means of diagram summation.

Equation (6) has an exact solution in the form of a monochromatic wave:

$$A(\mathbf{k}) = a\exp[-i\Omega(\mathbf{k}_0)t]\delta(\mathbf{k}-\mathbf{k}_0), \quad (8)$$

where $\Omega(\mathbf{k}_0) = T(\mathbf{k}_0, \mathbf{k}_0, \mathbf{k}_0, \mathbf{k}_0)|a|^2$ is the frequency shift due to the interaction. The solution in the form (8) is itself an approximation to the stationary nonperiodic wave.

The criterion for the applicability of Eq. (6) is

$$\left| \frac{V(2\mathbf{k}_0, \mathbf{k}_0, \mathbf{k}_0)a}{\omega(2\mathbf{k}_0) - 2\omega(\mathbf{k}_0)} \right| \ll 1. \quad (9)$$

When $\omega(\mathbf{k}) = c|\mathbf{k}|$, the denominator vanishes, so that Eq. (6) is inapplicable for waves with a linear dispersion law. For Eq. (6) to hold it is necessary that the effects of nonlinearity be less than the effects of dispersion.

Let us investigate the stability of Eq. (8) relative to excitation of wave pairs. We seek a solution of Eq. (6) in the form

$$A(\mathbf{k}, t) = a\delta(\mathbf{k} - \mathbf{k}_0)\exp[-i\Omega(\mathbf{k}_0)t]$$
$$+ \alpha\delta(\mathbf{k} - \mathbf{k}_1)\exp(-i\Delta\omega_1 t)$$
$$+ \beta\delta(\mathbf{k} - 2\mathbf{k}_0 + \mathbf{k}_1)\exp[-i\Delta\omega_2 t].$$

Here

$$\Delta\omega_1 = 2T(\mathbf{k}_1, \mathbf{k}_0, \mathbf{k}_0, \mathbf{k}_1)|a|^2,$$
$$\Delta\omega_2 = 2T(2\mathbf{k}_0 - \mathbf{k}_1, \mathbf{k}_0, \mathbf{k}_0, 2\mathbf{k}_0 - \mathbf{k}_1)|a|^2.$$

By linearizing the equation with respect to α and β, we get the set of equations

$$\partial\alpha / \partial t = -ia^2 T(\mathbf{k}_1, 2\mathbf{k}_0 - \mathbf{k}_1, \mathbf{k}_0, \mathbf{k}_0)\exp(i\gamma t)\beta^*,$$
$$\partial\beta / \partial t = -ia^2 T(\mathbf{k}_1, 2\mathbf{k}_0 - \mathbf{k}_1, \mathbf{k}_0, \mathbf{k}_0)\exp(i\gamma t)\alpha^*, \quad (10)$$

where

$$\gamma = \omega(\mathbf{k}_1) + \omega(2\mathbf{k}_0 - \mathbf{k}_1)$$
$$- 2\omega(\mathbf{k}_0) + \Delta\omega_1 + \Delta\omega_2 - 2\Omega(\mathbf{k}_0)$$

Equation (10) leads to an instability with increment

$$\nu = [|a|^4 T^2(\mathbf{k}_1, 2\mathbf{k}_0 - \mathbf{k}_1, \mathbf{k}_0, \mathbf{k}_0) - \gamma^2/4]^{1/2}.$$

In order that the unstable waves be of sufficiently small amplitude, there must exist wave vectors $\mathbf{k}_1$ for which $\gamma(\mathbf{k}_1) = 0$. We limit ourselves to consideration of the case in which $|\mathbf{k}_1 - \mathbf{k}_0| \gg |a|^2 T \times (d\omega/d|\mathbf{k}|)^{-1}$. Then the frequency shifts resulting from the interaction can be neglected. The condition $\gamma = 0$ is the same as for Eqs. (2); thus the waves with a convex-upward dispersion-law curve ($d\omega/d|\mathbf{k}| > 0$, $d^2\omega/d|\mathbf{k}|^2 < 0$) are unstable against decays into waves with wave vectors lying in a layer of thickness $\delta_k \sim \nu_{max}(d\omega/d|\mathbf{k}|)^{-1}$ close to the surface, and described by Eq. (2). On the other hand, waves with a convex-downward curve for the dispersion law ($d\omega/d|\mathbf{k}| > 0$, $d^2\omega/d|\mathbf{k}|^2 > 0$) are unstable.

The instability leads to a randomizing of the wave in a time of the order of $1/\nu$. In the one-dimensional case, this instability is impossible. The results obtained are valid for the study of the instability of only sufficiently narrow wave packets, the phase relations in which do not change appreciably during the time of development of the instability. This leads to a condition on the packet width $\delta_k \ll \nu_{max}(d\omega/d|\mathbf{k}|)^{-1}$. In the opposite case of wide packets, the instability is preserved, but the increment decreases to a value on the order of $a^4T^2/\omega(\mathbf{k})$.

3. INSTABILITY OF WAVE IN AN ISOTROPIC PLASMA

We now turn to the application of the results obtained above to specific problems. We consider Langmuir oscillations in an isothermal plasma without a magnetic field. The dispersion law of Langmuir waves has the form

$$\omega(\mathbf{k}) = \omega_0 + {}^3/_2 v_{Te}^2 |\mathbf{k}|^2 / \omega_0.$$

According to the criterion formulated above, the Langmuir wave is unstable against processes of the type (2). It is curious to note that for the Langmuir wave, the equations

$$n\omega(\mathbf{k}) = \sum_{s=1}^{n} \omega(\mathbf{k}_s),$$

$$n\mathbf{k} = \sum_{s=1}^{n} \mathbf{k}_s,$$

which correspond to the decay of n quanta, have a unique solution:

$$\mathbf{k}_1 = \mathbf{k}_2 = \ldots = \mathbf{k}_n = \mathbf{k}.$$

We can then conclude that the Langmuir wave is unstable against decay instabilities of arbitrary order.

We now consider ion-sound waves in a plasma with cold ions, which can be described by the hydrodynamic equations. Assuming the motion of the ions to be irrotational, we introduce u—the velocity potential of the ions, φ—the electrostatic potential, n, M—the density and mass of the ions, and T—the temperature of the electrons.

The equations have the form

$$\frac{\partial n}{\partial t} + \operatorname{div}(n\nabla u) = 0, \quad \frac{\partial u}{\partial t} + \frac{1}{2}(\nabla u)^2 = -\frac{e}{M}\varphi,$$
$$\Delta\varphi = -4\pi e(n - n_0 e^{e\varphi/T}). \quad (11)$$

Upon simultaneous satisfaction of the conditions

$$\delta n = \frac{n - n_0}{n_0} \ll 1, \quad (kr_D)^2 \ll 1,$$

we can use the last equation of (11) to determine φ

with an accuracy up to terms of fourth order:

$$\varphi = \frac{T}{e}\left(\delta n + \frac{T}{4\pi e^2 n_0}\Delta\delta n - \frac{1}{2}(\delta n)^2 + \frac{1}{3}(\delta n)^3\right).$$

We now transform to the variables a(k) by the formulas

$$\delta n = \left(\frac{1}{16\pi^3 M n_0}\right)^{1/2}\int \frac{|\mathbf{k}|}{\omega^{1/2}(\mathbf{k})}(a(\mathbf{k})e^{i\mathbf{kr}} + a^*(\mathbf{k})e^{-i\mathbf{kr}})\,d\mathbf{k},\ u$$

$$= -i\left(\frac{1}{16\pi^3 M n_0}\right)^{1/2}\int \frac{\omega^{1/2}(\mathbf{k})}{|\mathbf{k}|}(a(\mathbf{k})e^{i\mathbf{kr}} - a^*(\mathbf{k})e^{-i\mathbf{kr}})\,d\mathbf{k}.$$

When $kr_D \ll 1$ the dispersion law is

$$\omega(\mathbf{k}) = c_s|\mathbf{k}|(1 - {}^1/_2|\mathbf{k}|^2 r_D^2),$$

$c_s = \sqrt{T/M}$ is the velocity of the ion sound.

In the variables a(**k**), Eqs. (11) take the form (5), in which the coefficient functions, for $kr_D \ll 1$, are equal to

$$V(\mathbf{k}, \mathbf{k}_1, \mathbf{k}_2) = U(\mathbf{k}, \mathbf{k}_1, \mathbf{k}_2) = \frac{1}{16\pi^{3/2}(Mn_0)^{1/2}}\left(\frac{T}{M}\right)^{1/4}$$

$$\times\left[\frac{1}{2}|\mathbf{k}|^{1/2}|\mathbf{k}_1|^{1/2}|\mathbf{k}_2|^{1/2} + \frac{(\mathbf{k}_1\mathbf{k}_2)|\mathbf{k}|^{1/2}}{|\mathbf{k}_1|^{1/2}|\mathbf{k}_2|^{1/2}} + \frac{(\mathbf{k}\mathbf{k}_2)|\mathbf{k}_1|^{1/2}}{|\mathbf{k}|^{1/2}|\mathbf{k}_2|^{1/2}} + \frac{(\mathbf{k}\mathbf{k}_1)|\mathbf{k}_2|^{1/2}}{|\mathbf{k}|^{1/2}|\mathbf{k}_1|^{1/2}}\right]$$

$$W(\mathbf{k}, \mathbf{k}_1, \mathbf{k}_2, \mathbf{k}_3) = \frac{1}{32\pi^3 M n_0}|\mathbf{k}|^{1/2}|\mathbf{k}_1|^{1/2}|\mathbf{k}_2|^{1/2}|\mathbf{k}_3|^{1/2}$$

and possess the necessary symmetry properties (4).

The function $\omega(\mathbf{k})$ for ion-sound waves is convex upward; therefore, the ion-sound wave is unstable. The decay produces a pair of waves, almost parallel to **k**; this leads to the result that the terms in Eq. (7) containing a frequency difference in the denominator are shown to be $(kr_D)^{-2}$ times greater than the rest. Such terms are produced by virtual decays and coalescence of ion-sound plasmons; therefore, it is necessary to take into account only the first component describing this process in the Hamiltonian (3) for the ion-sound waves.

The increment of the decay is of the order $\nu \sim (\delta n/n)^2 \omega_i / kr_D$, ω_i is the ion plasma frequency, while the condition for the applicability of the approximate equation (6) is $\delta n/n \ll (kr_D)^2$.

4. INSTABILITY OF WAVES ON THE SURFACE OF A LIQUID

Waves on the surface of a liquid (without account of capillary phenomena) obey a dispersion law $\omega^2(\mathbf{k}) = g|\mathbf{k}|\tanh(|\mathbf{k}|h)$. The function $\omega(\mathbf{k})$ is convex upward; therefore, instability can take place. We compute the increment of instability for the case of an infinitely deep liquid. We choose as variables describing the oscillations of the liquid $\Phi(\mathbf{r}, z, t)$—the velocity potential, $\psi(\mathbf{r}, t) = \Phi(\mathbf{r}, z, t)|_{z=\eta}$, $\eta(\mathbf{r}, t)$ is the deviation of the surface from its equilibrium value. Here **r** is the radius vector along the undisturbed surface of the liquid, z the coordinate perpendicular to the surface; z increases with the depth of the liquid.

In these variables, the equations of the surface oscillations have the form

$$\partial\eta/\partial t - A = -\nabla\eta\nabla\psi + A(\nabla\eta)^2,$$

$$\partial\psi/\partial t + g\eta = -{}^1/_2(\nabla\psi)^2 + {}^1/_2A^2[1 + (\nabla\eta)^2],$$

$$\nabla^2\Phi + \partial^2\Phi/\partial z^2 = 0. \quad (12)$$

The density of the liquid is set equal to unity, $A = \partial\varphi/\partial z|_{z=\eta}$.

The general solution of Laplace's equation which satisfies the condition $\Phi \to 0$ as $z \to -\infty$ is given by the formula

$$\Phi(\mathbf{r}, z, t) = \int \Phi(\mathbf{k}, t)\exp(|\mathbf{k}|z)\exp(i\mathbf{kr})\,d\mathbf{k}.$$

Consequently,

$$\psi(\mathbf{r}, t) = \sum_{n=0}^{\infty}\frac{\eta^n(\mathbf{r}, t)}{n!}\int |\mathbf{k}|^n\Phi(\mathbf{k}, t)\exp(i\mathbf{kr})\,d\mathbf{k},$$

$$A(\mathbf{r}, t) = \sum_{n=0}^{\infty}\frac{\eta^n(\mathbf{r}, t)}{n!}\int |\mathbf{k}|^{n+1}\Phi(\mathbf{k}, t)\exp(i\mathbf{kr})\,d\mathbf{k}. \quad (13)$$

For the system (12) to be closed it is necessary to eliminate $\Phi(\mathbf{k}, t)$ from Eqs. (13). The transformation to the variables a(**k**) is brought about by the formulas

$$\eta(\mathbf{k}) = (|\mathbf{k}|/g)^{1/4}[a(\mathbf{k}) + a^*(-\mathbf{k})],$$
$$\psi(\mathbf{k}) = -i(g/|\mathbf{k}|)^{1/4}[a(\mathbf{k}) - a^*(-\mathbf{k})].$$

Here $\eta(\mathbf{k})$ and $\psi(\mathbf{k})$ are the Fourier transforms of $\eta(\mathbf{r}, t)$ and $\psi(\mathbf{r}, t)$.

The equations of the surface oscillations in the variables a(**k**) has the form (5), where

$$4U(\mathbf{k},\mathbf{k}_1,\mathbf{k}_2)=4V(-\mathbf{k},\mathbf{k}_1,\mathbf{k}_2)$$

$$=g^{1/4}\Big\{[(\mathbf{k}\mathbf{k}_1)+|\mathbf{k}|\,|\mathbf{k}_1|]\Big(\frac{|\mathbf{k}_2|}{|\mathbf{k}|\,|\mathbf{k}_1|}\Big)^{1/4}$$

$$+[(\mathbf{k}\mathbf{k}_2)+|\mathbf{k}|\,|\mathbf{k}_2|]\Big(\frac{|\mathbf{k}_1|}{|\mathbf{k}|\,|\mathbf{k}_2|}\Big)^{1/4}$$

$$+[(\mathbf{k}_1\mathbf{k}_2)+|\mathbf{k}_1|\,|\mathbf{k}_2|]\Big(\frac{|\mathbf{k}|}{|\mathbf{k}_1|\,|\mathbf{k}_2|}\Big)^{1/4}\Big\},$$

$$4W(\mathbf{k},\mathbf{k}_1,\mathbf{k}_2,\mathbf{k}_3)$$

$$=-|\mathbf{k}|\,|\mathbf{k}_1|\Big(\frac{|\mathbf{k}_2|\,|\mathbf{k}_3|}{|\mathbf{k}|\,|\mathbf{k}_1|}\Big)^{1/4}\Big\{\frac{1}{2}|\mathbf{k}-\mathbf{k}_2|+\frac{1}{2}|\mathbf{k}-\mathbf{k}_3|$$

$$+\frac{1}{2}|\mathbf{k}_1-\mathbf{k}_2|+\frac{1}{2}|\mathbf{k}_1-\mathbf{k}_3|-|\mathbf{k}|-|\mathbf{k}_1|\Big\}$$

$$-|\mathbf{k}_2|\,|\mathbf{k}_3|\Big(\frac{|\mathbf{k}_1|\,|\mathbf{k}_3|}{|\mathbf{k}|\,|\mathbf{k}_2|}\Big)^{1/4}$$

$$\times\Big\{\frac{1}{2}|\mathbf{k}-\mathbf{k}_2|+\frac{1}{2}|\mathbf{k}_1-\mathbf{k}_2|$$

$$+\frac{1}{2}|\mathbf{k}_1-\mathbf{k}_3|-|\mathbf{k}_2|-|\mathbf{k}_3|+\frac{1}{2}|\mathbf{k}-\mathbf{k}_3|\Big\}$$

$$+|\mathbf{k}|\,|\mathbf{k}_2|\Big(\frac{|\mathbf{k}_1|\,|\mathbf{k}_3|}{|\mathbf{k}|\,|\mathbf{k}_2|}\Big)^{1/4}\Big\{\frac{1}{2}|\mathbf{k}+\mathbf{k}_1|+\frac{1}{2}|\mathbf{k}_2+\mathbf{k}_3|$$

$$+\frac{1}{2}|\mathbf{k}-\mathbf{k}_3|+\frac{1}{2}|\mathbf{k}_1-\mathbf{k}_2|-|\mathbf{k}_2|-|\mathbf{k}_3|\Big\}$$

$$+|\mathbf{k}|\,|\mathbf{k}_3|\Big(\frac{|\mathbf{k}_1|\,|\mathbf{k}_2|}{|\mathbf{k}|\,|\mathbf{k}_3|}\Big)^{1/4}\Big\{\frac{1}{2}|\mathbf{k}+\mathbf{k}_1|+\frac{1}{2}|\mathbf{k}_2+\mathbf{k}_3|$$

$$+\frac{1}{2}|\mathbf{k}-\mathbf{k}_2|+\frac{1}{2}|\mathbf{k}_1-\mathbf{k}_3|-|\mathbf{k}|-|\mathbf{k}_3|\Big\}$$

$$+|\mathbf{k}_1|\,|\mathbf{k}_2|\Big(\frac{|\mathbf{k}|\,|\mathbf{k}_3|}{|\mathbf{k}_1|\,|\mathbf{k}_2|}\Big)^{1/4}\cdot\Big\{\frac{1}{2}|\mathbf{k}-\mathbf{k}_2|$$

$$+\frac{1}{2}|\mathbf{k}_1-\mathbf{k}_3|+\frac{1}{2}|\mathbf{k}+\mathbf{k}_1|$$

$$+\frac{1}{2}|\mathbf{k}_1+\mathbf{k}_3|-|\mathbf{k}_1|-|\mathbf{k}_2|\Big\}$$

$$+|\mathbf{k}_1|\,|\mathbf{k}_3|\Big(\frac{|\mathbf{k}|\,|\mathbf{k}_2|}{|\mathbf{k}_1|\,|\mathbf{k}_3|}\Big)^{1/4}\Big\{\frac{1}{2}|\mathbf{k}-\mathbf{k}_3|$$

$$+\frac{1}{2}|\mathbf{k}_1-\mathbf{k}_2|+\frac{1}{2}|\mathbf{k}+\mathbf{k}_1|$$

$$+\frac{1}{2}|\mathbf{k}_2+\mathbf{k}_3|-|\mathbf{k}_1|-|\mathbf{k}_3|\Big\}.$$

The functions U, V, and W satisfy the symmetry conditions (4); this shows that the variables a(**k**) are actually the classical analogs of the creation operators of wave quanta. The approximate equation (6) is valid under the condition of weak nonlinearity of the wave

$$\eta/\lambda\ll 1, \tag{14}$$

λ is the wavelength, η the characteristic amplitude of the wave.

Equation (6) has the solution

$$a(\mathbf{k})=a\exp\left[-it(\sqrt{g|\mathbf{k}|}+2|\mathbf{k}|^3|a|^2)\right]\delta(\mathbf{k}-\mathbf{k}_0), \tag{15}$$

which is an approximation of the well known solution of the equations of hydrodynamics—of the stationary wave of finite amplitude on a surface of a liquid of infinite depth.[4]

For the condition $\eta/\lambda \ll 1$, the frequency shift can be neglected. In accord with the criterion formulated above, we can conclude that a progressive periodic wave on the surface of a liquid is unstable relative to excitation of pairs of oscillations whose wave vectors lie close to the curve described by the equation

$$2\sqrt{|\mathbf{k}_0|}=\sqrt{|\mathbf{k}_1|}+\sqrt{|2\mathbf{k}_0-\mathbf{k}_1|}.$$

The increment of instability is computed in terms of U, V, and W by means of Eqs. (7) and (1).

In order of magnitude, $\nu\sim\sqrt{g/\lambda}(\lambda/\eta)^2$ and the wave should become random after a time on the order of $1/\nu$.

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