

DOUBLY-LOGARITHMIC ASYMPTOTIC FORM OF THE COMPTON EFFECT FOR LARGE-ANGLE SCATTERING

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The doubly-logarithmic asymptotic approximation of the Compton effect is obtained for scattering at large angles. The asymptotic form is calculated by summing the asymptotic contributions of Feynman diagrams. It is shown that both diagrams containing, and diagrams not containing, infrared divergences are important. The asymptotic amplitude is represented by $\exp(-\alpha(4\pi)^{-1}\ln^2s)$. The asymptotic form of the cross section is obtained for different ways of measuring the accompanying bremsstrahlung. The total cross section for elastic scattering plus bremsstrahlung with no restriction on the photon energy does not include doubly-logarithmic terms and equals the asymptotic form of the Klein-Nishina-Tamm cross section.

INTRODUCTION

At high energies \sqrt{s} the asymptotic forms of Feynman diagrams contain logarithms of \sqrt{s} , and the effective perturbation-theoretical expansion parameter becomes $\pi^{-1}\alpha \ln s$ in some cases.^[1,2] In several problems, such as that of bremsstrahlung, \ln^2s appears for each power of α , i.e., for each intermediate photon. Using the customary terminology we shall refer to Feynman diagram terms containing $(\pi^{-1}\alpha \ln^2s)$ as doubly-logarithmic (d.l.) terms. When

$$\pi^{-1}\alpha \ln s \ll 1, \tag{1}$$

which denotes the smallness of singly-logarithmic terms, the summation of d.l. terms alone will yield a correct asymptotic form which we shall designate as the d.l. asymptotic form.

Sudakov was the first to obtain the d.l. asymptotic form for the vertex part.^[3] Abrikosov^[4] and Yennie, Frautschi, and Suura^[5] have also studied the d.l. asymptotic form of the Compton effect and some other processes. They considered diagrams like Fig. 1, d and e, containing infrared divergences for all intermediate photons. However, the total number of d.l. terms is not exhausted by these diagrams. For example, let us consider the asymptotic form of the Compton effect at high energies s and at angles close to 180° in the c.m. system ($u = \text{const}$). The sixth-order diagrams a, b, and c of Fig. 1, which do not include infrared divergences, have been calculated by Gell-Mann et al.^[6] in this asymptotic form for finite photon mass λ .

The sum of the three diagrams is proportional to

$$\alpha^2 \ln^2 s \ln^2 \lambda^2. \tag{2}$$

When $\lambda \rightarrow 0$ this asymptotic form is incorrect because it contains an infrared divergence that is absent from the diagrams. The correct asymptotic form is obtained by replacing λ^2 with s^{-1} , thus converting (2) into a d.l. term. The other sixth-order diagrams, which contain d.l. terms, are infrared terms (Fig. 1).¹⁾

In the present work, by summing all d.l. perturbation-theoretical diagrams the d.l. asymptotic form of the Compton effect is obtained for back-scattering ($s \rightarrow \infty, u = \text{const}$); this also coincides with the asymptotic form for two-photon annihilation of electron-positron pairs. The latter process is also of great experimental interest, as it can be investigated in colliding-beam experiments. The energy at which the d.l. asymptotic form becomes important can be estimated from the condition $\pi^{-1}\alpha \ln^2s = 1$, which in the c.m. system gives $E = \sqrt{s} = 10^4 \text{ MeV} = 10 \text{ GeV}$ ($\pi^{-1}\alpha \ln s = 0.05$). The sum of all d.l. diagrams equals the amplitude of the pole diagram (Fig. 2a) multiplied by $\exp(-\alpha(4\pi)^{-1}\ln^2s)$.

It is clear from the foregoing that the appearance of d.l. terms is not associated with infrared divergences, i.e., with the possibility of emitting quanta with no lower energy limit. The d.l. terms

¹⁾Abrikosov [4] calculated the infrared diagrams of Fig. 1, d and e, incorrectly; as a result he obtained the correct result for the cross section (38a) given below.

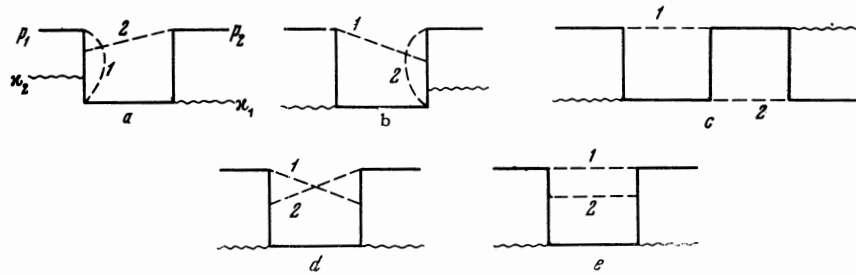


FIG. 1.

in our case have a very simple origin. It is well known that the cross section for bremsstrahlung in large-angle Compton scattering exhibits $\ln^2 s$ energy dependence. An increasing cross section of this competing process is accompanied by a correspondingly decreasing amplitude of elastic scattering. It is important to note that the d.l. contribution to the cross section for bremsstrahlung is certainly not attributable solely to soft quanta in the rest system of the emitting particles. It is easily proved, for example, that in the laboratory system an important contribution to bremsstrahlung is provided by all quanta having energies $\omega \ll s$ and perpendicular momenta $k_{\perp} = \omega \vartheta \ll m = 1$. The d.l. contribution of intermediate quanta is determined by this region.

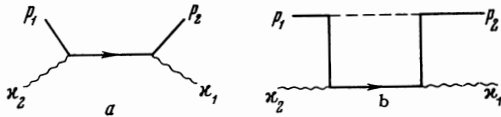


FIG. 2.

2. CALCULATION OF THE ASYMPTOTIC FORMS OF DIAGRAMS

Let us consider the free-electron Compton effect. Let p_1, p_2 and κ_1, κ_2 be the 4-momenta of the initial and final electrons and photons, respectively. The Mandelstam invariants are

$$s = (p_1 + \kappa_1)^2, \quad u = (p_1 - \kappa_2)^2 = q^2, \quad t = (p_1 - p_2)^2. \quad (3)$$

We shall obtain the asymptotic expression for the process for $s \rightarrow \infty$ and $u = \text{const}$ in the case of scattering at c.m. angles near 180° . The only Feynman diagrams that contribute to this asymptotic form are those of Fig. 1, a and b, containing the "small" momentum $q = p_1 - \kappa_2$ in the internal electron lines between external photons.²⁾ The diagrams obtained by interchanging these external

photons contain the "large" momentum $r = p_1 + \kappa_1$ ($r^2 = s$) in the internal electron lines, and are small of the order s^{-n} , where n is the number of internal electron propagators.^[4] We shall not also consider closed electron loops. It can be shown that when the photon self-energy is taken into account two logarithms are lost, when the scattering of light on light is taken into account three logarithms are lost etc.^[4]

To eliminate infrared divergences we follow Abrikosov^[4] in introducing a small virtual term for the initial and final electrons ($p_1^2 = p_2^2 = m^2 + \delta m^2$) in place of the photon mass λ . The asymptotic form of the amplitude is then calculated more simply but differs from the asymptotic form that is obtained when λ is used. Infrared divergences are removed, as is well known, by adding the elastic cross section to the cross sections for inelastic processes associated with the emission of any number of soft photons. The asymptotic forms for inelastic processes also differ depending on whether λ or δ is used. The result of the addition does not, of course, depend on the specific cutoff.

To calculate the integrals we employ Sudakov's procedure,^[3] which consists in replacing the integration over the 4-momentum k of intermediate quanta by an integration over α, β , and k_{\perp} , where³⁾

$$k = \alpha p + \beta p_1 + k_{\perp}, \quad k^2 = s\alpha\beta + k_{\perp}^2, \quad k_{\perp}^2 < 0, \quad d^4k = \frac{s}{2} d\alpha d\beta d^2k_{\perp}; \quad (4)$$

k_{\perp} is a space-like 4-vector that is perpendicular to the plane of the vectors p_1 and p_2 . For scattering near 180° ($u = \text{const}$), k_{\perp} is the component of the photon momentum that is perpendicular to the incident photon direction in both the c.m. and lab. systems. For $k^2 = 0$, $s\alpha(s\beta)$ represents the total c.m. energy of the intermediate photon and initial (final) electron.

Let us consider the single fourth-order diagram (Fig. 2b) that depends on s . From the Feynman rules we obtain

²⁾In all diagrams the external photons are represented by wavy lines and the intermediate photons by dashed lines.

³⁾All subsequent equations and formulas will be accurate to terms of the order $1/s$.

$$J = -\frac{-ie^4}{(2\pi)^4} \int \frac{\gamma_i(\hat{p}_2 + \hat{k} + 1)\hat{e}_1(\hat{q} + \hat{k} + 1)\hat{e}_2(\hat{p}_1 + \hat{k} + 1)\gamma_i}{[(p_2 + k)^2 - 1 + i\epsilon][(q + k)^2 - 1 + i\epsilon](k^2 + i\epsilon)[(p_1 + k)^2 - 1 + i\epsilon]} d^4k, \quad (5)$$

where $\hat{p} = p_1\gamma_1$ and $\hat{\hbar} = c = m = 1$. Taking into account that (5) is to be calculated between the free Dirac bispinors \bar{u}_{p_2} and u_{p_1} and transforming to the new variables given in (4), we obtain for the numerator:

$$2s[e_1(\hat{q} + \hat{k}_\perp + 1)\hat{e}_2 + O(\alpha, \beta, \alpha\beta)]. \quad (6)$$

We shall show subsequently that the unwritten terms containing α or β in the numerator do not contribute to the d.l. asymptotic form.

The factors of the denominator in (5) become

$$(p_2 + k)^2 - 1 + i\epsilon = s\beta + 2\alpha + \delta + k_\perp^2 + s\alpha\beta + i\epsilon, \quad (7a)$$

$$(q + k)^2 - 1 + i\epsilon = (q + k_\perp)^2 + s\alpha\beta - 1 + i\epsilon, \quad (7b)$$

$$k^2 + i\epsilon = s\alpha\beta + k_\perp^2 + i\epsilon, \quad (7c)$$

$$(p_1 + k)^2 - 1 + i\epsilon = s\alpha + 2\beta + \delta + k_\perp^2 + s\alpha\beta + i\epsilon. \quad (7d)$$

We have here omitted quadratic terms in α and β , which do not affect the asymptotic form of the integral.⁴⁾ We shall show subsequently that the principal contribution to the asymptotic form comes from the regions $|\alpha| \ll 1$ and $|\beta| \ll 1$; this justifies the dropping of terms of the order of α or β compared with unity. The second term in (7a) and (7d) is important only for $\beta < 1/s$ and $\alpha < 1/s$, respectively. We shall first integrate over α . It is easily shown that when $\beta \leq 0$ or $\beta \leq 1$ the zeros of all four denominator factors in (7) lie either above or below the real axis; therefore the integral vanishes. When $0 \leq \beta \leq 1$ the zeros of (7b) and (7c) lie below and the zero of (7d) lies above, while the zero of (7a) lies below for $\beta > 2/s$ and above for $\beta < 2/s$. When the integration contour is closed below the axis it encloses the poles of (7a), (7b), and (7c). It is easily seen that the contribution from the pole of (7a) does not contain logarithms of s .

The omitted terms of the numerator that are proportional to α (or β) compensate with $1/s$ accuracy the only important pole located above the axis in (7d) [or (7a)]; therefore the contribution from these terms vanishes. The contributions of the poles of (7b) and (7c) cancel for large k_\perp . We shall therefore assume $-k_\perp^2 \lesssim 1$. The contribution from the pole of (7b) is

$$J_b = -\frac{e^4}{(2\pi)^3} \int_{1/s}^1 \frac{d\beta}{\beta} \int \frac{da}{\alpha} \int d^2k_\perp \frac{\hat{e}_1(\hat{q} + \hat{k}_\perp + 1)\hat{e}_2}{k_\perp^2 + s\alpha\beta} \times \delta[(q + k_\perp)^2 - 1 + s\alpha\beta]. \quad (8)$$

In (7a) and (7d) we have dropped terms of the order $1/s$; accordingly the lower limit of integration over β has been replaced by $1/s$ (the coefficient of this limit is unimportant). We have also neglected quantities of the order of α or β compared with unity. We get rid of the δ function conveniently by integrating over k_\perp ; we then obtain a limitation on the variation of the product $s\alpha\beta$:

$$1 \leq s\alpha\beta \quad (9)$$

and, therefore,

$$J_b = -\frac{e^4}{(2\pi)^3} \int_{1/s}^1 \frac{d\beta}{\beta} \cdot \text{const} = \ln s \cdot \text{const}'. \quad (10)$$

We see that the contribution from this pole furnishes only a singly-logarithmic (s.l.) term and should therefore be dropped in our approximation. The loss of one logarithm although the original logarithm (8) is of d.l. form results from the condition (9) imposed on $s\alpha\beta$. We note that the regions $\beta \sim 1/s$ and $\beta \sim 1$ furnish no logarithmic contributions. We can therefore assume $1/s \ll \beta$ and $\alpha \ll 1$ or $\alpha \sim 1/s\beta$, thus justifying our foregoing procedure with regard to the dropping of terms.

The contribution from the last residue of (7c) corresponding to a real intermediate photon, can be divided conveniently into two parts, which are contributions from the regions $-k_\perp^2 > \sigma$ and $-k_\perp^2 < \sigma$, $1/s \ll \sigma \ll 1$. The first of these contributions is treated exactly as in the preceding case and leads to a fixed value of $s\alpha\beta$ while furnishing a s.l. term.

The contribution from the second region is

$$J_c = -\frac{e^4\pi}{(2\pi)^3} \int_0^1 \frac{d\beta}{\beta + 2\alpha/s + \delta/s} \int \frac{da}{\alpha + 2\beta/s + \delta/s} \times \int_0^\sigma dz \delta(z - s\alpha\beta) \frac{\hat{e}_1(\hat{q} + 1)\hat{e}_2}{q^2 - 1} = f(u) \left(-\frac{\alpha}{2\pi} A \right), \quad z = -k_\perp^2, \quad (11)$$

where

$$f(u) = e^2 \frac{\hat{e}_1(\hat{q} + 1)\hat{e}_2}{q^2 - 1},$$

$$A = 1/2 \ln^2 s - 2 \ln s \ln \delta, \quad \alpha = e^2/4\pi. \quad (12)$$

We see that this region gives a d.l. term that determines the asymptotic form of the integral in (5) as a whole. Without changing the d.l. contribution, we can, as in the preceding case, assume $\alpha, \beta \ll 1$ in (11). However, care must be exercised at the lower limits because of the infrared divergence. It is easily proved that the d.l. contributions from

⁴⁾By rotating the α and β axes we can arrive at $k = p_1'\beta' + p_2'\alpha + k_\perp$, where $p_1'^2 = p_2'^2 = 0$, and quadratic terms in α' and β' do not appear.^[3]

the regions ($\alpha \ll 1/s, \beta \gg 1/s$), ($\alpha \gg 1/s, \beta \ll 1/s$), and ($\alpha \ll 1/s, \beta \ll 1/s$) corresponding to a soft intermediate photon in (4) are mutually cancelled for $\delta \ll 1/s$; these regions make no contribution when $\delta \sim 1$. Therefore the entire d.l. contribution to (11) is determined completely by the region $\alpha > 1/s, \beta > 1/s$ ($\alpha, \beta \sim 1/s$ make no logarithmic contribution). This region corresponds to a hard photon in any system of coordinates such as (4). Cancellation of the d.l. contribution from soft photons occurs in all perturbation-theoretical orders, as we have shown in^[7]. For the sake of simplicity we shall subsequently assume $\delta \sim 1$ and determine the contribution of hard photons satisfying

$$1/s \ll \alpha, \beta \ll 1. \quad (12a)$$

The appearance of a d.l. term is associated with the possibility that the value of $s\alpha\beta$ can vary down to zero. If we introduce the photon mass with $\lambda^2 \gg 1/s$, which corresponds to introducing λ^2 as an additional term of the δ function in (11), then λ^2 will be the lower limit of $s\alpha\beta$ and we shall obtain only a s.l. term. This is the situation that arose in^[8], where the d.l. term appeared in the cross section only upon the addition of the inelastic cross section associated with the emission of an additional soft quantum.

When integrating higher order diagrams we shall henceforth retain the order of integration that has already been shown: We shall first integrate over α or β in the complex plane, close the contour on the side containing the pole of the photon Green's function, write the contribution of each pole as a δ function, and integrate the latter over k_{\perp} . The square of the logarithm of s for each photon can then arise only out of an integral having the form

$$\int \frac{d\alpha}{\alpha} \int \frac{d\beta}{\beta} \quad (13)$$

under the condition that $s\alpha\beta$ is not bounded in both directions while α and β satisfy condition (12a), for which $\delta \sim 1$. The diagram will furnish a d.l. contribution as $\ln^2 s$ appears for each intermediate photon.

3. ASYMPTOTIC FORM OF THE AMPLITUDE

We select the next order, which is the sixth order, as an example. The sixth-order diagrams making d.l. contributions are shown in Fig. 1. The remaining sixth-order diagrams are corrections to the vertex parts and self energies and do not furnish d.l. contributions, as can be shown directly.

We shall begin by analyzing diagrams d and e of Fig. 1. As in the fourth order, only the momenta \hat{p}_1 and \hat{p}_2 can remain in the numerators of the

vertical-line electron propagators. The other terms will contain corrections of the order of α or β in the integrand. By moving \hat{p}_1 and \hat{p}_2 towards the free bispinors u_{p_1} and u_{p_2} it becomes easy to show that the only important terms are the anticommutators of the momenta \hat{p}_i and the γ_l matrices at the vertices, which contribute $2s$ for each virtual photon. The resulting expression is

$$\begin{aligned} J_d = & - \frac{e^6}{(2\pi)^8} \int d^2 k_{1\perp} \int d^2 k_{2\perp} \\ & \times \int \frac{d\alpha_1 d\alpha_2 d\beta_1 d\beta_2}{(\alpha_1 + i\varepsilon)(\alpha_1 + \alpha_2 + i\varepsilon)(\beta_2 + i\varepsilon)(\beta_2 + \beta_1 + i\varepsilon)} \\ & \times [(\hat{e}_1(\hat{q} + \hat{k}_{1\perp} + \hat{k}_{2\perp} + 1)\hat{e}_2)] \\ & \times \{(k_{1\perp}^2 + s\alpha_1\beta_1 + i\varepsilon)(k_{2\perp}^2 + s\alpha_2\beta_2 + i\varepsilon) \\ & \times [(q + k_{1\perp} + k_{2\perp})^2 + s(\alpha_1 + \alpha_2)(\beta_1 + \beta_2) - 1 + i\varepsilon]\}^{-1}. \end{aligned} \quad (14)$$

All integrals over α_i and β_i are taken between the limits $1/s$ and unity. We note that the poles with respect to both α_1 and α_2 lie on different sides of the real axis only when $\beta_1 < 0$ and $\beta_2 < 0$.

It is easily seen that an integral such as (13) occurs for each intermediate photon when the following conditions are fulfilled:

$$\alpha_2 \gg \alpha_1, \quad \beta_1 \gg \beta_2, \quad (15a)$$

$$s(\alpha_1 + \alpha_2)(\beta_1 + \beta_2) \approx s\alpha_2\beta_1 \ll 1 \quad (|q^2 - 1| \sim 1), \quad (15b)$$

$$|k_{1\perp}^2| \ll 1, \quad |k_{2\perp}^2| \ll 1. \quad (15c)$$

No other region makes a d.l. contribution. Observing these conditions, calculating the contributions of the residues with respect to α_1 and α_2 , and integrating over $k_{i\perp}$, we obtain

$$J_d = f(u) \left(\frac{\alpha}{2\pi}\right)^2 \int \frac{d\alpha_1}{\alpha_1} \int \frac{d\beta_1}{\beta_1} \int \frac{d\alpha_2}{\alpha_2} \int \frac{d\beta_2}{\beta_2}, \quad \alpha = \frac{e^2}{4\pi}, \quad (16)$$

$$s\alpha_2\beta_1 \ll 1, \quad \alpha_2 \gg \alpha_1, \quad \beta_1 \gg \beta_2, \quad -k_{i\perp}^2 = s\alpha_i\beta_i \ll 1, \quad (17)$$

where $f(u)$ is the Born amplitude (12).

The d.l. contribution of diagram e in Fig. 1 differs from that of diagram d by the exchange $\beta_2 \rightleftharpoons \beta_1$ and has the form of the integral in (16) under the conditions

$$s\alpha_2\beta_2 \ll 1, \quad \alpha_2 \gg \alpha_1, \quad \beta_2 \gg \beta_1, \quad -k_{i\perp}^2 = s\alpha_i\beta_i \ll 1. \quad (18)$$

Let us now consider diagram a of Fig. 1. In the numerator we, as previously, obtain $2s$ from the anticommutators of \hat{p}_i and γ_l at the vertices of quantum 2. The anticommutator of \hat{p}_1 and γ_i at the upper vertex of quantum 1 leads to the replacement of γ_i by \hat{p}_1 at the lower vertex. Continuing the

movement of this \hat{p}_1 towards the free end with the momentum p_1 , we easily see that the only important term is the anticommutator of \hat{p}_1 and $\hat{k}_1 + \hat{k}_2$ in the numerator of the nearest electron propagator. This

term, $s(\alpha_1 + \alpha_2)$, compensates the denominator of the electron propagator between the photon lines having the momenta κ_1 and κ_2 . We finally obtain

$$J_a = -\frac{e^6}{(2\pi)^8} \int d^2k_{1\perp} \int d^2k_{2\perp} \int \frac{d\alpha_1 d\alpha_2 d\beta_1 d\beta_2}{(\alpha_1 + i\varepsilon)[(q + k_{1\perp} + k_{2\perp})^2 + s(\alpha_1 + \alpha_2)(\beta_1 + \beta_2) - 1 + i\varepsilon](\beta_2 + i\varepsilon)} \times \frac{\hat{e}_1(\hat{q} + k_{2\perp} + 1)\hat{e}_2}{[(q + k_{2\perp})^2 + s\alpha_2\beta_2 - 1 + i\varepsilon](k_{1\perp}^2 + s\alpha_1\beta_1 + i\varepsilon)(k_{2\perp}^2 + s\alpha_2\beta_2 + i\varepsilon)}. \quad (19)$$

We note that the poles with respect to α_1 and α_2 in (19) lie on opposite sides of the real axis when $\beta_1 < 0$, $\beta_2 > 0$, and $|\beta_1| > \beta_2$. To obtain residues of the photon propagator poles the contour must then be closed below the axis for integration over α_2 , and must be closed above the axis for integration over α_1 . Having determined these residues and having integrated over $k_{i\perp}$, we obtain (16) under the conditions

$$\alpha_2 \gg |\alpha_1|, |\beta_1| \gg \beta_2, |s(\alpha_1 + \alpha_2)(\beta_1 + \beta_2)| = |s\alpha_2\beta_1| \gg 1, \quad -k_{2\perp}^2 = s\alpha_2\beta_2 \ll 1. \quad (20)$$

We note that whereas $-k_{2\perp}^2 = s\alpha_2\beta_2 \ll 1$, the quantity $-k_{1\perp}^2 = s\alpha_1\beta_1$ can be either larger or smaller than unity; thus a d.l. contribution appears for both $-k_{1\perp}^2 \ll 1$ and $-k_{1\perp}^2 \gg 1$.

A completely analogous calculation of diagram b in Fig. 1 yields the integral (16) under the conditions

$$\alpha_2 \gg |\alpha_1|, |\beta_1| \gg \beta_2, |s\alpha_2\beta_1| \gg 1, -k_{1\perp}^2 = s\alpha_1\beta_1 \ll 1. \quad (21)$$

Diagram c is the most complex diagram. Transforming the numerator, beginning with the external electron propagators, we find that the momentum $2\hat{p}_1$ arrives at the upper vertex of the central vertical line and $2\hat{p}_2$ at the lower vertex. All terms in the numerator of the central vertical line that are proportional to \hat{p}_1 or \hat{p}_2 make no contribution. Thus only the perpendicular components of the momenta "survive" in the numerator; these include \hat{q} (\hat{q} is almost a perpendicular vector: $q = q_{\perp} + ap_2 + bp_1$; $a, b \sim 1/s$). The same applies to the lower (upper) horizontal line of the diagram, since the numerator of this line stands between u_{p_1} and \hat{p}_2 (\bar{u}_{p_2} and \hat{p}_1).⁵⁾ When we interchange \hat{p}_1 with the numerator of the central vertical line we reverse the sign of the vectors in the numerator. Then when \hat{p}_1 is commuted with \hat{p}_2 we obtain the factor $2s$; the other

terms do not contribute. The product of the three remaining numerators with sign changes taken into account is

$$-[(q + k_1)^2 - 1](\hat{q} + \hat{k}_{2\perp} + 1) \cdot \\ -[(q + k_2)^2 - 1](\hat{q} + \hat{k}_{1\perp} + 1) \\ + (\hat{q} + \hat{k}_{1\perp} + 1)(\hat{q} - 1)(\hat{q} + \hat{k}_{2\perp} + 1) \\ + s\alpha_1\beta_1(\hat{q} + \hat{k}_{2\perp} + 1) + s\alpha_2\beta_2(\hat{q} + \hat{k}_{1\perp} + 1). \quad (22)$$

We have converted the quadratic terms containing $k_{i\perp}$ to similar terms containing the whole vector k_i by adding and subtracting $s\alpha_i\beta_i$. By inserting the numerator (22) into the integral for diagram c it is easily shown that in the first two terms of (22) the quadratic factors cancel the corresponding denominators. Following this, the integrals containing the first and second terms of (22) differ only in sign and therefore cancel completely the integrals for diagrams a and b, respectively. We note that this cancelation can occur without imposing the conditions (20) and (21) that distinguish the d.l. contribution. The last two terms of (22) do not contribute because they compensate the pole with respect to α_1 or β_2 that lies above the real axis, after which the remaining poles all lie on one side of the real axis. The third term of the numerator leads to the integral (16), where the region determining the d.l. contribution is defined by

$$\alpha_2 \gg |\alpha_1|, |\beta_1| \gg \beta_2, |s\alpha_2\beta_1| \gg 1, -k_{i\perp}^2 = s\alpha_i\beta_i \ll 1. \quad (23)$$

We note that this remaining uncompensated term makes a d.l. contribution only in the region $-k_{1\perp}^2 \ll 1$ for both intermediate quanta; thus in addition to the canceled contributions of diagrams a and b the d.l. contribution from the region $-k_{1\perp}^2 \gg 1$ is also canceled (as has been observed in^[6] for the case of nonzero photon mass). The cancelation of the d.l. contribution from the region $-k_{1\perp}^2 \gg 1$ occurs in all perturbation-theoretical orders.^[7]

We note that (23) differs from (17) only in the reversal of the inequalities imposed on $s\alpha_2\beta_1$.

⁵⁾The anticommutators of \hat{p}_1 and \hat{e}_2 , $2p_1e_2$ and \hat{p}_2 , \hat{e}_1 and $2p_2e_1$ do not depend on s because of the transverse character of the photon and the condition $u = \text{const}$.

Adding the sum of diagrams a, b, and c to diagram d and taking into account that the region $|\alpha_2\beta_1| \sim 1$ [see (7)] makes no d.l. contribution, we find that the sum of diagrams a, b, c, and d is still given by the same integral (16) under the condition

$$-k_i^2 = s\alpha_i\beta_i \ll 1, \quad \alpha_2 \gg |\alpha_1|, \quad |\beta_1| \gg \beta_2, \quad i = 1, 2. \quad (24)$$

Adding this sum to the last diagram e [see (18)] and considering that the region $|\beta_1| \sim \beta_2$ also makes no d.l. contribution, we find that the sum of all sixth-order diagrams is determined by the conditions

$$-k_i^2 = s\alpha_i\beta_i \ll 1, \quad \alpha_2 \gg |\alpha_1|. \quad (25)$$

The foregoing result is clearly independent of the numerical indices assigned to the photons, i.e. when we reverse the condition imposed on α_2 and α_1 the integral yields the same contribution. We can therefore remove this condition and divide the result by 2. The final result is

$$\begin{aligned} \sum_{x=a}^e J_x &= \frac{1}{2} f(u) \left(\frac{\alpha}{2\pi} \right)^2 \int \frac{d\alpha_1}{\alpha_1} \int \frac{d\beta_1}{\beta_1} \int \frac{d\alpha_2}{\alpha_2} \int \frac{d\beta_2}{\beta_2} \Big|_{s\alpha_i\beta_i \ll 1} \\ &= f(u) \frac{1}{2} \left(\frac{\alpha}{2\pi} \int_{1/s}^1 \frac{d\alpha}{\alpha} \int_{1/s}^{\alpha} \frac{d\beta}{\beta} \right)^2 = f(u) \frac{1}{2} \left(\frac{\alpha}{4\pi} \ln^2 s \right)^2. \end{aligned} \quad (26)$$

By adding the second, fourth, and sixth orders we obtain the amplitude F in the form

$$F(s, u) = f(u) \left\{ 1 - \frac{\alpha}{4\pi} \rho^2 + \frac{1}{2} \left(\frac{\alpha}{4\pi} \rho^2 \right)^2 - \dots \right\}, \quad \rho = \ln s. \quad (27)$$

We derived this equation by assuming $\delta \sim 1$, but the result also is valid for $\delta \rightarrow 0$.^[7] In (15b) we have assumed $q^2 - 1 \sim 1$, so that our analysis cannot be applied for $q^2 \rightarrow 1$. We note that the asymptotic contribution of all the considered orders is proportional to the $f(u)$ -pole diagram a of Fig. 1, which contains the entire dependence of the amplitude on $q^2 = u$. The first three terms of (27) coincide with the expansion of $\exp(-\alpha\rho^2/4\pi)$.

The diagrams of all higher orders can be treated similarly, but for lack of space we refer those who are interested to^[7] and shall state only the result. It is customarily assumed that in the course of motion along an electron line intermediate photons are first emitted and then absorbed. Every d.l. diagram can be represented by a ladder (Fig. 3a) constructed out of an electron line, with

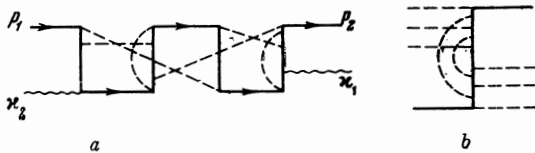


FIG. 3.

all photon lines beginning and ending only on the rungs of the ladder. On each rung (Fig. 3b) the emitted and absorbed photons are grouped to form two nonintersecting beams. The first beam along the electron line consists only of the emitted photon lines that are absorbed at some later location on the electron line (following the entire first beam). The second beam consists of the absorbed photon lines emitted from the same or from preceding rungs. The last emitted photon line in the first beam and the first absorbed photon line in the second beam must connect different rungs. External photons are to be associated with directions corresponding to the u channel, i.e., they are the reverse of the lines shown in Fig. 3. The sixth-order diagrams of Fig. 1 comprise the simplest illustration of the latter case.

The sum of all n -th order diagrams makes the d.l. contribution $f(u)(-\alpha A/2\pi)^n/n!$, which is equivalent to the exponential d.l. asymptotic amplitude $F(s, u)$:

$$F(s, u) = f(u) \exp(-\alpha A/2\pi), \quad (28)$$

where $f(u)$ and A are defined by (12).

In conclusion, we note the following interesting fact. It is easily verified, using the sixth order as an example, that we obtain the correct d.l. contribution from all diagrams when at the vertices of photon emission (absorption) we replace γ_i by \hat{k}/α (or \hat{k}/β) and divide d^4k by $s/2$. This also holds true for higher order diagrams. The resulting expression coincides with that obtained when photon Green's functions in the form $k_\mu k_\nu d_l/k^2$, where $d_l = 2/s\alpha\beta$, are inserted in the Feynman integrals. We know that the contribution of longitudinal photons to the renormalized scattering amplitude (neglecting photon loops in the external electron lines) is $\exp(\int d^4k d_l/k^2)$, in agreement with (28).

4. RELATION OF THE ASYMPTOTIC AMPLITUDE TO PARTIAL WAVES FOR COMPLEX l

It is interesting to understand our foregoing result from the point of view of an asymptotic form of the amplitude in the u channel for fixed u and large unphysical values of the transferred momentum s , as is done in the theory of complex angular momenta.

The Compton effect is characterized by three independent helicity amplitudes of determinate parity:

$$f_{ss} = \langle 1/2 | T | 1/2 \rangle, \quad f_{sn} = \langle 1/2 | T | 3/2 \rangle, \quad f_{nn} = \langle 3/2 | T | 3/2 \rangle, \quad (29)$$

where $1/2$ and $3/2$ are the sums of the photon and electron spin projections on the direction of electron momentum.

The asymptotic form of these helicity amplitudes is determined by the characteristics of the corresponding partial amplitudes f_{SS}^l , f_{SN}^l , and f_{NN}^l ($l = j - 1/2$). In the Born approximation we have

$$f_{ss}^l = -\frac{e^2 \delta_{l0}}{\sqrt{u}-1}, \quad f_{sn}^l = \frac{e^2}{\sqrt{l}}, \quad f_{nn}^l = \frac{e^2(\sqrt{u}-1)}{l}. \quad (30)$$

The pole in the amplitude f_{NN}^l for $l = 0$ comes from the diagram in Fig. 4a and, as is shown in^[6], in the case of massive photons it is the "nucleus" for the appearance of a moving Regge pole. To obtain f_{NN}^l in higher approximations we can first take into account the contributions coming to it from diagrams such as Fig. 4b having no singularities with respect to u , and then obtain the contributions of the other diagrams with the aid of the unitarity condition.

In the case of real photons, which are massless, we can unfortunately not make use of unitarity, because for this purpose we would have to be concerned with intermediate states populated by many quanta. This situation is associated with the fact that the virtual quanta with small k_\perp considered in the preceding sections are soft in the u channel. Nevertheless, the helicity amplitude $f_{NN}(u, s)$ can be calculated on the basis of the following simple but very useful fact. In a state with the projection 3/2 of its total spin on the direction of motion the product of the photon polarization vector $\hat{e} = e_\mu \gamma_\mu$ and the bispinor u_p vanishes:

$$u_{p_1} \hat{e}_1' = 0, \quad \bar{u}_{p_2} \hat{e}_2' = 0 \quad (e_1' = e_2, \quad e_2' = e_1). \quad (31)$$

These identities result from the impossibility of constructing a state with the projection 3/2 from a spinor. The amplitude f_{NN} in the asymptotic form is of the order $1/s$.^[6]

All diagrams of the type of Fig. 2, a and b, that we have considered make no contribution to the d.l. asymptote of f_{NN} . Indeed, in virtue of (31) the anti-commutators of \hat{p}_i and γ_i in the external electron lines vanish identically; this produces a loss of s . The result of commuting \hat{p}_i and γ_i vanishes because of the free Dirac equation $(\hat{p}_i - 1)u = 0$. The remaining terms of the numerator contain α or β and are small because of the condition $\alpha, \beta \ll 1$. The asymptotic behavior of f_{NN} is determined entirely by Fig. 4b diagrams. This indicates that a right-hand cut and unitarity are not essential for calcu-

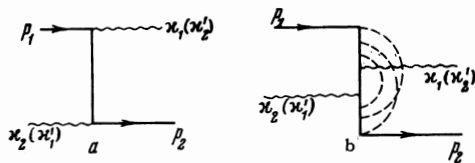


FIG. 4.

lating f_{NN} with d.l. accuracy. In the partial-wave treatment this results from the factor $\sqrt{u}-1$ in the expression for f_{NN}^l in (30); this factor causes vanishing of the contribution from states with $u \sim 1$ (soft quanta). The diagrams in Fig. 4b are proportional to $1/s$; for this reason we have not discussed them earlier. Diagrams of this kind containing more than one internal electron line are proportional to s^{-n} ($n \geq 2$); they must therefore be dropped.

The diagrams in Fig. 4 can be calculated in an elementary manner. For example, the n -th order contribution can be obtained by interchanging the ends of all photon lines, for which α_i and β_i then become independent variables. The result is the product of the pole diagram Fig. 4a by the expression

$$\frac{1}{n!} \left\{ -\frac{\alpha}{2\pi} A \right\}^n, \quad (32)$$

where $(n!)^{-1}$ arises from the interchange of the photon lines and A is defined by (12). Summation over n yields the exponential in (28).

The entire discussion concerning the calculation of f_{NN} can be applied to the calculation of f_{NS} . Thus we have

$$f_{nn} = \frac{e^2(\sqrt{u}-1)}{s} \exp\left(-\frac{\alpha}{2\pi} A\right),$$

$$f_{ns} = \frac{e^2}{s} \exp\left(-\frac{\alpha}{2\pi} A\right). \quad (33)$$

The situation is reversed for the calculation of f_{SS} . Diagrams like Fig. 4a, having no singularities with respect to u , yield contributions of the order $1/s$, while the amplitude itself is not diminished. The amplitude is determined by the diagrams considered in the preceding sections and therefore, with respect to the u channel, by intermediate states containing low-energy quanta. The actual calculation yielded

$$f_{ss} = \frac{e^2}{\sqrt{u}-1} \exp\left(-\frac{\alpha}{2\pi} A\right). \quad (34)$$

This result could have been predicted on the basis of (33) from the factorization requirement for helicity amplitudes.

In conclusion, let us consider the partial waves of the u channel. For $\delta \sim 1$ and $A = \rho^2/2$ a simple calculation yields

$$f_{nn}^l = \frac{e^2(\sqrt{u}-1)}{l} \varphi(z), \quad f_{sn}^l = \frac{e^2}{\sqrt{l}} \varphi(z),$$

$$f_{ss}^l = \frac{e^2}{\sqrt{u}-1} [\varphi(z) - 1], \quad \varphi(z) = \int_0^\infty e^{-x^2/z-x} dx, \quad z = \frac{4\pi l^2}{\alpha}. \quad (35)$$

Thus when we took account of d.l. terms, or, equivalently, the correct behavior of partial waves for $l \sim \sqrt{\alpha}$, the δI_0 nonanalyticity in the partial waves vanished. In this sense we can state that taking account of d.l. terms leads to Regge-like behavior of the partial amplitudes.

5. ASYMPTOTIC FORM OF THE CROSS SECTION

To obtain our final result the elastic-process cross section must be added to the cross section for inelastic processes associated with the emission of any number of additional bremsstrahlung photons. To preserve the kinematics of the process we shall assume that the total bremsstrahlung energy is much smaller than the initial energy: $\Sigma k_{0i} \ll \sqrt{s}$. This is equivalent to the conditions $\Sigma \alpha_i \ll 1$ and $\Sigma \beta_i \ll 1$, which are always satisfied when obtaining the d.l. contributions.

Let us consider the asymptotic cross section in the sixth order. Following Abrikosov,^[4] it is useful to represent inelastic cross sections by diagrams such as those in Fig. 5, which contains d.l. diagrams for inelastic cross sections involving the emission of two real bremsstrahlung photons. The d.l. cross-section diagrams of the corresponding order involving the emission of a single real bremsstrahlung photon are obtained from Fig. 5 by transposing the end of one bremsstrahlung photon from the right-hand or left-hand corner of the upper electron line to the corresponding free corner of the lower electron line. For example, diagram c becomes diagram c'. When the ends of

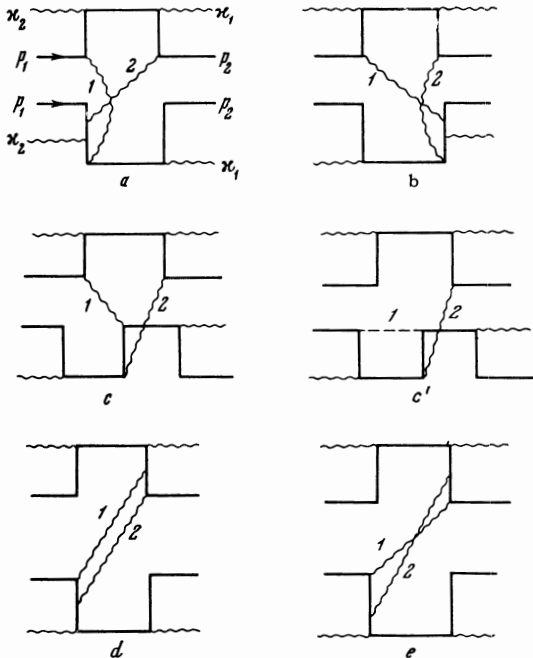


FIG. 5.

both bremsstrahlung photons are transferred from the upper to the lower electron line we obtain the unrelated diagrams of Fig. 1 and Fig. 2a corresponding to elastic scattering. Since the d.l. contribution to the amplitude arose out of real intermediate photons, the calculation of the d.l. contributions to the cross section from Fig. 5 diagrams is analogous to the corresponding calculation for Fig. 1 diagrams. The integrals yielding the d.l. contributions of Fig. 5 diagrams differ from the corresponding integrals for Fig. 1 diagrams only in that q is replaced by $q' = q - k_{1\perp}$ (which does not affect the result when $-k_{1\perp}^2 \ll 1$ and $q^2 \sim 1$), and that the integration region over all α_i and β_i is modified in accordance with experimental conditions.⁶⁾ Unlike intermediate photons, bremsstrahlung photons are emitted at both vertices, thus yielding the factor (-1) for each bremsstrahlung photon.

In the higher orders the diagrams yielding d.l. contributions to the cross section can also be represented by means of Fig. 5, where the upper and lower diagrams will have been replaced by any diagrams of Fig. 3; then one end of each bremsstrahlung photon belongs to any beam of emitted photons in the Fig. 3 diagrams, while the other end belongs to any beam of absorbed photons. The outside photons of a beam can then be either intermediate or bremsstrahlung photons.

The result of adding the elastic and inelastic cross sections is

$$\frac{d\sigma}{du} = \frac{d\sigma_0}{du} \exp\left(-\frac{\alpha}{\pi} A\right) \exp\left(+\frac{\alpha}{\pi} \bar{A}\right), \quad (36)$$

where

$$d\sigma_0/du = 2\pi\alpha^2/s(1-u) \quad (37)$$

is the asymptotic form of the Klein-Nishina-Tamm cross section. The first exponential is the square of the factor multiplying the elastic scattering amplitude in (28). The second exponential arises out of the inelastic cross sections. The values of A and \bar{A} , obtained from (12), determine the contributions of the intermediate and bremsstrahlung photons, respectively. The integration limits for

⁶⁾The removal of limitations such as $|s\alpha_2\beta_1| \gtrsim 1$ and $\alpha_2 \gtrsim |a_1|$ when summing diagrams (see Sec. 3) is not performed in exactly the same way for elastic and inelastic cross sections, as a general rule. For example, diagrams d' and e' separately in an inelastic cross section involving the emission of one bremsstrahlung photon do not contain the limitation $|\beta_1| \gtrsim \beta_2$ of (15a); this leads to the required doubling of their sum. Contributions from large $k_{1\perp}$ are compensated among the diagrams a, b, and c of Fig. 5 in the same way as in the amplitude of Fig. 1.

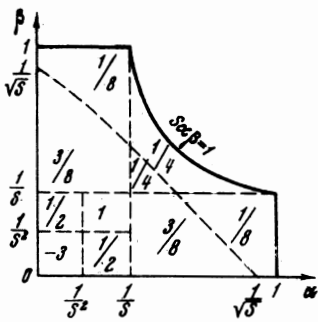


FIG. 6.

bremsstrahlung photons in \bar{A} depend on the experimental conditions. The virtual term δ that has been introduced for electrons, and which A and \bar{A} contain, cancels out.^[5]

In Fig. 6 the solid curve denotes the region of integration over intermediate quanta in A . The dashed lines define the integration regions for bremsstrahlung quanta in \bar{A} under determinate experimental conditions. The contributions of the different regions are indicated by numerals in units of $\rho^2 = \ln^2 s$.

We now present the resulting cross section formulas obtained with different methods of cutting off bremsstrahlung experimentally.

1. In the lab system ($p_1 = 0$) bremsstrahlung is emitted with the energy $k_0 = \beta + s\alpha/2 \leq \epsilon$. Then

a) $\epsilon \sim 1$, i.e., $\beta < 1$, $\alpha < 1/s$, and

$$\frac{d\sigma}{du} = \frac{d\sigma_0}{du} \exp\left(-\frac{\alpha\rho^2}{\pi}\right), \quad (38a)$$

b) $\epsilon \sim 1/s$, i.e., $\beta < 1/s$, $\alpha < 1/s^2$, and

$$\frac{d\sigma}{du} = \frac{d\sigma_0}{du} \exp\left(-\frac{4\alpha\rho^2}{\pi}\right). \quad (38b)$$

2. Bremsstrahlung is emitted with the energy $k_0 \leq \epsilon \sim 1$ in the rest system of either the initial or final electron. In this case we have either $\beta < 1$, $\alpha < 1/s$ or $\beta < 1/s$, $\alpha < 1$, and the cross section is

$$\frac{d\sigma}{du} = \frac{d\sigma_0}{du} \exp\left(-\frac{\alpha\rho^2}{2\pi}\right). \quad (39)$$

3. In the c.m. system bremsstrahlung is emitted with the energy $k_0 = \sqrt{s}\beta/2 + \sqrt{s}\alpha/2 < \epsilon \sim 1$. This

kind of experiment in which two-photon pair annihilation is measured can be performed with colliding beams; the cross section in this case coincides with (39).

We note in conclusion that, as already mentioned in our Introduction, the major d.l. contribution to the bremsstrahlung cross section is determined by quanta with $-k_1^2 \ll 1$, $\omega \ll E$. This indicates that if we consider the total cross section for elastic scattering and any number of bremsstrahlung quanta, this cross section will equal (36) with d.l. accuracy, with the integration region in \bar{A} determined by the conditions $-k_1^2 < \sigma$, $\sigma \ll 1$, thus coinciding with the integration region in A . Consequently, this total cross section equals $d\sigma_0/du$ with d.l. accuracy.

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