INDUCED BREMSSTRAHLUNG EFFECT IN THE RELATIVISTIC REGION

M. V. FEDOROV

P. N. Lebedev Physics Institute, Academy of Sciences, U.S.S.R.

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The induced bremsstrahlung effect on relativistic electrons is considered. An expression for the cross section for the stimulated emission of one photon is derived for the case of scattering by isolated ions. Its asymptotic form for high energies is derived. The case of electron scattering in a crystal is considered. The conditions for negative absorption are derived. Numerical estimates are presented. It is demonstrated that in principle an electromagnetic-wave amplification factor $\alpha \sim 3 \times 10^2\ \mathrm{cm}^{-1}$ can be obtained for narrow beams of ultrafast electrons ($\Delta\varphi \lesssim 10^{-5}$, $\epsilon > 10^2\ mc^2$) scattered in a crystal.

1. USUALLY negative absorption is obtained by using transitions between states of the discrete spectrum of some system. The necessary condition in this case is to produce population inversion, at least in some part of the energy spectrum. At the same time, it is possible in principle also to produce conditions for negative absorption on the basis of transitions between the states of the continuous spectrum. One such possibility is connected with the bremsstrahlung induced when free electrons are scattered by heavy particles (in particular, ions). This method has a number of advantages. First, in transitions between the states of the continuous spectrum the problem of producing the population inversion is greatly simplified. For example, a monochromatic electron beam always comprises a system with population inversion. The conditions under which radiation prevails over absorption are determined by the electron interaction with the scattering object. Second, analysis shows that as a rule these conditions are satisfied over a very wide range of frequencies in the case of the induced bremsstrahlung effect. This allows us to expect that in principle negative absorption may be obtained, for example, in the hard ultraviolet region.

Marcuse[1,2] has considered the induced bremsstrahlung effect in the scattering of slow electrons by isolated ions. He has shown that negative absorption is possible if the direction of the initial electron velocity lies inside a certain cone, the axis of which coincides with the polarization vector of the electric field of a plane wave. However, estimates show that at the presently attainable electron beam densities the effect is too weak in the optical frequency region of greatest interest. The amplification coefficient does not exceed $\sim 10^{-10}\ \mathrm{cm}^{-1}$ in this case.

In this paper we consider first induced bremsstrahlung effects in the scattering of relativistic electrons by isolated ions. The result shows, however, that in this case, on going to very high electron energies, the conditions for negative absorption can be satisfied only in a narrow frequency region, close to $\omega = 0$. This is followed by an analysis of the stimulated bremsstrahlung effect in the scattering of fast electrons in crystals. In this case negative absorption becomes possible under certain conditions. In addition, the presence of a periodic structure leads to a sharp increase in the effect, owing to interference of the radiation from the individual ions.

2. We consider the bremsstrahlung induced in scattering of electrons by isolated ions, without confining ourselves to the case of low electron energies. Using the results of Gluckstern et al.[3], we can write down immediately in the first Born approximation the final expressions for the differential cross sections of the stimulated emission ($d\sigma_e$) and absorption ($d\sigma_a$) of one polarized photon of frequency ω and momentum $\mathbf{k}$, summed and averaged over the polarizations of the electron in the initial and final states:

$$d\sigma_{e,a} = \frac{N}{V}\frac{2\pi Z^2 e^6 p_2}{\omega^3 v \varepsilon_1}\frac{1}{\mathbf{q}^4}F\,do, \tag{1a}$$

$$F = \frac{(\mathbf{p}_2\mathbf{e}_1)^2}{(\varepsilon_2 - \mathbf{p}_2\mathbf{e}_2)^2}(4\varepsilon_1^2 - \mathbf{q}^2) + \frac{(\mathbf{p}_1\mathbf{e}_1)^2}{(\varepsilon_1 - \mathbf{p}_1\mathbf{e}_2)^2}(4\varepsilon_2^2 - \mathbf{q}^2) - 2\frac{(\mathbf{p}_1\mathbf{e}_1)(\mathbf{p}_2\mathbf{e}_1)}{(\varepsilon_1 - \mathbf{p}_1\mathbf{e}_2)(\varepsilon_2 - \mathbf{p}_2\mathbf{e}_2)}(4\varepsilon_1\varepsilon_2 - \mathbf{q}^2) + \frac{\omega^2}{(\varepsilon_1 - \mathbf{p}_1\mathbf{e}_2)(\varepsilon_2 - \mathbf{p}_2\mathbf{e}_2)}[(\mathbf{p}_1 - \mathbf{p}_2)^2 - (\mathbf{p}_1\mathbf{e}_2 - \mathbf{p}_2\mathbf{e}_2)^2]. \tag{1b}$$

Here ϵ_1 and $\epsilon_2 = \epsilon_1 \mp \omega$ are the initial and final energies of the electron, $\mathbf{p}_1$ and $\mathbf{p}_2$ are the momenta of the electron before and after the interaction with the ion, Ze is the ion charge, $v = |\mathbf{p}_1|/\epsilon_1$, N/V is the photon density, $\mathbf{e}_1$ is the photon polarization vector, $\mathbf{e}_2 = \mathbf{k}/\omega$, $\mathbf{q} = \mathbf{p}_1 \mp \mathbf{k} - \mathbf{p}_2$ is the momentum transferred to the ion, and do is the solid-angle element in the direction of $\mathbf{p}_2$. In these formulas and throughout, the upper sign corresponds to photon emission and the lower to absorption. Integration of the cross sections (1a) over the directions of the scattered electrons will be carried out in perfect analogy with the procedure of Gluckstern and Hull[4] for the spontaneous bremsstrahlung cross section. As a result we obtain

$$\sigma_{e,a} = \sigma_{e,a}^{\parallel} \cos^2\chi + \sigma_{e,a}^{\perp} \sin^2\chi,$$

where χ is the angle between the vector $\mathbf{e}_1$ and the $(\mathbf{p}_1, \mathbf{k})$ plane, and the cross sections $\sigma_{e,a}^{\parallel}$ and $\sigma_{e,a}^{\perp}$ correspond to the motion of the electron in the $(\mathbf{p}_1, \mathbf{k})$ plane and perpendicular to it. The formulas for $\sigma_{e,a}^{\parallel}$ and $\sigma_{e,a}^{\perp}$ will not be written out here, since they are very cumbersome. We indicate only that they can be obtained from (4.2) and (4.3) of[4] by making due allowance for the difference between the conservation laws for the emission and absorption, by making the substitution $\mathbf{k} \to \pm\mathbf{k}$ throughout, and also by replacing the common factor which determines the photon density:

$$k^2 dk d\Omega_0 / (2\pi)^3 \to N / V.$$

For induced bremsstrahlung, interest attaches only to the total cross section of photon emission $\sigma_{br} = \sigma_e - \sigma_a$. Negative absorption is possible when $\sigma_{br} > 0$. We shall assume that the photon frequency is much lower than the kinetic energy of the electron:

$$\xi = \omega\varepsilon_1 / (\varepsilon_1^2 - m^2) \ll 1. \tag{2}$$

In practice it is just this condition which determines the range of values of the frequency ω and of the energy ϵ_1 at which the cross sections for emission and absorption can be comparable in magnitude, and consequently, it is meaningful to raise the question of the conditions for negative absorption. In the case when $\xi \geq \epsilon_1/(\epsilon_1 + m)$ we have $\sigma_e = 0$ and $\sigma_{br} = -\sigma_a < 0$.

Under condition (2), the expressions for the integrated cross sections $\sigma_{e,a}^{\parallel,\perp}$ simplify, so that

$$\sigma_{br}^{\parallel,\perp} = \frac{N}{V} \frac{4\pi^2 Z^2 e^6}{\omega^2 v^2 \varepsilon_1} f^{\parallel,\perp}, \tag{3a}$$

$$f^{\parallel} = \frac{2}{p_1^2}\left\{-2\frac{m^2\varkappa^2(\varepsilon_1^2+m^2)}{\varepsilon_1^4\Delta^4} + \frac{1}{\varepsilon_1^2\Delta}(p_1^2 - m^2 - p_1^2\varkappa^2) + \frac{1}{\Delta}\ln\frac{2}{\xi\delta}\times\left[1 - \frac{2\varepsilon_1^2+m^2}{\varepsilon_1^2\Delta} + \frac{2m^2\varkappa^2(p_1^2+3m^2)}{\varepsilon_1^4\Delta^3}\right] + \frac{1}{\varepsilon_1^2\Delta}\ln\frac{2}{\xi\Delta}\left[\frac{\mathbf{p}_1\mathbf{e}_2}{\varepsilon_1}\frac{m^2}{\Delta} - 2p_1^2\right] + \frac{1}{\varkappa^2}\left[\left(1 - \frac{m^2}{\varepsilon_1^2\Delta}\right)\times\ln\frac{2}{\xi\delta} + \frac{\mathbf{p}_1\mathbf{e}_2}{\varepsilon_1}\ln\frac{2}{\xi\Delta} - \frac{p_1}{\varepsilon_1}\ln\frac{\varepsilon_1+p_1}{m}\right]\right\} \tag{3b}$$

$$f^{\perp} = -\frac{\mathbf{p}_1\mathbf{e}_2}{\varepsilon_1^3\Delta^2} + \frac{2}{p_1^2}\left\{\frac{1}{\Delta}\left(\frac{p_1^2}{\varepsilon_1^2} - \frac{2}{\Delta}\right)\ln\frac{2}{\xi\delta} - \frac{1}{\varkappa^2}\left[\left(1 - \frac{m^2}{\varepsilon_1^2\Delta}\right)\ln\frac{2}{\xi\delta} + \frac{\mathbf{p}_1\mathbf{e}_2}{\varepsilon_1}\ln\frac{2}{\xi\Delta} - \frac{p_1}{\varepsilon_1}\ln\frac{\varepsilon_1+p_1}{m}\right]\right\}, \tag{3c}$$

where $\delta = m/\varepsilon_1$, $\Delta = 1 - \mathbf{p}_1\mathbf{e}_2/\varepsilon_1$, $\varkappa^2 = 1 - (\mathbf{p}_1\mathbf{e}_2)^2/p_1^2$.

In the nonrelativistic case ($p_1 \ll \epsilon_1$, $\delta \cong 1$, $\Delta \cong 1$), the obtained formulas coincide with already known results[1,2]. On going to higher energies, as expected, the cross section σ_{br} increases strongly in absolute magnitude, when the direction of the initial momentum of the electron approaches the direction of the photon momentum $\mathbf{k}$ ($\mathbf{p}_1 \cdot \mathbf{e}_2 \to p_1$, $\kappa \to 0$). The conditions defining the region of negative absorption are then appreciably altered. When $\epsilon_1^2 \cong 2m^2$, the plane $\mathbf{p}_1 \perp \mathbf{k}$ goes outside the region of negative absorption. $\sigma_{br}^{\perp}$ is negative throughout, and consequently for fixed values of κ and ϵ_1 the total cross section is maximal, if the initial momentum of the electron lies in the $(\mathbf{k}, \mathbf{e}_1)$ plane.

In the ultrarelativistic case ($\delta \ll 1$) for $\kappa \ll 1$, the following asymptotic expressions hold for the functions $f^{\parallel,\perp}$:

$$f^{\parallel} \cong \frac{1}{p_1^2}\left\{-\frac{2^6\delta^2\varkappa^2}{(\delta^2+\varkappa^2)^4} + \ln\frac{2}{\xi\delta}\left[-\frac{2^3(\varkappa^2-\delta^2)^2}{(\varkappa^2+\delta^2)^4} + 4\frac{\delta^8 + 9\delta^6\varkappa^2 + 33\delta^4\varkappa^4 - 5\delta^6\varkappa^2 + 2\varkappa^2}{(\delta^2+\varkappa^2)^5}\right]\right\},$$

$$f^{\perp} \cong -\frac{1}{p_1^2}\frac{2^4}{(\varkappa^2+\delta^2)^2}\ln\frac{2}{\xi\delta}. \tag{4}$$

Thus, negative absorption is possible in a small vicinity of the direction $\kappa = \delta$, $\chi = 0$, if the following inequality is satisfied

$$5\ln\frac{2}{\xi\delta} > \frac{4}{\delta^2}. \tag{5}$$

This condition imposes very stringent limitations on the frequency ω and the energy ϵ_1, and is not satisfied in practice at very high energies when $\omega \neq 0$. For a frequency $\omega \cong 3 \times 10^{15}$ sec^{-1} the condition (5) yields $\delta > \delta_0 \cong 0.22$. Assuming that the asymptotic representation (4) is still valid in this case, we get for $\kappa = \delta$ and $\chi = 0$ the following expression for the gain $\alpha = \sigma_{br}\rho_i\rho_e/2NV^{-1}$ (ρ_i and ρ_e are the ion and electron densities):

$$\alpha \cong \frac{2\pi^2 Z^2 e^6 \rho_i \rho_e}{\omega^2 m^2 \varepsilon_1 c^2}\left(1 - \frac{\delta_0^2}{\delta^2}\right)(4 - 5\delta_0^2). \tag{6}$$

It is easy to see, however, that the gain does not increase over the nonrelativistic case.

In the general case, on going over to relativistic electron energies, the cone that determines the region of negative absorption becomes deformed and its axis approaches the direction of the vector **k**.

The limits of the stimulated-emission cone (in the plane $\chi = 0$) are determined in the general case by the equation

$$[\Delta^3(\varkappa)(2-\delta^2) - \Delta^2(\varkappa)(2+\delta^2) + 2\delta^2\varkappa^2(1+2\delta^2)]\ln\frac{2}{\xi\delta} = 2\delta^2\varkappa^2(1+\delta^2). \tag{7}$$

The condition under which this equation has a solution (with respect to κ) defines the region of values (ϵ_1, ω) at which negative absorption is possible.

The main result of this analysis is that negative absorption at optical frequencies is possible only for relatively slow electrons. The emission cross section is in this case very small. In the case of high electron energies, absorption takes place everywhere with the exception of a narrow region of small frequencies, and the cross section is given by formulas (3a) and (5).

3. Let us consider the induced bremsstrahlung effect when electrons are scattered in a crystal. It is known[5,6] that after summing over the final states of the lattice, the interference part of the differential cross section for bremsstrahlung (absorption) of one photon in an infinite crystal with rectangular lattice can be expressed as follows in terms of the corresponding cross section for the scattering of electrons by an isolated ion:

$$d\sigma_{c,a} = d\sigma_{e,a}^{(1)} N_i \frac{(2\pi)^3}{a_x a_y a_z} e^{-\mathbf{q}^2\overline{u^2}} \sum_{l_x l_y l_z} \delta(\mathbf{q} - \mathbf{q}_{l_x l_y l_z}), \tag{8}$$

where $d\sigma_{e,a}^{(1)}$ is given by (1), N_i is the total number of ions in the crystal, a_x, a_y, and a_z are the lattice constants in the direction of the crystallographic axis (x, y, z), $\mathbf{q}_{l_x l_y l_z} = 2\pi\{l_x/a_x,\ l_y/a_y,\ l_z/a_z\}$ are the reciprocal lattice vectors, l_i are integers, and $\bar{u}^2$ is the mean square of the thermal oscillations. Henceforth we confine ourselves for simplicity to the case of the cubic lattice, $a_x = a_y = a_z = a$. We also use **q** to define just the vectors of the reciprocal lattice, leaving out the subscripts l_x, l_y, and l_z.

The cross section (8) should be integrated over the directions of the momenta of the scattered electrons. This leaves one unintegrated δ function in each term of formula (8). In the case of spontaneous bremsstrahlung, interest attaches to the total cross section integrated over the momenta of the emitted photons. This enables us to eliminate the remaining δ function. In the case of induced bremsstrahlung the problem consists in determining the radiation cross section σ_{br} for fixed ω and **k**, coinciding with the frequency and wave vector of the incident wave. It is necessary then to take into account the fact that actually the electron beam always has a finite divergence. Therefore the cross sections (8) should be averaged over the directions of the initial electron velocity:

$$\bar{\sigma}_{e,a} = \int do_1\, d\sigma_{e,a} f(\mathbf{n}_1), \tag{9}$$

where $\mathbf{n}_1 = \mathbf{p}_1/p_1$, do_1 is the solid-angle element in the direction of $\mathbf{n}_1$, and $f(\mathbf{n}_1)$ is the electron distribution function normalized to unity. We shall assume that it has a narrow maximum near some fixed direction. The width of the maximum is $\Delta\varphi \ll 1$.

The three dimensional δ functions in each term of the sum (8) thus determine the vector $\mathbf{n}_2 = \mathbf{p}_2/p_2$ and one of the parameters defining the vector $\mathbf{n}_1$. Consequently, each term has a certain single-parameter region of directions $\mathbf{n}_1$ in which it differs from zero. These regions are conical surfaces whose axes are directed along the vectors $\mathbf{q} \pm \mathbf{k}$.

Bearing this in mind, we go over in the δ functions from the variables q_i to the variables $n_{1\|}$, $n_{2\|}$, and φ_2', where $n_{i\|}$ are the projections of the vectors $\mathbf{n}_i$ on the directions $\mathbf{q} \pm \mathbf{k}$, and φ_i' are the angles between the planes ($\mathbf{n}_i$, $\mathbf{q} \pm \mathbf{k}$) and some initial plane passing through the vector $\mathbf{q} \pm \mathbf{k}$. The transition is carried out with the aid of the equation

$$\delta(\mathbf{q} - \mathbf{q}_{l_x l_y l_z}) = \Delta^{-1}\delta(n_{1\|} - c_1)\,\delta(n_{2\|} - c_2)\,\delta(\varphi_2' - \varphi_1'),$$

$$\Delta = \left|\frac{D(q_x, q_y, q_z)}{D(n_{1\|}, n_{2\|}, \varphi_2')}\right|. \tag{10}$$

Calculation yields

$$\Delta = p_1 p_2 |p_1 n_{1\|} - p_2 n_{2\|}|. \tag{11}$$

The projections of the vectors $\mathbf{n}_i$ on the directions $\mathbf{q} \pm \mathbf{k}$ are determined by the expressions

$$n_{1\|} = c_1 = \frac{1}{2}\left|\frac{|\mathbf{q}\pm\mathbf{k}|}{p_1} + \left(1 - \frac{p_2^2}{p_1^2}\right)\frac{p_1}{|\mathbf{q}\pm\mathbf{k}|}\right|,$$

$$n_{2\|} = c_2 = -\frac{p_1}{2p_2}\left|\frac{|\mathbf{q}\pm\mathbf{k}|}{p_1} - \left(1 - \frac{p_2^2}{p_1^2}\right)\frac{p_1}{|\mathbf{q}\pm\mathbf{k}|}\right|. \tag{12}$$

We shall henceforth confine ourselves to an examination of relativistic electrons, i.e., we shall assume that $\delta \ll 1$. We introduce the notation

$$\eta = (2\pi / ap_1) \cong (\delta\lambda_c / a) \ll 1,$$

where $\lambda_C = h/mc$ is the Compton wavelength of the electron. Taking into account the smallness of the parameters ξ and η, we can rewrite (12) in the form

$$n_{1\|} \cong \frac{1}{2}\left| l\eta \pm \frac{2\xi}{l\eta} \pm v\xi\cos(\widehat{\mathbf{k}, \mathbf{q}})\right|,$$

$$n_{2\|} \cong -\frac{1}{2}\left| l\eta \mp \frac{2\xi}{l\eta} \pm v\xi\cos(\widehat{\mathbf{k}, \mathbf{q}})\right|, \qquad (13)$$

where $l = (l_x^2 + l_y^2 + l_z^2)^{1/2}$.

Let us examine in detail the case when the direction of motion of the electrons is close to the direction of the crystallographic axis z. From (13) we see that for frequencies $\omega \sim 10^{15}$ sec^{-1} at not very large l we have $|n_{i\|}| \ll 1$, since $(\xi/\eta) \cong (a/\lambda) \ll 1$ (λ is the radiation wavelength). Consequently the apex angles of the emission and absorption cones are close to $\pi/2$, and when the electrons move in a direction close to the z axis the main contribution is made by the vectors $\mathbf{q}$ that lie in the (xy) plane. In a small vicinity of the z axis, the conical surfaces are planes passing at small angles to the z axis perpendicular to the (q, k) plane. We take into account the fact that the cross sections increase rapidly when $\mathbf{n}_1 \to \mathbf{k}/\omega$ and therefore confine ourselves to the case of vectors $\mathbf{k}$ close to the z axis, and consequently to small angles $\theta = \pi/2 - (\mathbf{k}, \mathbf{q})$.

We consider first the case of electrons with very high energies $\delta < \delta_1 = 2a^2/\lambda\lambda_C \sim 2 \times 10^{-2}$. The expressions for $n_{i\|}$ then depend on the relation between the quantities l and $l_1 = (2\xi)^{1/2}/\eta = (\delta_1/\delta)^{1/2}$:

$$n_{1\|} = \begin{cases} \xi/l\eta \pm {}^1/_2 l\eta + {}^1/_2\xi\theta, & l < l_1 \\ {}^1/_2(l\eta \pm \xi\theta) \pm \xi/l\eta, & l > l_1 \end{cases}. \qquad (14)$$

Recognizing that the axes of the radiation and absorption cones are determined by the vectors $\mathbf{q} \pm \mathbf{k}$, we obtain the following expressions for the angles ψ_e and ψ_a between the z axis and the emission and absorption planes:

$$\psi_e^{(l)} \cong \frac{1}{2}(l\eta + \xi\theta),$$

$$\psi_a^{(l)} \cong \begin{cases} 2\xi/l\eta - \dfrac{1}{2}l\eta, & l < l_1 \\ {}^1/_2(l\eta - \xi\theta), & l > l_1 \end{cases}. \qquad (15)$$

Thus, when $l < l_1$ we have $\psi_a^{(l)} > \psi_e^{(l)}$ and the angular distance between the emission and absorption planes (for specified l), which is equal to $\eta(l_1^2 - l^2)/l$, is larger than or at least of the same order as the angles between the planes corresponding to different l ($\gtrsim \eta$). With increasing l, the emission and absorption planes come closer together. When $l > l_1$ we get $\psi_e^{(l)} - \psi_a^{(l)} = \xi\theta$ and consequently $|\psi_e^{(l)} - \psi_a^{(l)}| \ll \eta/2$.

In the case when $\delta > \delta_1$, there is no $l < l_1$ region, since $l_1 < 1$. The angular distance between the emission and absorption planes for specified l is always much smaller than the angles between the planes corresponding to different l. More strictly speaking, the quantities $\psi_e^{(l)}$ and $\psi_a^{(l)}$ define the angles in the ($\mathbf{k}$, $\mathbf{q}_l$) plane between the perpendicular to the vector $\mathbf{q}$ and the generators of the emission and absorption cones.

The summation in (8) is cut off as a result of the thermal factor at $l \sim l_0 = a/2\pi(\overline{u^2})^{1/2}$. In the case when the condition $l_0 \ll (a/\lambda_C\delta)^{1/2}$ is satisfied, there is no need for taking into account the terms connected with the vectors $\mathbf{q}$ and not lying in the (x, y) plane. These terms make a contribution different from zero only when the electrons move at large angles to the z axis ($\sim l_0^{-1} \gg l_0\eta$). We shall assume also that the condition $(\overline{u^2})^{1/2} \gg l_C$ is satisfied. In this case $\delta \gg |n_{i\|}|$ for all $l_1 < l \ll l_0$, which simplifies the calculations.

The induced bremsstrahlung effect depends essentially on the divergence $\Delta\varphi$ of the electron beam. Let $\Delta\varphi \ll \xi|\theta|$. The cross section for emission (absorption) differs from zero only if the direction that determines the maximum of the electron distribution function lies in one of the emission (absorption) surfaces. With this, only one term in (8) differs from zero. The conditions for negative absorption are realized, for example, if the vector $\mathbf{k}$ and the direction of motion of the electron beam lie in the (xz) (or (yz)) plane, and the angle between the z axis and the electron propagation direction is $(1/2)(l\eta + \xi\theta)$, where l are integers. The emission cross section for $l > l_1$ is given by the expression

$$\bar{\sigma}_{br} = \bar{\sigma}_e \cong \frac{N}{V}\,\frac{2^4 Z^2 e^6 N_i a}{\omega^3 p_1 l^3} \times \eta\,\frac{[\gamma(\theta^2 + \varphi^2 - \delta^2) + 2\sqrt{1-\gamma^2}\,\theta\varphi]^2}{(\theta^2 + \varphi^2 + \delta^2)^4}\,\frac{1}{\Delta\varphi}, \qquad (16)$$

where γ is the cosine of the angle between the polarization vector $\mathbf{e}_1$ and the ($\mathbf{k}$, $\mathbf{q}$) plane, φ is the angle between the ($\mathbf{n}_1$, $\mathbf{q}$) and ($\mathbf{k}$, $\mathbf{q}$) planes, and the vector $\mathbf{n}_1$ determines in this case the direction of the maximum of the electron beam.

Going over to the amplification coefficient, we obtain

$$\alpha \cong \frac{\pi 2^4 Z^2 e^6 \rho_i \rho_e \varepsilon_1^2}{l^3 \omega^3 \hbar m^4 c^6} \frac{1}{\Delta\varphi}\left[\delta^2 \frac{\gamma(\theta^2+\varphi^2-\delta^2)+2\sqrt{1-\gamma^2}\,\theta\varphi}{(\theta^2+\varphi^2+\delta^2)^2}\right]^2. \quad (17)$$

From this, for $\rho_e \sim 10^9$ cm^{-3}, $\rho_i \sim 10^{23}$ cm^{-3}, $\omega \sim 3 \times 10^5$ sec^{-1}, $\delta \sim 10^{-3}$, and $l = 4$, we get $\alpha \sim 2 \times 10^{-5}$ $(\Delta\varphi)^{-1}$. This result was obtained under the assumption that $\Delta\varphi \ll \xi|\theta| \lesssim 10^{-12}$ and consequently the amplification coefficient is very large in this case. However, such a stringent limitation on the divergence of the electron beam can hardly be realized.

Let us consider the case of a less stringent limitation $\Delta\varphi \ll \eta \sim 10^{-2}\delta$. If $\delta < \delta_1$, then the electrons radiate when their propagation direction coincides with one of the lower-order emission surfaces $l < l_1$. The emission cross section is in this case

$$\bar{\sigma}_{\text{br}} = \bar{\sigma}_e = \frac{N}{V}\frac{4Z^2e^6N_i a^2}{\pi\omega^2 p_1 l^5 v}\cdot\frac{1}{\Delta\varphi} \times\left[\frac{\gamma(\delta^2+\varphi^2-\theta^2+2(\xi/l\eta)^2)+2\sqrt{1-\gamma^2}(\theta+\xi/l\eta)\varphi}{(\delta^2+\theta^2+\varphi^2)(\delta^2+\varphi^2+(\theta+2\xi/l\eta)^2)}\right]^2. \quad (18)$$

Actually when $\delta < \delta_1$ negative absorption is possible under even less stringent assumptions concerning the divergence $\Delta\varphi$. This is connected with the fact that in this case there exists near the z axis a certain region which contains only the emission directions. In the simplest case, when the vector **k** is directed along the z axis, we can state that the electrons radiate if the beam does not go beyond the limits of a cone whose axis coincides with the z axis, and whose angular aperture is $(1/2)(l_1\eta) = (\xi/2)^{1/2}$, i.e., if the condition $\Delta\varphi \lesssim (1/2)\, l_1\eta \sim 10^{-3}\delta^{1/2}$ is satisfied. We assume that in this case $\Delta\varphi \ll \delta$ and, taking further account of the fact that the main contribution is made by the surfaces with $l = 1$, we obtain from (18)

$$\bar{\sigma}_{\text{br}} \cong \frac{N}{V}\frac{4Z^2e^6N_i a^2\varepsilon_1^3}{\pi\omega^2\hbar^2 m^4 c^8}\frac{1}{\Delta\varphi}\left[\frac{\delta^2+2(\xi/\eta)^2}{\delta^2+4(\xi/\eta)^2}\right]^2. \quad (19)$$

Going over to the amplification coefficient and estimating under the same assumptions as before, we obtain $\alpha \sim 10^{-2}(\Delta\varphi)^{-1} \gtrsim 3 \times 10^2$ cm^{-1} at 3×10^{-5}.

If the direction of motion of the electron beam coincides with higher-order surfaces $l > l_1$ (or when $\delta > \delta_1$), then in the case $\xi/\theta \ll \Delta\varphi \ll \eta/2$, as before, only one term differs from zero in the sum (8), but both emission and absorption are possible. Taking into account the smallness of the parameter ξ, we have (see formula (1))

$$F \cong 4p_1^2\Big\{\Big(\frac{\mathbf{n}_1\mathbf{e}_1}{1-v\mathbf{n}_1\mathbf{e}_2}-\frac{\mathbf{n}_2\mathbf{e}_1}{1-v\mathbf{n}_2\mathbf{e}_2}\Big)^2 \mp 2v\xi\Big(\frac{v\mathbf{n}_1\mathbf{e}_1}{1-v\mathbf{n}_1\mathbf{e}_2}-\frac{\delta^2(\mathbf{n}_2\mathbf{e}_1)(\mathbf{n}_2\mathbf{e}_2)}{(1-v\mathbf{n}_2\mathbf{e}_2)^2}\Big) \times\Big(\frac{\mathbf{n}_1\mathbf{e}_1}{1-v\mathbf{n}_1\mathbf{e}_2}-\frac{\mathbf{n}_2\mathbf{e}_1}{1-v\mathbf{n}_2\mathbf{e}_2}\Big)\Big\}. \quad (20)$$

Under the assumption that $\varphi \ll 1$, $\theta \ll 1$, $\delta \ll 1$, $\delta \gg l\eta/2$, and $l > l_1$, we obtain the following relations:

$$\mathbf{n}_i\mathbf{e}_1 \cong (-1)^{i+1}\cdot\frac{1}{2}(l\eta\pm\xi\theta)\left[\gamma\mp\frac{\xi}{l\eta}\sqrt{1-\gamma^2}\,\varphi\right] +\left[\sqrt{1-\gamma^2}\,\varphi-\gamma\theta\pm\frac{1}{2}\frac{\xi}{l\eta}\gamma(\delta^2+\theta^2+\varphi^2)\right], \quad (21)$$

$$1-v\mathbf{n}_i\mathbf{e}_3 \cong \frac{1}{2} \times\left[\delta^2+\varphi^2+\theta^2-(-1)^{i+1}\theta(l\eta\pm\xi\theta)\mp\theta\frac{\xi}{l\eta}(\delta^2+\varphi^2+\theta^2)\right]. \quad (22)$$

Starting from these formulas and taking into account relations (11), we obtain as a result

$$\bar{\sigma}_{\text{br}} = \bar{\sigma}_e - \bar{\sigma}_a \cong \frac{N}{V}\frac{2^5Z^2e^6N_i a\varepsilon_1}{\omega^2 l^4 p_1^3}[\gamma(\delta^2+\theta^2+\varphi^2) +2\sqrt{1-\gamma^2}\,\varphi\theta]\times[\gamma\theta(5\delta^2-\theta^2+9\varphi^2)-2\sqrt{1-\gamma^2}\,\varphi \times(\delta^2-2\theta^2+3\varphi^2)](\delta^2+\varphi^2+\theta^2)^{-4}(\Delta\varphi)^{-1}. \quad (23)$$

In the general case the condition $\bar{\sigma}_{\text{br}} > 0$ defines the certain region of angles φ and θ. In particular, if the direction of motion of the electron beam lies in the (**k**, **q**) plane, i.e., $\varphi = 0$, then the region of negative absorption is determined from the condition that the function $\theta(5\delta^2 - \theta^2)$ be positive. Thus, the electrons radiate when $0 < \theta < \delta\sqrt{5}$, and when $\theta < -\delta\sqrt{5}$. It is easy to see that the cross section σ_{br} reaches in this case a maximum at $\gamma = 1$ and $\theta \cong 0.4\delta$. Estimating the amplification coefficient, we obtain $\alpha \sim 10^{-11}(\Delta\varphi)^{-1}$, and the condition for the applicability of the result becomes $10^{-12} \ll \Delta\varphi \ll 10^{-4}$.

If $\Delta\varphi \gg l\eta$, it is necessary to take into account the contributions from many terms in the sum (8) over the vectors of the reciprocal lattice **q**. It is then possible to change over from summation to integration. It is easy to see that in this case there are no interference effects, and negative absorption is impossible. Thus, the foregoing analysis indicates that it is possible in principle to obtain negative absorption at different energies and divergences of the electron beams. Apparently greatest interest attaches to the case of ultrarelativistic electrons ($\delta < \delta_1$) and small angles between their propagation direction and the crystallographic z axis ($l < l_1$), for in this case the emission cross section is sufficiently large and rather large divergences are permissible (formulas (18) and (19)).

All the estimates were made for the case of optical frequencies, $\omega \cong 3 \times 10^{15}$ sec^{-1}. As seen from these formulas, the cross sections decrease with increasing frequency. The limits of applicability of the results with changing frequency are determined by the requirement that the parameter ξ/η be small, i.e., by the condition $\lambda \gg a$.

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