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ON THE SINGULARITY OF THE ELECTRON GREEN'S FUNCTION

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It is shown that in the gauge with $d_l = 3$ the Green's function of the electron contains no infrared divergences and has a pure pole singularity at the point corresponding to the physical electron mass. The character of the singularity of the pole-free part is also determined.

1. IT is well known that because of the emission of soft virtual photons the Green's function of the electron does not have a simple pole for $p^2 \to m^2$ (m is the electron mass) but a branch point of the power type $(p^2 - m^2)^{-1+\gamma}$, where $\gamma = -\alpha/\pi$ in the Feynman gauge.[1] It is established that up to terms $\sim\alpha$, $\gamma = \alpha(d_l - 3)/2\pi$ for an arbitrary gauge. It is seen from this that the singularity of the Green's function becomes a simple pole for the choice $d_l = 3$. This result follows, however, only in lowest order of perturbation theory, and it is therefore of interest to find out what happens when the higher orders are taken into account. We shall show in the present paper that the choice of a gauge with $d_l = 3$ guarantees the absence of infrared divergences in the Green's function of the electron in all orders of perturbation theory, and that in this gauge the Green's function has a simple pole at $p^2 = m^2$ in addition to terms which remain finite at $p^2 = m^2$. The singularity of the pole-free part of the Green's function is of the type $(p^2 - m^2) \times \ln(m^2 - p^2)$.

This result is of interest for the formal theory of scattering in the sense, say, of Lehmann, Symanzik, and Zimmerman, where the pole character of the Green's function is of fundamental importance. The result may have practical value in the calculation of quantities related to the electron Green's function near the mass shell.

2. Let us write the Källén-Lehmann representation for the electron Green's function:

$$S(p) = \int_{m^2}^{\infty} dz\,(z - p^2 - i0)^{-1}(w_1 + \hat{p}w_2), \qquad (1)$$

$$w_1 + \hat{p}w_2 \equiv w = (2\pi)^3 \sum_\alpha \langle 0|\psi(0)|\alpha p\rangle\langle\alpha p|\bar{\psi}(0)|0\rangle. \qquad (2)$$

The states with $|\alpha p\rangle$ contain for $p^2 < 4m^2$ one electron with momentum q ($q^2 = m^2$) and n photons with momenta k_i, where $q + \Sigma k_i = p$. This picture corresponds to the Feynman gauge where the contraction of the electromagnetic field is proportional to $g_{\alpha\beta}$.

The transition to an arbitrary gauge is carried out in the following way. Let us assume that besides the photons there are other scalar neutral particles with arbitrary masses μ_s (s = 1, 2, ...) which interact with the electron current. If the s-th particle is described by the field B_s, then its interaction with the current is written in the form $\xi_s e j^\alpha \partial_\alpha B_s$, where ξ_s is some number. We shall assume, moreover, that the norm of the single-particle state for the s-th particle has the sign $\eta_s = \pm 1$.

The physical picture is evidently not changed by the introduction of the particles B_s, since they do not in fact participate in the interaction. It is easily seen that their role reduces to a change in the gauge of the electromagnetic field. The effective contraction of the electromagnetic field with account of the fields B_s will be equal to

$$ig_{\alpha\beta}\Delta_0^c(x-y) + i\partial_\alpha\partial_\beta \sum_s \xi_s^2\eta_s\Delta_{\mu_s}^c(x-y) \qquad (3)$$

($\Delta_{\mu s}^c$ is the causal function for mass μ_s). This result is unaltered if we assume that the auxiliary particles B have a continuous distribution in the mass. Then

$$\sum_s \xi_s^2\eta_s \to \int d\mu^2\sigma(\mu^2),$$

where σ is an arbitrary real weight function. The gauge with $d_l = 3$, which was discussed in Sec. 1, corresponds to the choice $\sigma(\mu^2) = 2\,\delta'(\mu^2)$.

In an arbitrary gauge we shall thus assume that the states $|\alpha p\rangle$ contain the particles B besides the electron and the photons. Having in mind the special gauge with $d_l = 3$, we may assume the masses of the auxiliary particles to be very small.

We are interested in the behavior of S(p) and w(p) for $p^2 - m^2 \equiv p_1^2 \ll m^2$. In the c.m.s.

$$\mathbf{q} = -\Sigma\mathbf{k}_i, \quad q_0 \approx m + (\Sigma\mathbf{k}_i)^2/2m.$$

From the law of conservation of energy we have

$$\Sigma k_{i0} \approx p_1^2/2m, \quad k_{i0} \leqslant p_1^2/2m.$$

We shall assume in the following that the masses of the auxiliary particles are $\mu_S \ll p_1^2/2m$.

Let us consider the matrix element

$$\langle 0|\psi(0)|qk_i\rangle = (2k_{10})^{-1/2} \ldots (2k_{n0})^{-1/2} M^{(n)}(qk_i)u(q).$$

The momenta k_i refer to the photons as well as to the particles B. Evidently

$$M^{(n)}(qk_i) = \varepsilon_1^{\alpha_1}(k_1)\ldots\varepsilon_n^{\alpha_n}(k_n)M^{(n)}_{\alpha_1\ldots\alpha_n}(qk_i).$$

where $\epsilon(k)$ are the polarization vectors for the photons and $\epsilon(k) = ik\xi_S$ the polarization vectors for the B_S particles. Our task is to investigate $M^{(n)}(qk_i)$ for small k_i.

3. We use the generalized Ward identity which gives for M[2]

$$M^{(n)}(\varepsilon_i \to k_i) = -eM^{(n-1)}, \qquad (4)$$

where $M^{(n-1)}$ does not contain the vector k_i. For the case $n = 1$

$$M^{(1)}(\varepsilon \to k) = -e. \qquad (5)$$

Let us now choose some relativistically invariant function $L(q, k, \epsilon)$ which is linear in ϵ and reduces to $-e$ when $\epsilon \to k$. A convenient choice is

$$L = -e[2(q\varepsilon) + (k\varepsilon)]/[2(qk) + k^2]. \qquad (6)$$

Then we conclude from (5)

$$M^{(1)} = L + T^{(1)}, \qquad (7)$$

where $T^{(1)}(\epsilon \to k) = 0$.

Let us now consider (4) for $n = 2$. We find that

$$M^{(2)} = L_1 M^{(1)}(k_2) + \tilde{M}^{(2)},$$

where $\tilde{M}^{(2)}(\epsilon_1 \to k_1) = 0$. Furthermore, using (7) and the symmetry of $M^{(2)}$ under the interchange of $k_1\epsilon_1$ and $k_2\epsilon_2$, we obtain

$$M^{(2)} = L_1L_2 + L_1T_2^{(1)} + L_2T_1^{(1)} + T^{(2)}, \qquad (8)$$

where $T^{(2)}(\epsilon_1 \to k_1) = 0$ and $T^{(2)}(\epsilon_2 \to k_2) = 0$. Continuing this procedure we represent the matrix element $M^{(n)}$ in the form of a sum of terms containing either L or expressions T which are orthogonal to their vectors k_i:

$$M^{(n)} = \prod_{i=1}^{n} L_i + \sum_{j=1}^{n} T_j^{(1)} \prod_{i=1\neq j}^{n} L_i + \sum_{j_1\neq j_2} T^{(2)}_{j_1j_2} \prod_{i\neq j_1\neq j_2} L_i + \ldots + T^{(n)}, \qquad (9)$$

where $T^{(n)}$ has the following property: $T^{(n)}(\epsilon_i \to k_i) = 0$.

Let now all k_i tend uniformly to zero: $k_i = \rho\tilde{k}_i$ and $\rho \to 0$. Then $L \sim \rho^{-1}$ and the first term in (9) behaves like ρ^{-n}. It is essential that the remaining terms increase less rapidly by at least one power of ρ.

Indeed, let us consider the graphs of perturbation theory. For small ρ we make in the Feynman integrals a substitution of the variable of integration: $l_i = \rho\tilde{l}_i$, where the l_i are the momenta of the virtual photons connected with the electron line q. It is easy to estimate the general behavior of the Feynman integral for $\rho \to 0$. The electron line connected with the electron q gives the factor ρ^{-N}, where N is the total number of emitted and absorbed photons. Assume that m real photons ($m \leq n$) are absorbed on this line, N_1 photons are connected with other spinor lines and N_2 photons are emitted and absorbed by the same line. Then $N = m + N_1 + 2N_2$. The photon lines give the factor $\rho^{-2(N_1+N_2)}$, and the volume elements in the region of integration contribute $\rho^{4(N_1+N_2)}$. Actually the number of integrations will be less: for each closed spinor loop there will be a δ function which eliminates four integrations. On the other hand, each closed fermion loop contributes at least four additional powers of ρ and this compensates for the reduction in the number of integrations. The over-all dependence on ρ is determined by the factor ρ^a, where $a = 4(N_1+N_2) - 2(N_1+N_2) - m - N_1 - 2N_2 = -m + N_1$. We have maximal increase for $m = n$ and $N_1 = 0$. In the Feynman numerators we include only the leading terms, i.e., the factors $\hat{q} + m$. Each γ matrix is sandwiched between two factors $\hat{q} + m$ and is replaced by q/m.

After summing all graphs we obtain the main term in the form $\rho^{-n}(q\epsilon_1)\ldots(q\epsilon_n)f(q, \tilde{k}_i)$. The corrections increase by one power of ρ less. The leading term cannot vanish for $\epsilon_i \to k_i$ and hence, must be identified with the first term in (9). It follows from this that all terms in (9) except the first have the order ρ^{1-n}.

It is easy to find also the general structure of the corrections increasing like ρ^{1-n}. Owing to the denominators in the Feynman integral there will be corrections of the form

$$(q\varepsilon_1)\ldots(q\varepsilon_n)\left(f_1 + \sum_{ij}(k_ik_j)f_{ij}\right),$$

where f_1 and f_{ij} depend only on (qk_i). The numerators give corrections which may contain one factor of the type $(k_i\epsilon_j)$, $(\epsilon_i\epsilon_j)$, $[\hat{k}_i, \hat{\epsilon}_j]$, or $[\hat{\epsilon}_i, \hat{\epsilon}_j]$, besides $(q\epsilon_i)$ and functions which depend on (qk_i). This follows from the fact that, in taking account of the terms $\sim\rho^{1-n}$ all γ matrices in the vertices except the two outer ones can be replaced by q/m.

One can show by direct calculation that it is impossible to construct from all these terms an expression which vanishes under the replacement of more than two ϵ by their k. Hence $T^{(n)}$ with $n \geq 3$ increases no more rapidly than ρ^{2-n}.

For n = 1 we have with our choice of L

$$T^{(1)} = \left\{\frac{\lambda}{2}(\hat{q} + m)[\hat{k}, \hat{\varepsilon}] - \frac{e}{2m}(qk)[\hat{q}, \hat{\varepsilon}] + \frac{e}{2m}(q\varepsilon)[\hat{q}, \hat{k}]\right\} \Big/ [2(qk) + k^2], \tag{10}$$

where λ is the magnetic moment of the electron.[2]

For n = 2 we may construct $T^{(2)}$ from the terms mentioned in the form

$$T^{(2)} = [(q\varepsilon_1)(q\varepsilon_2)(k_1k_2) - (q\varepsilon_1)(qk_2)(k_1\varepsilon_2) - (q\varepsilon_2)(qk_1)(k_2\varepsilon_1) + (\varepsilon_1\varepsilon_2)(qk_1)(qk_2)]f, \tag{11}$$

where f depends on (qk_i) and increases like ρ^{-3}. We note that the explicit form of f has been determined by Soloviev.[2] However, this author effectively used the assumption that f contains no terms which are finite when only one of the two k_i goes to zero. The validity of this assumption is unclear.

We must still find out how $T^{(n)}$ behaves when only part of the momenta, say $k_1, \ldots, k_m$ (m < n) go to zero. A consideration of the graphs of perturbation theory shows as before that in this case the maximal increase is connected with the graphs which when summed up give a contribution to $M^{(n)}$ of the form

$$M_1^{(n-m)}(q + k_1 + \ldots + k_m, k_{m+1}, \ldots, k_n) M^{(m)}(q, k_1, \ldots, k_m),$$

where $M_1^{(n-m)} \to M^{(n-m)}$ for $k_i \to 0$ (i = 1, ..., m). If $k_i = \rho\tilde{k}_i$ (i = 1, ..., m) and $\rho \to 0$, this contribution increases like ρ^{-m}. The corrections increase less rapidly by one power of ρ. Since the part which increases like ρ^{-m} is equal to

$$M^{(n-m)} \prod_{i=1}^{m} L_i,$$

it does not enter in $T^{(n)}$. It is clear that $T^{(n)}$ increases like ρ^{1-m} for $\rho \to 0$. In sum we find that the transverse expression $T^{(n)}(q\tilde{k}_i)$ behaves like ρ^{1-m} when any m vectors $k_i = \rho\tilde{k}_i$ and $\rho \to 0$, where $1 \leq m \leq n$. For $m = n \geq 3$ the behavior is determined by the factor ρ^{2-n}.

4. Let us now turn to the quantity w. Substituting in this quantity the expansion (9) of the matrix elements, we obtain a sum of a number of terms. The emission or absorption of particles corresponds either to the factors L or to the transverse expressions T. Let us consider first those terms in w in which there are no particles emitted or absorbed by the factors L. Since each particle is here connected with the transverse T these terms do not depend on the gauge, and the contribution of the B particles to these terms is zero. A typical term has the form (after summing over the spins of the electron)

$$\sum_{\varepsilon}\int \frac{d^3k_1}{2k_{10}} \cdots \frac{d^3k_n}{2k_{n0}} \frac{1}{2q_0} \delta\left(\sum k_{i0} - \frac{p_1^2}{2m}\right) F_{nml}(q, k_i, \varepsilon_j), \tag{12}$$

where

$$F_{nml} = L_1 \ldots L_m L_{l+1}^* \ldots L_n^* T_{m+1\ldots n}^{(n-m)} (\hat{q} + m) \overline{T}_{1\ldots l}^{(l)}, \tag{13}$$

with $0 \leq m \leq l \leq n$. The symbol Σ_ϵ denotes the summation over the polarizations of the photons.

Let us show first of all that this expression exists. In estimating the convergence of the integral (12) we may disregard the δ function and assume that the integration over each momentum goes from zero to about $p_1^2/2m$. If any s momenta go to zero $[k_i = \rho\tilde{k}_i$, $i = i_1, \ldots, i_s \subset (1, \ldots, n)$ and $\rho \to 0]$ the dependence of the integrand on ρ will be composed of ρ^{3s-1} from the differentials of these momenta, ρ^{-s} from the factors $2k_{i0}$, and $\rho^{-2s+\nu_s}$ from F_{nml}. The quantity ν_s is equal to 1 or 2, depending on whether the set $(i_1, \ldots, i_s)$ belongs in (1, ..., m) or (l + 1, ..., n) or not. The integral over ρ will have the form $\int d\rho \cdot \rho^{-1+\nu_s}$ and will exist for arbitrary s and $i_1, \ldots, i_s$, including the case s = n. This guarantees the existence of the integral (12).

With account of the δ function the general dependence of the integral (12) on p_1^2 will be of the form

$$(p_1^2)^{-1+\nu_n}(\text{const} + O(p_1^2)). \tag{14}$$

The quantity ν_n depends on the numbers n, m, and l. If $m \neq n$ and $l \neq 0$, then $\nu_n \geq 2$ ($\nu_n > 2$, if either n − m or $l \geq 3$ owing to the asymptotic form of $T^{(n-m)}$ or $T^{(l)}$). If m = n or l = 0, then $\nu_n \geq 2$ if $n \geq 3$, and possibly $\nu_n = 1$ if $n \leq 2$. We shall show now that even in the last case $\nu_n \geq 2$.

Indeed, the summation over the polarizations of the photons reduces for m = n or l = 0 in first approximation to the replacement of all ϵ_i by q in $T^{(n)}$. As is seen from (10) and (11) a factor appears in $T^{(1,2)}$ which gives an additional power of ρ after the integration over angles. Therefore the total contribution to w of the terms of the type considered is finite and vanishes at least as rapidly as p_1^2 for $p_1^2 \to 0$.

5. Let us now turn to the case where part of the particles are emitted and absorbed by the factors L. Let there be one such particle with momentum

k and polarization ϵ. The corresponding contribution to w is written in the form of a sum of terms of the form

$$\sum_{\varepsilon_i}\sum_{\varepsilon}\int\frac{d^3k_1}{2k_{10}}\cdots\frac{d^3k_n}{2k_{n0}}\int\frac{d^3k}{2k_0}\frac{1}{2q_0}\times\delta\Big(\sum_{i=1}^{n}k_{i0}+k_0+q_0-p_0\Big)|L|^2F_{nml}, \quad (15)$$

where F_{nml} is again given by (13).

Let us first carry out the summation over the polarizations (and kinds) of the particles with momentum k. We use here a special choice of the gauge with $d_l = 3$. An elementary calculation (which is conveniently done in the system where $\mathbf{q}+\mathbf{k}=0$) yields in this gauge

$$\sum_{\varepsilon}|L|^2=\frac{e^2}{(qk)^2}\sum_{a}((aq)(kq)-m^2(ak))\frac{\partial}{\partial(aq)}. \quad (16)$$

The sum is taken over all four-vectors a on which F_{nml} depends including the γ matrices.

Let us denote the result of operating with the right-hand side of (16) on F_{nml} by $F^{(1)}_{nml}$. We call attention to the fact that $F^{(1)}$ behaves no worse than F if the vectors k, $k_1, \ldots, k_n$ go to zero. Indeed, when any s of these vectors are small ($\sim\rho\to 0$) then the behavior of $F^{(1)}$ will be characterized by the factor $\rho^{-2s+\nu^{(1)}_s}$, where $\nu^{(1)}_s \geq 1$. Therefore the entire integral (15) will converge. Its dependence on p_1^2 is determined as $(p_1^2)^{-1+\nu^{(1)}_{n+1}}$. It is easy to see that $\nu^{(1)}_{n+1} = 1+\nu_n$, where ν_n determines the behavior of the function F. It is clear that $\nu^{(1)}_{n+1}\geq 2$ except possibly in the case when n = 0, and there are only particles connected with the factors L. This corresponds to $F_0=\hat{q}+m$. After summing over ϵ we find in this case

$$e^2(\hat{q}(qk)-m^2\hat{k})/(qk)^2.$$

After integration over the angles of **k** this expression behaves for $k=\rho\tilde{k}$ and $\rho\to 0$ not like ρ^{-1} but like a constant, which corresponds to $\nu_1^{(1)}=2$. Therefore the entire contribution to w from the terms corresponding to the emission and absorption of one particle by the factors L is finite and vanishes at least as rapidly as p_1^2 for $p_1^2\to 0$.

The case of two or more such particles is treated analogously. We then obtain functions $F^{(2)}$, $F^{(3)}$, etc., which arise from the twofold, threefold, etc., application of the differential operators (16) on F. They all behave no worse than F if any momenta go to zero, and therefore their integrals, which enter in w, exist. The behavior of $F^{(r)}$ under the condition that all momenta go to zero will be characterized by the factor $\rho^{-2(n+r)+\nu^{(r)}_{n+r}}$, where $\nu^{(r)}_{n+r}=r+\nu_n$. Correspondingly, the contribution of these terms to w will vanish at least as rapidly as $(p_1^2)^r$ for $p_1^2\to 0$.

6. Summarizing all results, we find that the contribution to w from the states with photons in the chosen gauge with $d_l = 3$ is finite (contains no infrared divergences) and behaves like p_1^2 for $p_1^2\to 0$. After integrating w according to (1) we find the Green's function of the electron. The one-electron term gives rise to a pole term $1/(m-\hat{p})$. The states with photons give an expression which is finite for p^2-m^2. The singularity of the pole-free term evidently has the character $p_1^2\ln(-p_1^2)$.

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