LIGHT ABSORPTION IN SEMICONDUCTORS IN CROSSED ELECTRIC AND MAGNETIC FIELDS

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The effect of an electric and magnetic field on the light absorption coefficient in semiconductors is considered. It is shown that the Franz–Keldysh effect occurs in a magnetic field when $cE_x/sH_z > 1$, where $s = (\epsilon_g/2m)^{1/2}$ and when the electron motion is infinite and the spectrum is continuous; in this case, when the magnetic field is increased, the absorption coefficient decreases more rapidly with decreasing frequency than when $H = 0$. The spectrum is discrete for $cE_x/sH_z < 1$, and with increasing electric field the absorption edge shifts towards low, frequencies, and the possibility of the allowed transition decreases while that of the forbidden ones increases.

1. INTRODUCTION

As is well known, in a strong magnetic field the stated density of the electrons and holes has singularities near the edge of the Landau sub-bands. Oscillations of the absorption coefficient of light, connected with these singularities, were first observed in 1957.[1-4] Since that time, the study of magneto-optic effects—absorption,[1-9] reflection,[10] interband Faraday and Voigt effects[11]—has become one of the most widely used and reliable methods for investigating the band structure of semiconductors. Aronov[12] has shown in 1963, that measurement of the same effects in crossed electric and magnetic fields makes it possible to observe forbidden transitions and to determine separately, both from the distances between lines and from their shift in the electric field, the effective masses and the g factors in each band, something impossible in measurements in magnetic fields alone. This effect was observed in Ge by Vrehen and Lax.[13, 14] Lax[15] noted in his review that the expressions given in [12] are valid only in a sufficiently weak electric field, and that to solve the problem in the general case it is necessary to take into account the non-parabolicity of the bands. We shall show, however, that considerations advanced in [15] with regards to the behavior of the absorption coefficient in a strong electric field, are apparently in error.

The change in the absorption coefficient in a strong electric field was theoretically predicted by Keldysh[16] and by Franz[17] in 1958, and was observed by now in many crystals.[18-24] This effect has already been the subject of a large number of theoretical[25-28] and experimental[18-24] papers. The influence of magnetic fields on the Franz–Keldysh effect, insofar as we know, has never been considered before. The purpose of the present study is to calculate the dependence of the light absorption coefficient connected with interband transitions, for an arbitrary ratio of the magnetic and electric fields.

We consider here the simplest case, when both bands are not degenerate and have an extremum in the same point of k-space. In this case the two-band equation, as indicated in [29-31], can be reduced to a form that differs from the Dirac equation only in that c is replaced by

$$s = (\varepsilon_g / 2m)^{1/2}, \tag{1}$$

where ϵ_g is the width of the forbidden band, and m is either the effective mass for a spherical band or the reduced state-density mass $(m_x m_y m_z)^{1/3}$ for an anisotropic band. In the latter case the true fields E and H in the Dirac equation must be replaced by the reduced fields

$$E_i' = E_i \left(\frac{m}{m_i}\right)^{1/2}, \qquad H_i' = \frac{s}{c} H_i \left(\frac{m_i}{m}\right)^{1/2}, \tag{2}$$

where E_i and H_i are the components of the fields along the axis of the given ellipsoid. By using a suitable transformation that also differs from the Lorentz transformations in that c is replaced by s, it is possible to eliminate in this equation the magnetic field when $E' > H'$ in the moving coordinate system, and the electric field when $E' < H'$. Therefore the level quantization remains when $E' < H'$, and transitions take place between the Landau levels. The shift of the absorption edge in the electric field is in this case connected with the fact that in the direct transitions, when the momentum of the electron is conserved, the center of gravity of the oscillator shifts along the field and accordingly the energy required for the transition is reduced. In indirect transitions, to the contrary, when the position of the center of gravity of the oscillator does not change, the absorption edge shifts towards higher frequencies, as shown in [32].

When $E' > H'$, the motion of the electron is infinite and no quantization takes place, while the shift of the absorption edge is due to the penetration of the electron into the forbidden band—an analog of the quasiclassical penetration through a potential barrier. Therefore in the former case one should speak of the influence of the electric field on the magnetooptical transitions, and in the latter of the influence of the magnetic field on the Franz–Keldysh effect. We shall consider both cases below.

2. THE FRANZ-KELDYSH EFFECT IN A MAGNETIC FIELD

The two-band equation in crossed fields takes in the presence of an alternating field

$$\mathbf{E}_{\sim}' = 2\mathbf{E}_0' \sin(\omega_0 t' - \mathbf{q}_0\mathbf{r}') \tag{3}$$

the form

$$\hat{\mathscr{H}}\Psi = \{ms^2\rho_3 + \rho_1 s(\mathbf{P}'\sigma) - e\Phi'\}\Psi = i\hbar\partial\Psi/\partial t', \tag{4}$$

where

$$P_i' = \mathscr{P}_i' + \frac{e}{s}A_i', \quad \hat{\mathscr{P}}_i' = -i\hbar\frac{\partial}{\partial x'},$$

$$A_i' = \frac{s}{c}A_i\left(\frac{m}{m_i}\right)^{1/2}. \tag{5}$$

Here ρ_i and σ_i are the corresponding 4×4 matrices,[33] and

$$\mathbf{E}_0' = \omega_0 s^{-1}\mathbf{A}_0' - \mathbf{q}_0\Phi_0' \tag{6}$$

is the transformed amplitude, the projections of which are connected with the true values of the field E_{0i} by relations similar to (2). For the alternating components $\mathbf{A}' = 2\mathbf{A}_0' \cos(\omega_0 t' - \mathbf{q}_0 \cdot \mathbf{r}')$ and $\Phi_{\sim} = 2\Phi_0' \cos(\omega_0 t' - \mathbf{q}_0 \cdot \mathbf{r}')$ we choose a gauge such that $\Phi_{\sim} = 0$ in the moving frame. In order for the magnetic field to vanish in this reference frame when $E' > H'$, the velocity in the y direction, perpendicular to the magnetic field $\mathbf{E}$ ($\parallel x$) and $\mathbf{H}$ ($\parallel z$) should be equal to βs, where $\beta = H'/E'$. In this frame, the effective electric field is

$$E = (E'^2 - H'^2)^{1/2} = E'(1-\beta^2)^{1/2}.$$

To abbreviate the notation, all the quantities in the moving coordinate system will henceforth be designated by unprimed indices. If we now go over to the k-representation, then in this coordinate system

$$\hat{\mathscr{H}} = \hat{\mathscr{H}}_0 + \hat{\mathscr{H}}_{\text{int}}, \ \hat{\mathscr{H}}_0 = \{ms^2\rho_3 + \rho_1\hbar s(\mathbf{k}\sigma) + ieE\partial/\partial k_x\}, \tag{7}$$

$$\hat{\mathscr{H}}_{\text{int}} = e\rho_1\{(\mathbf{A}_0\sigma)e^{i(\omega t - \mathbf{qr})} + \text{h.c.}\}, \tag{8}$$

where in accordance with the Lorentz transformations and the chosen potential gauge

$$A_{0x} = A_{0x}' = \frac{s}{\omega_0}E_{0x}', \quad A_{0z} = A_{0z}' = \frac{s}{\omega_0}E_{0z}',$$

$$\Phi = \frac{\Phi_0' + \beta A_{0y}'}{(1-\beta^2)^{1/2}} = 0,$$

$$A_{0y} = \frac{A_{0y}' + \beta\Phi_0'}{(1-\beta^2)^{1/2}} = (1-\beta^2)^{1/2}A_{0y}'$$

$$= (1-\beta^2)^{1/2}\frac{s}{\omega_0}E_{0y}' \tag{9}$$

and accordingly

$$\omega = \frac{\omega_0}{(1-\beta^2)^{1/2}}, \quad q_y = \frac{\beta}{s}\frac{\omega_0}{(1-\beta^2)^{1/2}}, \quad q_x = q_z = 0. \tag{10}$$

We assume here that the photon momentum q_0 in the stationary system of coordinates is equal to zero, neglecting by the same token the relativistic terms of order s/c.

Just as in the calculation of the tunnel current in our earlier paper,[31] it is necessary, for the calculation of the probability of the interband optical transitions, to go over to a representation describing the state of the electron in the valence band and in the conduction band, for which purpose we carry out a transformation similar to the Foldy-Wouthuysen transformation. As shown in [31], in the new representation

$$\tilde{\mathscr{H}}_0 = e^{iS(\mathbf{k})}\mathscr{H}_0 e^{-iS(\mathbf{k})} = \rho_3 ms^2\eta + ieE\frac{\partial}{\partial k_x}, \tag{11}$$

where

$$\eta^2 = (1 + \hbar^2k^2/m^2s^2), \tag{12}$$

and

$$\tilde{\mathscr{H}}_{\text{int}} = e(\widetilde{\rho_1\mathbf{A}_0\sigma})e^{i(\omega t - \mathbf{qr})} + \text{h.c.}, \tag{13}$$

where

$$(\widetilde{\rho_1\mathbf{A}_0\sigma}) = e^{iS(\mathbf{k})}(\rho_1\mathbf{A}_0\sigma)e^{iS(\mathbf{k}-\mathbf{q})}. \tag{14}$$

We took account here of the fact that

$$e^{i\mathbf{q}\hat{\mathbf{r}}}F(\mathbf{k}) = F(\mathbf{k}+\mathbf{q})e^{i\mathbf{q}\hat{\mathbf{r}}}. \tag{15}$$

Here, according to [31],

$$e^{\pm iS(\mathbf{k})} = \left(\frac{\eta+1}{2\eta}\right)^{1/2} \pm i\rho_2\frac{(\sigma\mathbf{k})}{k}\left(\frac{\eta-1}{2\eta}\right)^{1/2}. \tag{16}$$

We neglect in (11) the interband matrix elements proportional to E, as well as the small additional intraband term

$$\delta\mathscr{H} = -e\mathbf{E}\frac{\eta-1}{2\eta}\frac{[\sigma\mathbf{k}]}{k^2}. \quad (17)^*$$

As will be shown in the appendix, allowance for this term has little effect on the absolute value of the absorption coefficient but the term $\delta\mathscr{H}$ must be taken into account when calculating its dependence on the light polarization direction.

The eigenfunctions of the operator $\widetilde{\mathscr{H}}_0$ are

$$\Psi_{\mu_i}(\mathbf{k}) = (eEL_x)^{-1/2}\exp\left\{\frac{i}{eE}\int_0^{k_x}(\varepsilon_i - \varepsilon_{0i}(k))\,dk_x\right\}$$

$$\times\delta_{k_y k_{yi}}\delta_{k_z k_{zi}}, \quad (18)$$

where ϵ_{0i} are the eigenvalues of the energy at E = 0:

$$\varepsilon_{0i}(k) = \mp ms^2\eta. \quad (19)$$

The index i = 1 pertains here to the valence band, and i = 2 to the conduction band, while $\mu_i = (k_{yi}k_{zi}\epsilon_i, l_i)$, with l_i running through two values corresponding to the two values of the spin. The probability of the interband transition is

$$W_{12} = \frac{2\pi}{\hbar}\sum_{l_1 l_2}|M_{\mu_1\mu_2}|^2\delta(\varepsilon_2 - \varepsilon_1 - \hbar\omega), \quad (20)$$

where according to (13), (15), (18), and (10)

$$M_{\mu_1\mu_2} = e\sum_{\mathbf{k}}\Psi_{\mu_2}^*(\mathbf{k})(\widetilde{\rho_1\mathbf{A}_0\sigma})_{l_1 l_2}\Psi_{\mu_1}(\mathbf{k}+\mathbf{q})$$

$$= \frac{1}{2\pi E}\delta_{k_{y_2}k_{y_1}+q}\delta_{k_{z_1},k_{z_2}}\int_{-\infty}^{+\infty}dk_x(\widetilde{\rho_1\mathbf{A}_0\sigma})_{l_1 l_2}$$

$$\times\exp\left\{i\frac{ms^2}{eE}\int_0^{k_x}\left[\eta(k) + \eta(k+q) - \frac{\hbar\omega}{ms^2}\right]dk_x\right\}. \quad (21)$$

Here $(\widetilde{\rho_1\mathbf{A}_0\sigma})_{l_1 l_2}$ are the interband matrix elements of the corresponding matrices. Since the matrix element $(\widetilde{\rho_1\mathbf{A}_0\sigma})_{l_1 l_2}$ has no singularities below the saddle point k_{x0}, the integral in (21) can be calculated by the usual saddle-point method when $a \gg 1$ and $\hbar\omega_0 < \epsilon_g$. To this end we introduce the dimensionless variables

$$a = \frac{m^2s^3}{eE\hbar}, \quad \Delta = \frac{\hbar\omega}{2ms^2}, \quad x = -i\frac{\hbar k_x}{ms},$$

$$y = \frac{\hbar(k_y + q/2)}{ms}, \quad z = \frac{\hbar k_z}{ms}, \quad u = \frac{\hbar q}{2ms} = \beta\Delta. \quad (22)$$

Then, carrying out the integration, we find that

$$|M_{\mu_1\mu_2}|^2 = \frac{1}{2\pi}\left(\frac{ms}{\hbar E}\right)^2|(\widetilde{\rho_1\mathbf{A}_0\sigma})_{l_1 l_2}|^2\frac{1}{\varphi_x''(x_0, y, z)}$$

$$\times e^{-2\varphi(x_0, y, z)}\delta_{y_1-u,\, y_2+u}\delta_{z_1 z_2}, \quad (23)$$

where

$$\varphi(x, y, z) = a\int_0^x(\eta_1 + \eta_2 - 2\Delta)\,dx, \quad (24)$$

$$\eta_1^2 = 1 - x^2 + (y+u)^2 + z^2,$$

$$\eta_2^2 = 1 - x^2 + (y-u)^2 + z^2. \quad (25)$$

The saddle point x_0 is determined by the condition $\varphi'_x(x_0) = 0$, i.e.,

$$\eta_1(x_0) + \eta_2(x_0) - 2\Delta = 0. \quad (26)$$

The total number of transitions can be calculated immediately in the moving system of coordinates, since the product Vt remains invariant under the Lorentz transformations. To this end it is necessary to sum (20) over k_{y1}, k_{y2}, k_{z1}, k_{z2}, l_1, and l_2 and to integrate over ϵ_1 and ϵ_2. After summing over k_{y2} and k_{z2} and integrating over ϵ_2, we replace the summation over k_{y1} and k_{z1} by integration, which is then carried out by the saddle point method, while the integration with respect to ϵ_1 yields eEL_x. We then obtain finally for the number of transitions per cm^3 and per second

$$I = \frac{(ms)^4}{4\pi eE\hbar^5}S[\varphi_y''(x_0, y_0, z_0)\varphi_z''(x_0, y_0, z_0)]^{-1/2}\varphi_x''^{-1}(x_0, y_0, z_0)$$

$$\times\exp\{-2\varphi(x_0, y_0, z_0)\}, \quad (27)$$

where

$$S = e^2\sum_{l_1 l_2}|(\widetilde{\rho_1\mathbf{A}_0\sigma})^2_{x=x_0,\, y=y_0,\, z=z_0}. \quad (28)$$

The saddle points y_0 and z_0 are determined from the vanishing of the first derivatives φ'_z and φ'_y. From (24) we find that $z_0 = 0$, and y_0 is determined from the equation

$$(y_0 + u)\arcsin\frac{x_0}{\sqrt{1 + (y_0+u)^2}}$$

$$+ (y_0 - u)\arcsin\frac{x_0}{\sqrt{1 + (y_0-u)^2}} = 0,$$

the root of which is $y_0 = 0$. According to (26) we have

$$\eta_1(x_0, y_0, z_0) = \eta_2(x_0, y_0, z_0) = (1 + u^2 - x_0^2)^{1/2} = \Delta \quad (29)$$

and consequently, according to (29) and (10),

$$x_0^2 = 1 - \Delta^2(1-\beta^2) = 1 - \Delta_0^2, \quad y_0 = z_0 = 0, \quad (30)$$

where

$$\Delta_0 = \hbar\omega_0/2ms^2. \quad (31)$$

*$[\sigma\mathbf{k}] \equiv \sigma\times\mathbf{k}$.

Formula (27) is valid so long as the saddle point k_{x0} lies on the imaginary axis, i.e., so long as x_0 is real. From (30) we see that the absorption edge, i.e., the frequency ω_0 below which the absorption coefficient attenuates exponentially in accordance with (27) is determined by the condition $\Delta_0 = 1$ or $\hbar\omega_0 = \epsilon_g$. Consequently, the absorption edge does not shift in the magnetic field if the electric field is strong. According to (24), (25), (29), and (30) we have

$$\varphi(x_0, y_0, z_0) = a\left\{(1+\beta^2\Delta^2)\arcsin\left(\frac{1-\Delta_0^2}{1+\beta^2\Delta^2}\right)^{1/2} - \Delta(1-\Delta_0^2)^{1/2}\right\}, \quad (32)$$

$$\varphi_x''(x_0, y_0, z_0) = 2a(1-\Delta_0^2)^{1/2}/\Delta, \quad (33)$$

$$\varphi_y''(x_0, y_0, z_0) = 2a\left\{\arcsin\left(\frac{1-\Delta_0^2}{1+\beta^2\Delta^2}\right)^{1/2} - \beta^2\Delta\frac{(1-\Delta_0^2)^{1/2}}{1+\beta^2\Delta^2}\right\}, \quad (34)$$

$$\varphi_z''(x_0, y_0, z_0) = 2a\arcsin\left(\frac{1-\Delta_0^2}{1+\beta^2\Delta^2}\right)^{1/2}. \quad (35)$$

At the same point x_0, y_0, and z_0 we have in accord with (16)

$$e^{iS(x_0,\, y_0\pm u,\, z_0)} = \left(\frac{\Delta+1}{2\Delta}\right)^{1/2} + i\rho_2\frac{\sigma_y u \mp \sigma_x x_0}{[2\Delta(\Delta+1)]^{1/2}} \quad (36)$$

Consequently, in accordance with (14) and (36), the interband part of the operator $\widetilde{(\rho_1\mathbf{A}_0\sigma)}$ is equal to

$$\widetilde{(\rho_1\mathbf{A}_0\sigma)} = \rho_1\left[\frac{A_{0x}\sigma_x}{\Delta} + A_{0y}\sigma_y + A_{0z}\left(\sigma_z\frac{\Delta+\Delta_0^2}{\Delta(\Delta+1)} + i\beta\frac{(1-\Delta_0^2)}{1+\Delta}\right)\right] \quad (37)$$

and from (28)

$$S = 2e^2\left\{A_{0x}^2\frac{(1-\beta^2)}{\Delta_0^2} + A_{0y}^2 + A_{0z}^2(1-\beta^2) \times\frac{1+\Delta_0(1-\beta^2)^{1/2}+\beta^2(1-\Delta_0^2)}{[(1-\beta^2)^{1/2}+\Delta_0]^2}\right\} \quad (38)$$

When $a(1-\Delta_0)^{7/2} \ll 1$ we can expand (32)–(37) in powers of $(1-\Delta_0)$, retaining the first nonvanishing terms. Then

$$\varphi(x_0, y_0, z_0) = \frac{4\sqrt{2}}{3}a_0(1-\Delta_0)^{3/2}\times\left[1+\frac{1}{5}(1-\Delta_0^2)\left(2\beta^2-\frac{3}{4}\right)\right], \quad (39)$$

$$a_0 = a\sqrt{1-\beta^2} = m^2s^3/eE'\hbar,$$

$$\varphi_x''(\varphi_y''\varphi_z'')^{1/2} = 2(2a)^2(1-\Delta_0)(1-\beta^2)^{3/2} \quad (40)$$

and according to (9) and (38)

$$S = 2s^2\omega_0^{-2}(1-\beta^2)e^2E_0'^2. \quad (41)$$

Consequently, in this approximation the anisotropy of the absorption coefficient, connected with I by the relation

$$\alpha = \frac{2\pi\hbar\omega_0}{cnE_0^2}I, \quad (42)$$

is determined only by the anisotropy of the effective masses. If the vector $\mathbf{E}_0$ is directed along the principal axis of the ellipsoid then, in accordance with (2), we have $E_0'^2/E_0^2 = (m_x^2/m_ym_z)^{1/3}$, and according to (27) and (39)–(42)

$$\alpha(H) = \frac{1}{8}\left(\frac{m_x^2}{m_ym_z}\right)^{1/3}\frac{e^3E'\varepsilon_g}{\hbar^2cn\omega_0(\varepsilon_g-\hbar\omega_0)} \times\exp\left\{-\frac{8\sqrt{2}}{3}a_0(1-\Delta_0)^{3/2} \times\left[1+\frac{1}{5}(1-\Delta_0)\left(2\beta^2-\frac{3}{4}\right)\right]\right\}. \quad (43)$$

Thus, the relative change in the absorption coefficient in the magnetic field is

$$\frac{\alpha(H)}{\alpha(0)} = \exp\left\{-\frac{4}{15}\frac{(\varepsilon_g-\hbar\omega_0)^{5/2}}{em^{1/2}c^2\hbar}\frac{\sum_i H_i^2m_i/m}{\left(\sum_i E_i^2m/m_i\right)^{3/2}}\right\}. \quad (44)$$

We see from (43) and (44) that the relative change in the absorption coefficient in a magnetic field increases with increasing distance from the edge of the band, whereas near the absorption edge, i.e., at $\hbar\omega_0 \approx \epsilon_g$, the magnetic field has practically no effect. The advantage of measurements in crossed fields, compared with ordinary measurements of the Franz-Keldysh effect, may apparently consist primarily in the fact that in this case it is possible to determine simultaneously both the band parameters and the values of the field E, which are usually not given with sufficient accuracy by measurements in p-n junctions or in near-boundary layers. Therefore such a method may be of interest as a direct method for measuring these fields.

In conclusion let us dwell briefly on the limits of applicability of the expressions derived above. The saddle-point method used in the integration of (23) and (27) is valid when $\varphi_x'' \gg 1$, $\varphi_y'' \gg 1$, and $\varphi_z'' \gg 1$. The most stringent criterion is connected with the integration with respect to y and is of the form

$$\frac{\varepsilon_g^{3/2}(\varepsilon_g-\hbar\omega_0)^{1/2}}{\sqrt{2}\,seE'\hbar}(1-\beta^2) \gg 1, \quad (45)$$

showing that the derived expressions become generally speaking incorrect when $\beta^2 \to 1$. On the other hand, as in the calculation of the tunnel current,[31] the electric field must be strong enough to be able to neglect the energy uncertainty connected with the scattering of the electrons and holes:

$$eE'(1-\beta^2)^{1/2} \gg m^{1/2}(\varepsilon_g - \hbar\omega_0)^{1/2}(1/\tau_p + 1/\tau_n). \quad (46)$$

3. MAGNETO-OPTIC EFFECT IN CROSSED FIELDS

If $H' > E'$ in a coordinate system moving with velocity βs, where now $\beta = E'/H'$ (as against H'/E' in Sec. 2), then the electric field vanishes and the magnetic field is

$$H = (H'^2 - E'^2)^{1/2} = H'(1-\beta^2)^{1/2}. \quad (47)$$

We direct the z axis along the magnetic field, choose in the moving system of coordinates the components of the reduced vector potential, defined by Eq. (5), in the usual form

$$A = (0, Hx, 0), \quad (48)$$

and choose the gauge of the potentials of the light-wave field, as in Sec. 2, such that $\Phi_0 = 0$ in the moving coordinate system. Then Eqs. (8)–(10) remain in force in this case, too.

Since the wave function is the bispinor $\Psi = \begin{pmatrix}\chi_1\\ \chi_2\end{pmatrix}$, the equation

$$(\mathcal{H} - \varepsilon)\Psi = [ms^2\rho_3 + \rho_1 s(\mathbf{P}\sigma) - \varepsilon]\Psi = 0 \quad (49)$$

can be represented as two equations for χ_1 and χ_2:

$$(ms^2 - \varepsilon)\chi_1 + s(\mathbf{P}\sigma)\chi_2 = 0, \quad (50a)$$

$$-(ms^2 + \varepsilon)\chi_2 + s(\mathbf{P}\sigma)\chi_1 = 0. \quad (50b)$$

Substituting χ_2 from (50b) in (50a), we get

$$\left\{\left(\mathbf{p} + \frac{e}{s}\mathbf{A}\right)^2 + \frac{e\hbar}{s}(\sigma\mathbf{H})\right\}\chi = \frac{\varepsilon^2 - m^2s^4}{s^2}\chi. \quad (51)$$

(In (51) we have left out the subscript of χ, since (51) is valid for both χ_1 and χ_2). This equation differs from the usual Pauli equation only in that $\epsilon - ms^2$ is replaced by $(\epsilon^2 - m^2s^4)/2ms^2$. We recall that here all the energies are measured from the center of the forbidden band. Since $H = H_z$ and σ_z is a diagonal matrix, the eigenvalues and the eigenfunctions of (51) are :[34]

$$\chi^{(\mu)} = \varphi_\alpha e^{ik_z z + ik_y y}\Phi_n\left(\frac{x - x_0}{r_m}\right), \quad (52)$$

$$\varepsilon_{2,1} = \pm\left[(ms^2)^2 + (\hbar s k_z)^2 + 2\hbar e s H\left(n + \frac{1}{2} + \frac{\alpha}{2}\right)\right]^{1/2}, \quad (53)$$

where φ_α is the eigenfunction of the matrix σ_z, $\alpha = \pm 1$ is the spin index, and $\Phi_n(x)$ is the normalized functions of the harmonic oscillator, $x_0 - r_m^2 k_y$—position of center of gravity of the oscillator and $r_m^2 = hs/eH$, the indices 1 and 2 pertain to the number of the band, and μ is the aggregate of the quantum numbers (n, α, k_y, k_z).

The wave functions for the carriers in a given band should be chosen such as to go over to the correct wave functions at the bottom of the band, i.e., such that $\chi_2 \to 0$ for Ψ_2 when $p \to 0$ and $\chi_1 \to 0$ when $p \to 0$ for Ψ_1. Then we obtain from (50a), (50b), (52) and (53), taking the normalization conditions into account,

$$\Psi_1 = \left(\frac{\varepsilon_1 - ms^2}{2\varepsilon_1}\right)^{1/2}\begin{vmatrix} -\dfrac{s(\mathbf{P}\sigma)}{ms^2 - \varepsilon_1} \\ 1 \end{vmatrix}\chi^{(\mu_1)}, \quad (54)$$

$$\Psi_2 = \left(\frac{\varepsilon_2 + ms^2}{2\varepsilon_2}\right)^{1/2}\begin{vmatrix} 1 \\ \dfrac{s(\mathbf{P}\sigma)}{ms^2 + \varepsilon_2} \end{vmatrix}\chi^{(\mu_2)}. \quad (55)$$

According to (8), the matrix element of the transition from the state μ_1 of the valence band to the state μ_2 of the conduction band is, in the moving system of coordinates,

$$M_{12}{}^i = \langle\Psi_2|e\rho_1 A_i\sigma_i e^{-i\mathbf{qr}}|\Psi_1\rangle, \quad (56)$$

and, substituting in (56), (54), and (55), we obtain

$$M_{12}{}^i = eA_i\left[\frac{(\varepsilon_2 + ms^2)(\varepsilon_1 - ms^2)}{4\varepsilon_1\varepsilon_2}\right]^{1/2}\left\{\langle\chi^{(\mu_2)}|\sigma_i e^{-i\mathbf{qr}}|\chi^{(\mu_1)}\rangle - \frac{s^2}{(ms^2 - \varepsilon_1)(ms^2 + \varepsilon_2)}\langle\chi^{(\mu_2)}|(\mathbf{P}\sigma)\sigma_i e^{-i\mathbf{qr}}(\mathbf{P}\sigma)|\chi^{(\mu_1)}\rangle\right\}. \quad (57)$$

The second term in (57) is of the order of $(\tilde{p}s/2m^2)^2 \sim \Delta\epsilon/2ms^2$, where $\Delta\epsilon$ is the characteristic distance from the given level to the edge of the corresponding band. Therefore for transitions between levels with not very large n, this term can be neglected. Then, calculating the integrals which enter in (57), and recognizing that $q_x = q_z - 0$ and $q_y = q$, we obtain

$$|M_{12}{}^i|^2 = e^2|\langle\varphi_{\alpha_2}|\sigma_i A_i|\varphi_{\alpha_1}\rangle|^2 |Q_{min(n_1, n_2)}^{|n_1 - n_2|}(a_0^2/2)|^2 \times \delta_{k_{z_1} k_{z_2}}\delta_{k_{y_2} k_{y_1+q}} \quad (58)$$

Here Q is the Leguerre function normalized to unity, and

$$a_0 = r_m q. \quad (58a)$$

From (58) we see that in the isotropic case, at polarizations parallel to x and y, the selection rules with respect to the spin are of the form

$\Delta\alpha = \pm 1$, and $\Delta\alpha = 0$ for polarization of light along the z axis. From (58) we see also that in crossed fields, when $a_0 \neq 0$, there are no selection rules with respect to the Landau quantum number, as was found in [12]. The number of transitions per cm^3 per second, which is invariant against the Lorentz transformations, is

$$I = \frac{1}{2\pi r_m^2} \sum_{\substack{n_1, n_2 \\ \alpha_1, \alpha_2}} |M_{12}|^2 \int_{-\infty}^{+\infty} dk_z \delta(\hbar\omega - \varepsilon_2 + \varepsilon_1). \quad (59)$$

From (9) and (59) it follows immediately that the transition probabilities for different polarizations are related like

$$|M_{12}^y|^2 / |M_{12}^x|^2 = 1 - \beta^2, \quad |M_{12}^x|^2 = |M_{12}^z|^2. \quad (60)$$

The dependence on the electric field will be determined by the function $|Q|^2$ from (58). From the energy conservation law it follows that transitions are possible only between states for which

$$\hbar\omega = [(ms^2)^2 + (\hbar s k_z)^2 + 2\hbar e s H(n_1 + 1/2 + \alpha_1/2)]^{1/2} + [(ms^2)^2 + (\hbar s k_z)^2 + 2\hbar e s H(n_2 + 1/2 + \alpha_2/2)]^{1/2}. \quad (61)$$

Since in a magnetic field the density of the states has singularities at $k_z = 0$, it follows from (61) and (10) that the positions of the absorption maxima is determined by the condition

$$\hbar\omega_0 = (1 - \beta^2)^{1/2} \{ [(ms^2)^2 + 2\hbar e s H'(1 - \beta^2)^{1/2}(n_1 + 1/2 + \alpha_1/2)]^{1/2} + [(ms^2)^2 + 2\hbar e s H'(1 - \beta^2)^{1/2}(n_2 + 1/2 + \alpha_2/2)]^{1/2} \}. \quad (62)$$

Consequently, for transitions between levels with small n we have

$$\hbar\omega_0 = (1 - \beta^2)^{1/2} \left\{ 2ms^2 + \hbar \frac{eH'}{ms} (1 - \beta^2)^{1/2} \left(n_1 + \frac{1}{2} + \frac{\alpha_1}{2} \right) + \hbar \frac{eH'}{ms} (1 - \beta^2)^{1/2} \left(n_2 + \frac{1}{2} + \frac{\alpha_2}{2} \right) \right\}. \quad (63)$$

For small β, formula (63) differs from the expression given in [12] in that it takes into account not only the shift of all the levels, but also the decrease in the distance between them with increasing electric field. In strong fields, when $\beta - 1$, the energy difference for given k_y, k_z and n_1, n_2 tends to zero like $(1 - \beta^2)^{1/2}$, and the distances between the Landau levels, and accordingly the cyclotron frequency ω_c, decrease and also tend to zero like $(1 - \beta^2)$. Since $a \to \infty$ when $\beta \to 1$, the probability of transition between given levels decreases rapidly. The decrease in the transition probability, as a result of the shift Δx_0 of the center of the oscillator is $r_m^2 q = r_m a_0$. According to (63), for small n

$$a_0^2 = \frac{4ms^2}{\hbar\omega_c'} \frac{\beta^2}{(1 - \beta^2)^{1/2}},$$

where

$$\omega_c' = eH' / ms.$$

When $\beta \ll 1$ this formula goes over into the expression given in [12]. We see that with increasing β the value of a_0 increases more rapidly than β^2, owing to the decrease in the denominator. When $\beta \to 1$ this effect is significant, and failure to allow for it and for the non-parabolicity effect have led the author of [36] to incorrect conclusions.

It follows from (58) that the entire analysis of the dependence of $|M_{12}|^2$ on a_0, given in [12], is fully valid in the general case. Thus, the number of zeroes on the $|M_{12}(a_0)|^2$ curve, as follows from the properties of Leguerre functions, is equal to the smaller of the quantum numbers n_1 or n_2. The transition probability decreases with increasing a_0 like $\exp(-a_0^2/2)$ for small n, and like $1/a_0$ for large n.

As indicated earlier, the appearance of forbidden transitions is connected with the shift of the centers of the oscillators on going from band to band. In the moving frame this shift is connected with the change in the wave vector k_y as a result of the Doppler increase in the wave vector of light, just as the shift in the resonant frequency is connected with the Doppler change of the frequency of light. In a stationary coordinate system, the transitions take place without a change in the electron momentum k_y, but the centers of gravity of the oscillators, for a given k_y, shift in both bands in opposite directions in the electric field.

We note that the shift of the terms and the decrease of the transition probability in crossed fields can be of great importance, for example, in the analysis of the operation of a semiconductor laser in a magnetic field. Under the operating conditions of the laser, the fields in the p-n junction are weak,[37] and the case $H' > E'$ is realized in magnetic fields of the order of 10^4 Oe, but the field E' is still sufficiently strong. Therefore in a transverse field a decrease in the transition probability can lead to an increase in the threshold current, whereas in a longitudinal magnetic field, where this effect is missing, the threshold current will decrease.

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APPENDIX

DEPENDENCE OF THE ABSORPTION COEFFICIENT ON THE POLARIZATION DIRECTION

As follows from (43), accurate to higher terms of the expansion in the quantity $\delta = (\epsilon_g - \hbar\omega)/\epsilon_g$, the dependence of the absorption coefficient on the polarization is determined only by the anisotropy of the effective masses. However, if we take into account the next higher terms in the expansion of $[\varphi''_x(\varphi_y\varphi_z)^{1/2}]^{-1}$ in powers of δ, then the ratio of the absorption coefficients turns out to be different from unity even in the spherical case. Thus, when account is taken of the terms of first order in δ, this ratio, according to (33)–(35), (38), is equal to

$$\alpha_x' : \alpha_y' : \alpha_z' = (1 + 2\delta) : 1 : \left[1 + \frac{4\beta^2\delta}{[1 + (1 - \beta^2)^{1/2}]^2}\right]. \quad (A.1)$$

The intraband term (17) also leads to anisotropy of the absorption coefficient. To take this term into account, it is necessary to diagonalize $\tilde{\mathcal{H}}_0$ with accuracy to the interband terms, for which, as in [31], we can use the transformation $T = e^{i\tilde{S}}$, where

$$\tilde{S} = -i\frac{\sigma_x}{2}\tan^{-1}\frac{k_z}{k_y}.$$

As a result we obtain an equation that coincides with Eq. (18) of [31], which takes into account the "spin" splitting of the levels in the electric field when $k \neq 0$. The perturbation operator (13) actually does not change in the new representation, since according to (30) its value is taken at the point $k_z = 0$. Then, for transitions without spin flip, i.e., when $\mathbf{A}_0 \parallel z$, the absorption coefficient remains the same as before, since the transition energy does not change. For transitions with spin flip, there appears in the exponential the additional terms

$$+ i\left\{\int_0^{k_x} \frac{\eta - 1}{2\eta}\frac{k_\perp}{k^2}\,dk_x + \left[\int_0^{k_x} \frac{\eta - 1}{2\eta}\frac{k_\perp'}{k'^2}\,dk_x\right]_{\mathbf{k}'=\mathbf{k}+\mathbf{q}}\right\}. \quad (A.2)$$

It is further necessary to integrate this expression with respect to k_x, substitute the significant values of k_y and k_z and expand the value obtained for the transition probability with allowance for (A.2) in powers of δ. These calculations lead to the following relation for the absorption coefficients:

$$\alpha''_{x,y} : \alpha_z'' = \left[1 + \frac{\beta^2(1 - \beta^2)\delta}{[1 + (1 - \beta^2)^{1/2}]^2}\right] : 1. \quad (A.3)$$

Simultaneous allowance for both effects in accord with (A.1) and (A.3) leads to the following ratio of the absorption coefficients

$$\alpha_x : \alpha_y : \alpha_z = \left\{1 + 2\delta + \frac{\beta^2(1 - \beta^2)\delta}{[1 + (1 - \beta^2)^{1/2}]^2}\right\} : \left\{1 + \frac{\beta^2(1 - \beta^2)\delta}{[1 + (1 - \beta^2)^{1/2}]^2}\right\} : \left\{1 + \frac{4\beta^2\delta}{[1 + (1 - \beta^2)^{1/2}]^2}\right\} \quad (A.4)$$

From (A.4) we see that the absorption coefficient turns out to be anisotropic also when $H = 0$:

$$\alpha_x : \alpha_y : \alpha_z = (1 + 2\delta) : 1 : 1. \quad (A.5)$$

From (A.4) and (A.5) it follows that the anisotropy of the Franz–Keldysh effect, both in a magnetic field and without one, is determined in practice by the anisotropy of the effective masses and is small for spherical bands, vanishing on approaching the absorption edge.

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