A REMARK ON THE MULTI-DIMENSIONAL COULOMB PROBLEM

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The Fock problem is extended to the multi-dimensional case. Explicit expressions (with all normalization factors) are obtained for the wave functions of the discrete as well as the continuous spectrum of an f-dimensional "hydrogen atom." An "additional theorem" for the functions of the continuous spectrum is derived. These functions form a convenient representation basis for the rotation group O(f + 1) and the Lorentz group L(f + 1). A method for obtaining a different form of solution is indicated. As an example it is shown that for the hydrogen atom (f = 3) there exists a solution in Fock space which coincides with the function for the symmetric top.

INTRODUCTION

As has been shown by Fock,[1] the "accidental" degeneracy of the levels of the hydrogen atom ($E < 0$) is connected with a hidden symmetry of the system, viz., the symmetry with respect to the group O(4). The case of the continuous spectrum has been considered by Perelomov and Popov.[2] Alliluev[3] and Györgyi and Révai[4] have studied the f dimensional Kepler problem ($E < 0$). In[3,4] the spectrum of eigenvalues was obtained, and it was also shown that the Hamiltonian of the system has the symmetry of the group O(f + 1).

Our task is to obtain, together with the spectrum, the explicit solutions for the f dimensional Coulomb potential in the discrete and continuous regions.

1. DISCRETE SPECTRUM

1. Let a system be given with the Hamiltonian

$$H = \sum_{i=1}^{f} \frac{p_i^2}{2m} + V, \quad (1.1)$$

where

$$V = -b/r, \quad r = \left(\sum_{i=1}^{f} x_i^2 \right)^{1/2}, \quad p_i = -i\frac{\partial}{\partial x_i}, \quad (1.2)$$

satisfying the Schrödinger equation

$$\hat{H}\Psi = E\Psi. \quad (1.3)$$

An obvious symmetry group of Eq. (1.3) is the group O(f). It is easy to see that the orthonormal eigenfunctions and the eigenvalues of the operator (1.1) are

$$\tilde{\Psi}_{n,l_1} = C(p_0 r)^{l_1} e^{-p_0 r} L_{n-l_1}^{2l_1+f-2}(2p_0 r)\tilde{Y}^{(f)}_{l_1,\ldots,l_{f-1}},$$

$$C = \left\{ \frac{2^{2l_1}(2p_0)^f \Gamma(n-l_1+1)}{(2n+f-1)\Gamma(n+l_1+f-1)} \right\}^{1/2},$$

$$\frac{mb}{p_0} = n + \frac{f-1}{2} \quad (1.4)$$

where

$$p_0 = \sqrt{-2mE}, \quad l_1 = 0, 1, 2, 3, \ldots, n; \quad n = 0, 1, 2, 3, \ldots,$$

$L_{n-l_1}^{2l_1+f-2}$ is the generalized Laguerre polynomial, and $\tilde{Y}^{(f)}_{l_1}$ is the normalized solution of the angular part of the Laplacian belonging to the eigenvalue $l_1(l_1 + f - 2)$. Let us take the Fourier transform of (1.3) and consider, following Fock,[1] the coordinates of the vector $\mathbf{p}/p_0$ as the stereographic projection of the points of an (f+1)-dimensional hypersphere. As a result we obtain the integral equation

$$Y(\xi) = \frac{mb\Gamma((f-1)/2)}{p_0 2\pi^{(f+1)/2}} \int \frac{Y(\xi')\,d\Omega'}{\{2(1-\cos\omega)\}^{(f-1)/2}}. \quad (1.5)$$

Here ξ is a point on the hypersphere, and

$$2(1-\cos\omega) = \sum_{i=0}^{f} (\xi_i - \xi_i')^2.$$

is the square of the distance between the points ξ and ξ' on the hypersphere, and $d\Omega$ is a surface element of the hypersphere. The function $Y(\xi)$ is connected with the Fourier transform $\Psi(\mathbf{p})$ by

$$Y(\xi) = \frac{1}{2^{(f+1)/2} p_0^{f/2+1}} (p_0^2 + p^2)^{(f+1)/2} \Psi(\mathbf{p}). \quad (1.6)$$

Using the methods of mathematical physics,[5] one can show that $Y(\xi)$ is the solution of the angular part of the Laplace equation with the eigenvalue $n(n + f - 1)$ under the condition

$$mb/p_0 = n + (f-1)/2. \qquad (1.7)$$

The quantity $Y_n(\xi)$ for fixed n forms an irreducible representation $D_n^{(f+1)}$ of the group $O(f + 1)$.

2. The solutions of the Laplace equation in different coordinate systems given on the hypersphere have been studied by Vilenkin et al.[6] We therefore write down at once the orthonormal solution for a coordinate system of the type shown in Fig. 1:

$$\tilde{Y}_{n,l_2,\dots,l_f} = \prod_{i=1}^{f-2} N_i^{-1/2} \sin^{l_{i+1}}\theta_i P_{n_i}^{(\nu_i)}(\cos\theta_i)\,\tilde{Y}_{l_{f-1}}^{m}(\theta_{f-1},\varphi), \qquad (1.8)$$

where

$$2\nu_i = 2l_{i+1} + f - i, \qquad n_i = l_i - l_{i+1},$$

$$\theta_f = \varphi, \qquad l_f = m, \qquad i_1 = n.$$

$$N_i = \frac{\pi 2^{1-2l_{i+1}-f+i}\,\Gamma(l_{i+1}+l_i+f-i)}{(l_i-l_{i+1})!\,(l_i+(f-i)/2)\,\{\Gamma(l_{i+1}+(f-i)/2)\}^2}$$

is the square of the norm, $P_n^{(\nu)}$ is a Gegenbauer polynomial, and $\tilde{Y}_{l_{f-1}}^{m}$ is the usual spherical function. Although the solution of the form (1.8) is given in [7] we have quoted it again for simplicity. Other normalized solutions can easily be written down using the algorithm of Vilenkin et al.[6] We know the explicit form of $\Psi(\mathbf{r})$ and $\Psi(\mathbf{p})$, and since one of them is the Fourier transform of the other, we thus obtain new integral representations of the Laguerre polynomials.

3. Finally, let us show that the hydrogen atom ($f = 3$) is a top in the Fock space. According to [6] we can introduce, on the hypersphere, a coordinate system of the type shown in Fig. 2:

$$\begin{aligned} \xi_0 &= \cos\theta\cos\varphi_1, \quad & \xi_2 &= \sin\theta\cos\varphi_2,\\ \xi_1 &= \cos\theta\sin\varphi_1, \quad & \xi_3 &= \sin\theta\sin\varphi_2. \end{aligned} \qquad (1.9)$$

The solution of the Laplace equation with the eigenvalue $n(n + 2)$ can, with the parametrization (1.9) be written as (cf. [6])

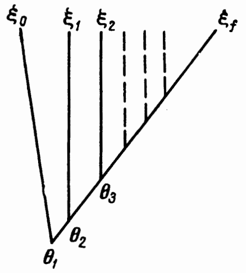

FIG. 1

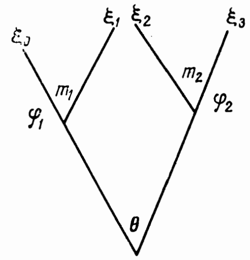

FIG. 2

$$\Psi_{n,k_1,k_2}(\alpha,\beta,\gamma) = D_{k_1k_2}^{(n/2)}(\alpha,\beta,\gamma), \qquad (1.10)$$

where

$$\alpha = \varphi_1 - \varphi_2, \quad \gamma = \varphi_1 + \varphi_2, \quad \beta = 2\theta,$$
$$2k_1 = m_1 - m_2, \quad 2k_2 = m_1 + m_2.$$

The function D is a generalized spherical function or function of the top, i.e., it is a representation of the group SO(3).

2. CONTINUOUS SPECTRUM

1. In the continuous region ($E > 0$) we must, instead of a hypersphere, consider a two-sheeted hyperboloid, where the upper sheet of the hyperboloid corresponds to the momenta p in the interval $0 < p_{in} < p_0 = \sqrt{2mE}$ and the lower sheet corresponds to the momenta in the interval $p_0 < p_e < \infty$[7] (or maybe vice versa), i.e., the upper sheet is projected into the interior of the sphere of radius p_0 and the lower sheet, into the exterior of the sphere. The integral equation (1.5) reduces to a system of two integral equations:

$$\mp Y_\pm(\xi) = \lambda\frac{\Gamma((f-2)/2)}{2\pi^{(f+1)/2}}\left\{\int\frac{Y_+(\xi')\,d\Omega'}{[2(\operatorname{ch}\omega \mp 1)^{(f-1)/2}]} + \int\frac{Y_-(\xi')\,d\Omega'}{[2(\operatorname{ch}\omega \pm 1)^{(f-1)/2}]}\right\} \qquad (2.1)^*$$

Here $\lambda = mb/p_0$, $d\Omega' = \sinh^{f-1}a'da'd\Omega'(f)$, $d\Omega'(f)$ is the measure on the hypersphere, and

$$Y_+(\xi) = \frac{(p_0^2 - p_{in}^2)^{(f+1)/2}}{p_0^{f/2+1}2^{(f+1)/2}}\Psi(\mathbf{p}_{in}),$$

$$Y_-(\xi) = \frac{(p_e^2 - p_0^2)^{(f+1)/2}}{p_0^{f/2+1}2^{(f+1)/2}}\Psi(\mathbf{p}_e).$$

The indices ± indicate that the function $Y(\xi)$ is given on the upper or lower sheets of the hyperboloid, respectively.

2. Let us turn to the construction of an orthonormal system of functions given on the hyperboloid in an $(f + 1)$ dimensional pseudo-Euclidean space which are solutions of the Laplace equation with eigenvalue $\sigma(\sigma + f - 1)$. Here σ is a complex number equal to $(f - 1)/2 + i\rho$ in the unitary case.

Using the integral representations of the scalar function $f(\xi)$ defined on the multi-dimensional hyperboloid $\xi_0^2 - \boldsymbol{\xi}^2 = 1$[8,9] and the explicit form of the function (1.8), we obtain an expansion of $f(\xi)$ for $f = 2n + 1$ and $f = 2n$, respectively:[10]

$$f(\xi) = \frac{1}{(2\pi)^{f/2}}\int_0^\infty d\rho\prod_{k=1}^{(f-1)/2}\left\{\rho^2 + \left[\frac{f-(2k+1)}{2}\right]^2\right\} \times\sum_\alpha a_\alpha(\rho)\,\Psi_{\rho,l_1}^{(f)}(a)\,\tilde{Y}_\alpha(\Omega), \qquad (2.2)$$

*ch ≡ cosh.

$$f(\xi) = \frac{1}{(2\pi)^{f/2}} \int_0^\infty d\rho \cdot \rho \operatorname{th} \pi\rho \times \prod_{k=1}^{f/2-1} \left\{ \rho^2 + \left[\frac{f-(2k+1)}{2} \right]^2 \right\} \sum_\alpha a_\alpha(\rho) \Psi^{(f)}_{\rho, l_1}(a) \tilde{Y}_\alpha(\Omega), \qquad (2.3)^*$$

where α is the set of indices $\{l_1, \dots, l_{f-2}, m\}$, Ω is the set of angles $\{\theta_1, \dots, \varphi\}$, and

$$\Psi^{(f)}_{\rho, l_1}(a) = \frac{\Gamma(f-2+l_1)\Gamma(i\rho-(f-1)/2+1)}{\Gamma(f-2)\Gamma(l_1+1)\Gamma(i\rho-(f-1)/2+1-l_1)} \times \frac{P^{-l_1-(f-2)/2}_{-1/2+i\rho}(\operatorname{ch} a)}{\operatorname{sh}^{(f-2)/2} a}. \qquad (2.4)^\dagger$$

The formulas for the inverse transformation have the form[10]

$$a_{l_1, \dots, m}(\rho) = (2\pi)^{f/2} \int f(\xi) \Psi^{(f)}_{-\rho, l_1}(a) \tilde{Y}^*_{l_1, \dots, m}(\theta_1, \dots, \varphi) \frac{d^f\xi}{\xi_0}. \qquad (2.5)$$

It follows from (2.2), (2.3), and (2.5) that the function

$$\Psi^{(f)}_{\rho, \alpha}(\xi) = \frac{\Gamma(f-2+l_1)\,|\Gamma(i\rho-(f-1)/2+1)\,|P^{-l_1+(f-2)/2}_{-1/2+i\rho}(\operatorname{ch} a)}{\Gamma(f-2)\Gamma(l_1+1)\,|\Gamma(i\rho-(f-1)/2+1-l_1)\,|\operatorname{sh}^{(f-2)/2} a} \times \tilde{Y}_\alpha(\Omega) \begin{cases} A, & f = 2n+1, \\ B, & f = 2n, \end{cases} \qquad (2.6)$$

where

$$A = \left[\prod_{k=1}^{(f-1)/2} \left\{ \rho^2 + \left[\frac{f-(2k+1)}{2} \right]^2 \right\} \right]^{1/2},$$

$$B = \left[\prod_{k=1}^{f/2-1} \left\{ \rho^2 + \left[\frac{f-(2k+1)}{2} \right]^2 \right\} \rho \operatorname{th} \pi\rho \right]^{1/2},$$

satisfies the normalization condition

$$\int \tilde{\Psi}^*_{\rho, \alpha}(\xi) \tilde{\Psi}_{\rho', \alpha'}(\xi) \frac{d^f\xi}{\xi_0} = \delta(\rho - \rho')\delta_{\alpha\alpha'}. \qquad (2.7)$$

Here

$$\frac{d^f\xi}{\xi_0} = \operatorname{sh}^{f-1} a \, da \, d\Omega(f).$$

Using the explicit form of the functions (2.6) and the "addition theorem" for the higher spherical functions,[7] we can at once write down an "addition theorem" for the functions (2.6):

$$\sum_\alpha \tilde{\Psi}^*_{\rho, \alpha}(\xi') \tilde{\Psi}_{\rho, \alpha}(\xi) = \frac{1}{(2\pi)^{f/2}} \frac{P^{-(f-2)/2}_{-1/2+i\rho}(\operatorname{ch} \omega)}{\operatorname{sh}^{(f-2)/2} \omega} \begin{cases} A^2, & f = 2n+1, \\ B^2, & f = 2n, \end{cases} \qquad (2.8)$$

*th ≡ tanh.

†sh ≡ sinh.

where $\cosh \omega = (\xi\xi')$.

In analogy to the "addition theorem" for the higher spherical functions,[7] we express (2.8) in terms of the Gegenbauer function:

$$\sum_\alpha \tilde{\Psi}^*_{\rho, \alpha}(\xi') \Psi_{\rho, \alpha}(\xi) = \frac{(\sigma + (f-1)/2)\Gamma((f-1)/2)}{2\pi^{(f+1)/2}} \times P_\sigma^{((f-1)/2)}(\operatorname{ch} \omega) \begin{cases} (-)^{(f-1)/2}, & f = 2n+1, \\ (-)^{f/2-1} \cot \pi\sigma, & f = 2n. \end{cases} \qquad (2.8')$$

We note that (2.6) is valid for the four-dimensional case, beginning with f = 4. The case f = 2 is discussed in [9, 11] The functions $\Psi^-_{\rho,\alpha}(\xi)$ form a complete system on each of the two sheets of the hyperboloid and are solutions of the Laplace equation with eigenvalue $\sigma(\sigma + f - 1)$. The latter is easily seen by considering directly the solutions of the Laplace equation. The set $\Psi_{\rho,\alpha}(\xi)$ forms an infinite-dimensional unitary representation of class I of the Lorentz group $T_\chi(g)$, $\chi = (i\rho, i\rho)$.

The kernels of the system of integral equations (2.1) are expressed in terms of $\tilde{\Psi}_{\rho,\alpha}(\xi)$ by the formulas

$$\frac{\Gamma((f-1)/2)}{2\pi^{(f+1)/2}[2(\operatorname{ch} \omega \mp 1)]^{(f-1)/2}} = \int_0^\infty d\rho F_{1,2}(\rho) \sum_\alpha \tilde{\Psi}^*_{\rho, \alpha}(\xi') \tilde{\Psi}_{\rho, \alpha}(\xi), \qquad (2.9)$$

where

$$F_1 = \frac{\coth \pi\rho}{\rho}, \qquad F_2 = \frac{1}{\rho \operatorname{sh} \pi\rho}. \qquad (2.10)$$

Expressions for the spectral densities $F_i(\rho)$ are obtained by integrating the kernels with account of formulas (2.6) to (2.8). Since the expressions (2.10) agree with those obtained in [2] for f = 3, we have

$$Y_{E, \alpha}(\xi) = \begin{cases} Y_{+, E}(\xi), & \xi_0 > 0 \\ Y_{-, E}(\xi), & \xi_0 < 0 \end{cases} = \begin{cases} c_1 \tilde{\Psi}_{\rho, \alpha}(\xi), & \xi_0 > 0, \\ c_2 \tilde{\Psi}_{\rho, \alpha}(\xi), & \xi_0 < 0, \end{cases} \qquad (2.11)$$

where

$$c_1 = (1 + e^{2\pi\rho})^{-1/2}, \quad c_2 = -(1 + e^{-2\pi\rho})^{-1/2}, \quad \pm\rho = mb / p_0.$$

Since $\Psi_{E,\alpha}(\mathbf{p})$ is expressed through the known function $\Psi_{\rho,\alpha}(\xi)$, the characteristics of $\Psi_{E,\alpha}(\mathbf{p})$ are completely determined.

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