FORMATION OF VORTEX NUCLEI IN SUPERCONDUCTORS OF THE SECOND KIND

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Owing to stability of the superconducting state with respect to small fluctuations of the ordering parameter, transitions to the mixed state in superconductors of the second kind occur in magnetic fields as a result of sufficiently large fluctuations. The vortex lines in this case are produced near the surface and penetrate into the superconductor. A minimal critical size exists, at which the vortex nucleus does not "close up" and continues to develop further. It is shown that the shape of the vortex line and the magnitude of the energy barrier for the nucleus are determined by two main factors: by the interaction between the vortex and magnetic field and by the elasticity of the vortex line. In this case the vortex line forms a semicircle the ends of which touch the superconductor surface. The energy barrier surmounted during formation of the vortex is calculated and it is shown that the barrier vanishes for a critical current on the surface, that is, when the superconducting state becomes unstable with respect to small fluctuations.

TURNING on of a magnetic field $H_0 > H_{c1}$ in superconductors of the second kind is usually accompanied by a phase transition from the superconducting state to the mixed one.[1] The magnetic flux partially penetrates in this case into the bulky superconductor in the form of quantized vortex filaments, which in the equilibrium state make up a regular lattice. At the same time, the metastable superconducting state continues to remain stable relative to slow fluctuations of the superconducting ordering parameter Δ, creating by the same token the possibility of "overheating" of this state for fields $H_0 > H_{c1}$. This stability is retained up to a certain field $H'_{c1} > H_{c1}$, the magnitude of which can be calculated from the condition for the vanishing of the second variation ($\delta^2 F = 0$) of the free energy of the superconductor as a functional of Δ.[2] For fields $H_0 > H'_{c1}$, we have $\delta^2 < 0$ and small fluctuations destroy the superconducting state. Obviously, the instability occurs on the boundary of the superconductor, where the magnetic field is maximal.

For extremely hard superconductors ($\kappa \gg 1$)[1)] the stability limit of the superconducting state H'_{c1} corresponds, as can be verified by direct calculations, to a critical current state[4] on the boundary of the superconductor, that is, in this case $dj/dv_s = 0$ and the current j is maximal. Here $\mathbf{v}_s = (\nabla\chi - 2e\mathbf{A})/2m$ is the velocity of the superconducting condensate, $\mathbf{A}$ is the vector potential of the magnetic field ($H = \operatorname{curl} \mathbf{A} = -e^{-1} m \operatorname{curl} \mathbf{v}_s$), and χ is the phase of the ordering parameter $\Delta = |\Delta| e^{i\chi}$. Physically, this is connected with the fact that when the fluctuation scale is $\sim\delta/\kappa \ll \delta$, the changes of the magnetic energy are insignificant, and the superconducting current can be regarded at such distances as a homogeneous current state, the thermodynamic-stability criterion of which is the inequality $dj/dv_s > 0$.

When $H_0 < H'_{c1}$ the penetration of the vortices into the superconductor and the transition to the mixed state are due to fluctuations. A detailed description of this process is a problem in physical kinetics, but within the framework of equilibrium statistical physics[5] we can calculate the critical dimensions and the shape of the vortex nucleus that is produced near the surface of the superconductor, and also the energy barrier corresponding to the nucleus.

Since the barrier should be minimal, the state containing the nucleus realizes the minimax of the free energy $F\{\Delta\}$ and consequently it can be described by the equilibrium equations of electrodynamics, which give an extremum for $F\{\Delta\}$. The

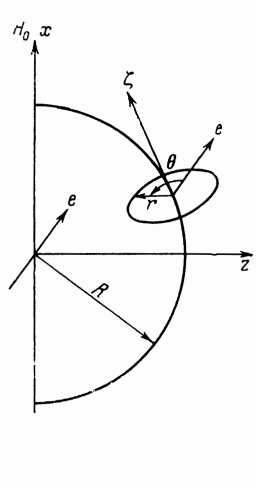
1) Here $\kappa \sim \delta_0(0)/\xi_0 \sim eH_c\delta^2$ is the parameter of the Ginzburg–Landau theory [3], δ is the depth of penetration of the field, $\delta_0(0)$ the depth of penetration of the weak field at temperature $T = 0$, $\xi_0 \sim v_0/T_c$ the correlation radius of the electrons (v_0 – Fermi velocity, T_c – critical temperature), H_c the thermodynamic critical field, and e and m the charge and mass of the electrons; all quantities are given in units $\hbar = c = 1$.

solution of these equations under the same boundary conditions is non-unique, in view of the possibility of "superheating" of the superconducting state. Formally this non-uniqueness is connected with the violation of the singly-connected character of the superconductor, owing to the occurrence of vortices, which are specified mathematically as lines of singularities of the quantity v_s, such that the velocity circulation over an infinitesimally small contour enclosing the filament obeys the relation[1]

$$\oint \mathbf{v}_s d\mathbf{l} = \frac{\pi}{m}.$$

The position, form, and the number of these filaments should be determined from the following supplementary conditions: in the case of the equilibrium state—from the condition of minimality of the free energy $F\{\Delta\}$, and in the case of a state containing a nucleus—from the conditions of minimax of $F\{\Delta\}$. In the latter case it is clear from energy considerations that there should be one vortex. By virtue of the continuity of the magnetic flux, the ends of the vortex will emerge to the surface, and from symmetry considerations it follows that the vortex filament should lie in a plane perpendicular to the surface of the superconductor and parallel to the external magnetic field H_0 (see the figure). Solving the equations of electrodynamics under these conditions and calculating the free energy F, we can then determine the form of the filament from the condition that F be stationary.

An essential circumstance for what is to follow is that for extremely hard superconductors the circulation $\oint \mathbf{v}_s \cdot d\mathbf{l}$ and accordingly the magnetic field of the filament are small ($\sim 1/\kappa$ in dimensionless variables[1]). Therefore in a sufficiently strong external field H_0 the solution describing the superconducting state with a vortex filament differs by only a small correction from the solution without the vortex. Let us expand in powers of this correction the free energy of the superconductor in an external field[6, 7] $\mathbf{H}_0$:



$$F = F_G + \int dV \frac{(\mathbf{H}-\mathbf{H}_0)^2}{8\pi}, \quad F_G = -T \ln \mathrm{Sp}\, e^{-\mathcal{H}/T}$$

($\mathcal{H}$ is the Hamiltonian of this system). Accurate to second-order terms we have

$$\Delta F = \delta F + \delta^2 F,$$

$$\delta F = \int dV \left\{ \frac{\mathbf{H}-\mathbf{H}_0}{4\pi} \delta\mathbf{H} - \mathbf{j}\delta\mathbf{A} \right\}$$

$$= \int dV \left\{ \frac{\mathbf{H}-\mathbf{H}_0}{4\pi} \delta\mathbf{H} + \frac{m}{e} \mathbf{j}\delta\mathbf{v}_s \right\},$$

$$\delta^2 F = \int dV \left\{ \frac{(\delta\mathbf{H})^2}{8\pi} + \frac{m}{2e} (\delta\mathbf{j}) (\delta\mathbf{v}_s) \right\}$$

$$(\operatorname{div} \mathbf{j} = \operatorname{div}(\delta\mathbf{j}) = 0). \tag{1}$$

Here $\mathbf{H}$ and $\mathbf{j}$ are the field and the current describing the superconducting state without the vortex (curl $\mathbf{H} = 4\pi\mathbf{j}$, div $\mathbf{H} = 0$; $\mathbf{H}(z) = 0) = \mathbf{H}_0$, $\mathbf{H}(z = \infty) = 0$); $\delta\mathbf{H}$, $\delta\mathbf{j}$, and $\delta\mathbf{v}_s$ are the corrections to this solution, due to the vortex (curl $(\delta\mathbf{H}) = 4\pi(\delta\mathbf{j})$, div $(\delta\mathbf{H}) = 0$, $\delta\mathbf{H}(\infty) = 0$).

As will be made evident shortly, the characteristic dimension R of the filament satisfies the inequalities $\delta/\kappa \ll R \ll \delta$, where δ/κ is the radius of the core of the filament, within which the nonlocality of the electrodynamic equations for the superconductor is essential. Taking these inequalities into account and integrating in (1) by parts with the aid of the relation $\mathbf{H} = -e^{-1} m$ curl $\mathbf{v}_s$, we obtain, owing to the presence of the singularity

$$\Delta F = \frac{1}{4e} \int_0^1 dt \left\{ \left(\frac{dH}{dz} \right)_0 z \frac{dx}{dt} + \frac{1}{2} (\delta H_m) \left[\left(\frac{dx}{dt} \right)^2 + \left(\frac{dz}{dt} \right)^2 \right]^{1/2} \right\}. \tag{2}$$

Here δH_m is the field in the center of the vortex, and $(dH/dz)_0 = -4\pi j$. The integration in (2) is carried along the filament in such a way that $dx/dt > 0$.

Variation of (2) under the condition $z(0) = z(1) = 0$ yields

$$\frac{dz}{dl} - R \frac{d^2x}{dl^2} = 0, \quad \frac{dx}{dl} + R \frac{d^2z}{dl^2} = 0;$$

$$\frac{dx}{dl}(0) = \frac{dx}{dl}(l) = 0, \tag{3}$$

where $R = \delta H_m / 8\pi j$ and $dl = \sqrt{(dx)^2 + (dz)^2}$ is the element of length of the filament. From (3) it follows that the filament forms a semicircle of radius R, with $x^2 + z^2 = R^2$.

It remains to determine the field δH_m in the center of the filament. At distances satisfying the inequality $\delta/\kappa \ll r \ll R$, the following equations are valid

$$\operatorname{rot}(\delta\mathbf{H}) = 4\pi(\delta\mathbf{j}), \quad \operatorname{div}(\delta\mathbf{H}) = 0;$$

$$\operatorname{div}(\delta\mathbf{j}) = 0, \quad \operatorname{rot}(\delta\mathbf{v}_S) \approx 0; \qquad (4)^*$$

$$\delta\mathbf{j} = \frac{j}{v_S}\delta\mathbf{v}_S + \left(\frac{dj}{dv_S} - \frac{j}{v_S}\right)\mathbf{e}(\mathbf{e}\delta\mathbf{v}_S), \quad \mathbf{e} = \frac{\mathbf{v}_S}{v_S}. \qquad (5)$$

The latter equality (5) for the current was obtained neglecting the nonlocality, that is, $\delta j_i = \delta v_{sk} dj_i(v_s)/dv_{sk}$.

From the second pair of equations in (4) it follows that**

$$\delta\mathbf{v}_S = \frac{1}{2m}\nabla\chi, \quad \chi = \operatorname{arctg}\left(\sqrt{\frac{v_S}{j}\frac{dj}{dv_S}}\operatorname{tg}\theta\right),$$

where the polar angle θ of the cylindrical coordinate system r, θ, ζ is measured from the $\mathbf{e}$ direction, and the ζ axis is directed along the filament. Obviously,

$$\delta v_{sr} = \delta v_{s\zeta} = 0,$$

$$\delta v_{s\theta} = \frac{1}{2mr}\sqrt{\frac{j}{v_s}\frac{dj}{dv_s}}\left(\frac{dj}{dv_s}\sin^2\theta + \frac{j}{v_s}\cos^2\theta\right)^{-1}.$$

Let us substitute these values in (5) and in the first pair of equations of (4) (the equation $\operatorname{div}(\delta\mathbf{H}) = 0$ is identically satisfied because $\delta H_r = \delta H_\theta = 0$):

$$\frac{\partial(\delta H_\zeta)}{\partial r} = -\frac{2\pi}{m}\sqrt{\frac{j}{v_s}\frac{dj}{dv_s}}\frac{1}{r},$$

$$\frac{\partial(\delta H_\zeta)}{\partial\theta} = -\frac{2\pi}{m}v_s\frac{d}{dv_s}\left(\frac{j}{v_s}\right)\sin\theta\cos\theta\frac{\partial\chi}{\partial\theta}.$$

From this it follows with logarithmic accuracy[1] that

$$\delta H_m = \frac{2\pi}{m}\sqrt{\frac{j}{v_s}\frac{dj}{dv_s}}\ln\varkappa.$$

An estimate of the radius of the filament

$$R = \frac{1}{4m}\sqrt{\frac{dj}{dv_s}\frac{1}{v_s j}}\ln\varkappa$$

yields

$$R \sim \frac{\ln\varkappa}{mv_s} \sim \frac{\ln\varkappa}{\varkappa}\delta.$$

Substituting the obtained values of $z(l)$, $x(l)$, R, and δH_m in (2) and integrating, we obtain the energy barrier surmounted when the nucleus is produced:

$$\Delta F = \frac{\pi^2}{32em^2}\left(\frac{1}{v_s}\frac{dj}{dv_s}\right)(\ln\varkappa)^2.$$

As seen from this formula, the barrier actually vanishes for the external field H'_{c1} corresponding to the critical current on the surface: $dj/dv_s = 0$. Away from the field H'_{c1} we have

$$\Delta F \sim H_c^2(\delta/\varkappa)^3(\ln\varkappa)^2.$$

For $T = 0$ we have

$$\Delta F \sim \mu\left(\frac{\mu}{T_c}\right)(\ln\varkappa)^2 \qquad \left(\mu = \frac{mv_0^2}{2}\right).$$

The physical picture contained in the foregoing calculation is simple. The critical dimensions and shape of the nucleus are determined by two fundamental factors: the interaction of the vortex with the external field, which repels the filament into the superconductor, and the elasticity of the filament,[2)] which is equal to the self-energy $\delta H_m/8e$ of the vortex per unit length. The interaction with the field is proportional to the difference $H_0 - H(z)$ per unit length of the projection of the filament on the direction on the magnetic field. Since the elasticity of the filament is relatively small, the radius of the nucleus is small compared with depth of penetration of the field. From this we get that in first approximation the magnetic energy of the vortex is proportional to the area under the $z(x)$ curve, whereas the elastic energy is proportional to the length of the filament. This indeed determines that the nucleus has a semicircular shape. In view of the nonlinearity of the equations of electrodynamics, the elasticity of the filament depends greatly on the magnitude of the homogeneous current in which the vortex is produced. When the current reaches the critical value, the elasticity vanishes and the radius of the nucleus vanishes together with the value of the energy barrier.

The foregoing raises the question regarding the character of the phase transition from the superconducting to the mixed state. Because of the continuity of the vortex-density function on the phase transition line ($H_0 = H_{c1}(T)$), this transition is obviously second-order.[1] Therefore the possibility of "superheating" of the superconducting state is unexpected. Actually, however, the following must

*rot ≡ curl.

**tg ≡ tan, arctg ≡ tan⁻¹.

2)We note that the calculation of the "delay" of the penetration of the vortex in the superconductor and the corresponding physical picture as presented in the paper by Bean and Livingston[8] are incorrect. The interaction between the vortex and its image, being small compared with the elastic energy of the filament, does not play any role in the formation of the energy barrier that separates the superconducting and the mixed states.

be taken into account:[3)] in ordinary phase transitions the formation of a new phase takes place homogeneously over the entire volume. If the very process of nucleation of a new phase is continuous (second order transition), then the energy barrier between the phases, naturally, is equal to zero and no "superheating" or "supercooling" is possible. An important circumstance in the case of transition from the superconducting state into the mixed state is that the formation of the new phase takes place from the surface of the sample, due to penetration of quantum vortex filaments. Therefore in a bulky semiconductor, where it is possible to introduce the concept of a homogeneous mixed phase, the change in the state proceeds continuously and the transition as a whole is of second order. On the other hand, because of the finiteness of the magnetic-flux quantum, there exists a certain minimal energy barrier, which must be overcome by the vortex penetrating into the superconductor.

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[3)]Formally the presence of "superheating" of one phase does not contradict the continuity of the first derivatives of the thermodynamic potential in second-order transitions, if the "supercooling" of the second phase is impossible. In this case the mixed state is stable in the entire region of its existence, and its "supercooling" boundary coincides formally with the line of equilibrium transition $H_0 = H_{c_1}(T)$.

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