# *INSTABILITY OF NONLINEAR STATIONARY POTENTIAL OSCILLATIONS IN ELECTRON-ION BEAMS*

V. M. SMIRNOV

Moscow Engineering-physics Institute

Submitted to JETP editor October 23, 1965

J. Exptl. Theoret. Phys. (U.S.S.R.) **50**, 1005–1012 (April, 1966)

A simple criterion is found for the instability of nonlinear stationary potential oscillations in one-dimensional ion-electron beams emitted by a plane in a bounded volume at a fixed potential difference between the emitter and the collector. It is assumed that short-range collisions can be neglected. Cases with a monoenergetic electron beam in vacuum and against a homogeneous ion background are considered. Some conclusions are drawn regarding the stability of nonlinear stationary modes for arbitrary initial ion and electron distribution functions.

## INTRODUCTION

PIERCE[1] considered in the linear approximation the instability of the zero-amplitude stationary mode of stationary oscillations, for the case of monoenergetic electrons against a homogeneous ion background at a constant potential difference $\varphi_0$ between the emitter and the collector ($\varphi_0 = 0$). The electrons are emitted with constant initial velocity $v_0$ and have an initial density $n_0$ equal to the density of the ion background. He has shown that the mode is stable if the distance $l$ between the emitter and the collector is smaller than the "Debye radius" $\pi v_0/\omega_0$ ($\omega_0 \equiv \sqrt{4\pi e^2 n_0/m}$ is the Langmuir electron frequency). On the other hand if $l$ is constant and the Debye radius becomes smaller ($n_0$ increases or $v_0$ decreases), then when the condition

$$\pi v_0 / \omega_0 = l \tag{1}$$

is satisfied the mode becomes unstable. With further decrease of the Debye radius the mode will change from unstable to stable and vice versa at the points $n\pi v_0/\omega_0 = l$ (n = 2, 3, 4, ...).

In this paper we generalize the result obtained in [1] to include arbitrary nonlinear stationary models in one-dimensional ion-electron beams emitted from a plane. The classification of such modes is given in [2].

We denote by $\varphi$, E, and v respectively the electrostatic potential (with negative sign), the electric field, and the particle velocity. The instability of the stationary modes in a bounded volume at fixed potential difference $\varphi_0$ between the emitter and the collector is due to the feedback via the voltage source that maintains $\varphi_0$ constant. Assume that at the initial instant of time there is a certain stationary mode with emitter field $E_0$, with the emitted particles having distribution functions $f_i(v)$ and $f_e(v)$ on the emitter ($v > 0$), and with a potential difference $\varphi_0$ between the emitter and the collector (Fig. 1). The voltage source maintains $\varphi_0$ constant with a certain delay time $\tau$. Assume that some stationary mode goes over within a time $\tau_1 < \tau$ in quasistationary fashion into a neighboring stationary mode with different $E_0$ and the same functions $f_i(v)$ and $f_e(v)$. Then, generally speaking, the potential $\varphi_c$ on the collector will differ from $\varphi_0$ in the new mode (we assume the emitter potential to be identically equal to zero). In order to restore the previous potential difference, the source must produce at the emitter and at the collector equal and opposite additional charges, which produce between the emitter and the collector a homogeneous compensating additional field $E_a = (\varphi_0 - \varphi_c)/l$. This field, superimposed on the emitter field, will decrease the fluctuation of $E_0$ if $\varphi_E > 0$ and increase the fluctuation of $E_0$ if $\varphi_E < 0$. We put here

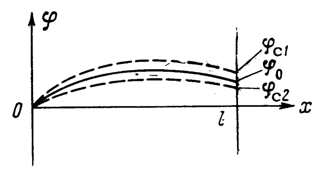


FIG. 1

$$\varphi_E \equiv d\varphi_c / dE_0 |_{\varphi_c = \varphi_0}; \tag{2}$$

$\varphi_c(E_0)$ is the dependence of the collector potential on the field at the emitter for stationary modes with specified functions $f_i(v)$ and $f_e(v)$ and for specified distances $l$ from the emitter to the collector. Accordingly, the initial stationary mode will be stable or unstable against sufficiently slow changes of $E_0$ (the changes of $E_0$ must be slow enough so that the initial mode while changing in a quasistationary manner, cannot experience a noticeable change within a time $\tau_1 < \tau$).

In Secs. 1 and 2 we consider the stability of the nonlinear modes formed by monoenergetic electrons in vacuum and against a homogeneous ion background. In Sec. 3 we draw certain general conclusions regarding the instability of the modes with arbitrary initial functions $f_i(v)$ and $f_e(v)$.

## 1. MONOENERGETIC ELECTRONS IN VACUUM

Assume that a plane emitter located at $x = 0$ emits in the x direction electrons with initial velocity $v_0$ and with initial density $n_0$. The system of equations

$$nv = n_0 v_0, \quad d^2\varphi / dx^2 = -4\pi e n, \quad v^2 = v_0^2 - 2e\varphi / m, \tag{3}$$

describing the behavior of the electrons in vacuum, has an integral

$$\frac{1}{8\pi}\left(\frac{d\varphi}{dx}\right)^2 = m n_0 v_0 \sqrt{v_0^2 - \frac{2e\varphi}{m}} + C, \tag{4}$$

which in dimensionless variables

$$\varphi' \equiv 2e\varphi / m v_0^2, \quad x' \equiv 2\sqrt{2}\omega_0 x / v_0 \quad (\omega_0^2 \equiv 4\pi e^2 n_0 / m)$$

can be rewritten in the form

$$(d\varphi'/dx')^2 = \sqrt{1-\varphi'} - 1 + E_0'^2; \quad E_0' \equiv d\varphi'/dx'|_{x'=0}. \tag{5}$$

Integrating (5) with boundary condition $\varphi'(0) = 0$, we obtain a solution of the system (3) (see Fig. 2)

$$^3/_4 x' = \pm[\sqrt{1-\varphi'} - 1 + E_0'^2]^{1/2}(\sqrt{1-\varphi'} + 2 - 2E_0'^2) + 3E_0' - 2E_0'^3,$$

$$-\infty < E_0' < \infty, \quad \varphi_c' < 1;$$

$$^3/_2 x' = -[2\sqrt{1-\varphi'} - 2 + E_0'^2]^{1/2}(\sqrt{1-\varphi'} + 2 - E_0'^2) + 3E_0' - E_0'^3,$$

$$E_0' \geqslant \sqrt{2}, \quad \varphi' \leqslant 1, \quad \varphi_c' > 1; \tag{6}$$

$$\frac{3}{2}x' = (E_0'^2 - 2)^{3/2} - E_0'^3 + 3E_0' + \frac{3}{2}\frac{\varphi' - 1}{\sqrt{E_0'^2 - 2}},$$

$$E_0' > \sqrt{2}, \quad \varphi' > 1, \quad \varphi_c' > 1.$$

For $0 < E_0' \leq 1$ there exists an envelope of the family of solutions

$$\varphi_{cr}'(x') = (^3/_4 x')^{2/3}[2 - (^3/_4 x')^{2/3}], \tag{7}$$

which for large x approaches asymptotically the solution for $E_0' = 0$. As seen from Fig. 2, modes with $E_0' < 0$ are stable for arbitrary position of the collector $x = x_c$, since

$$\left.\frac{d\varphi_c}{dE_0}\right|_{\varphi_c} = \left.\frac{\partial\varphi(x, E_0)}{\partial E_0}\right|_{x=x_c,\, E_0<0} > 0. \tag{8}$$

The modes for $0 < E_0' < 1$ are stable only if the collector is located ahead of the point of tangency of the envelope to the curve $\varphi'(x')$ corresponding to the given mode; the coordinates of this point, for specified $E_0'$, are defined by the relations

$$x_{cr}'(E_0') = \frac{4}{3E_0'^3}, \qquad \varphi_{cr}'(E_0') = \frac{1}{E_0'^2}\left(2 - \frac{1}{E_0'^2}\right). \tag{9}$$

For $1 < E_0' < \sqrt{2}$ the stationary modes exist only when the collector is located sufficiently close to the emitter, when the electrons reach the collector before the potential has time to grow to unity. These modes are stable, provided the collector potential is not too close to unity.

When $E_0' > \sqrt{2}$ all the modes are stable for any collector position. When the collector is sufficiently far from the emitter, all the electrons are reflected to the emitter in such modes. The total electron density near the emitter will in this case be equal to $2n_0$.

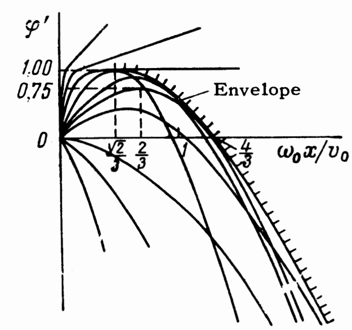


FIG. 2. Distribution of $\varphi'$ with respect to $x'/2\sqrt{2}$ for various $E_0$; $\varphi' \equiv 2e\varphi/mv_0^2$.

## 2. MONOENERGETIC ELECTRONS AGAINST A HOMOGENEOUS ION BACKGROUND

Assume that the emitting electrons have as before an initial velocity $v_0$ and an initial density $n_0$; the density of the ion background is denoted $n_{0i}$. Solving the system of equations

$$nv = n_0 v_0, \quad d^2\varphi / dx^2 = -4\pi e(n - n_{0i}),$$
$$v^2 = v_0^2 - 2e\varphi / m \tag{10}$$

in analogy with the system (3), we obtain in the same dimensionless variables x' and $\varphi'$ an integral for the periodic solutions (when the electrons are not reflected from the potential barrier):

$$\left(\frac{d\varphi'}{dx'}\right)^2 = \alpha\varphi' + \sqrt{1-\varphi'} - 1 + E_0'^2,$$

$$\alpha \equiv \frac{n_{0i}}{2n_0}, \quad E_0^2 \leqslant 1 - \alpha, \quad \alpha \leqslant 1. \tag{11}$$

Integrating (11), we obtain the periodic solution of the system (10) [see Fig. 3]:

$$x' = \alpha^{-3/2}\Big[\pm\Big(\frac{\pi}{2} - \arcsin\frac{1-2\alpha\sqrt{1-\varphi'}}{[(1-2\alpha)^2+4\alpha E_0'^2]^{1/2}}$$
$$-2\sqrt{\alpha}[E_0'^2+\alpha\varphi'+\sqrt{1-\varphi'}-1]^{1/2}\Big)$$
$$+\frac{\pi}{2} - \arcsin\frac{1-2\alpha}{[(1-2\alpha)^2+4\alpha E_0'^2]^{1/2}} - 2\sqrt{\alpha}E_0'\Big],$$
$$0 \leqslant E_0' \leqslant \sqrt{1-\alpha}, \quad \alpha \leqslant 1. \tag{12}$$

The period is the same for all the admissible values of $E_0'(|E_0'| < \sqrt{1-\alpha})$ and is equal to $2\pi\alpha^{-3/2}$. The plots for $E_0' < 0$ are symmetrical to the plots for $E_0' > 0$ about the line $x' = \pi\alpha^{-3/2}$. The periodic solutions are stable only up to the point of tangency with the envelope; the dependence of the potential $\varphi'_{cr}$ at this point on the value of $E_0'$ corresponding to the given mode is determined by the relation

$$\varphi'_{cr}(E_0') = 2/(\alpha+E_0'^2) - (\alpha+E_0'^2)^{-2}. \tag{13}$$

The extremal values of the potential are unity and $(\alpha^2 - 1)/\alpha^2$ for the modes with $|E_0'| = \sqrt{1-\alpha}$ and 0 and $(2\alpha - 1)/\alpha^2$ for the modes with $E_0' = 0$.

During the start of a new period ($x' > 2\pi\alpha^{-3/2}$), all the modes again become stable until the new encounter with the envelope, and the same happens in each period. For values of $E_0'$ satisfying the condition

$$1-\alpha < (E_0')^2 < 2-\alpha, \qquad 2 > \alpha \geqslant 0, \tag{14}$$

the stationary modes exist and are stable only when the collector is situated sufficiently close to the emitter ($\varphi_c < 1$). For values of $E_0'$ satisfying the condition

$$E_0'^2 - 2 + \alpha > 0, \quad \alpha \geqslant 0, \tag{15}$$

and for a sufficient distance from the collector there exist stationary modes with reflecting electrons, which are solutions of the equation

$$(d\varphi'/dx')^2 = \alpha\varphi' + 2\sqrt{1-\varphi'} - 2 + E_0'^2. \tag{16}$$

When $E_0' > 0$ these modes are always stable. When $E_0' < 0$, for modes satisfying the condition

$$\alpha(\alpha + E_0'^2 - 1)\sqrt{\alpha - 2 + E_0'^2} < -E_0', \tag{17}$$

there exists an envelope for sufficiently large values of $\varphi$; on the other hand, if (17) is not satisfied, then the mode is stable for all collector positions.

Figure 4 shows in the ($E_0'$, $\alpha$) plane the regions of periodic modes (I), modes with reflecting electrons which are stable everywhere (II), and those having an envelope (III), and also the region where stationary modes are impossible for a suitable distance between the collector and the emitter (IV).

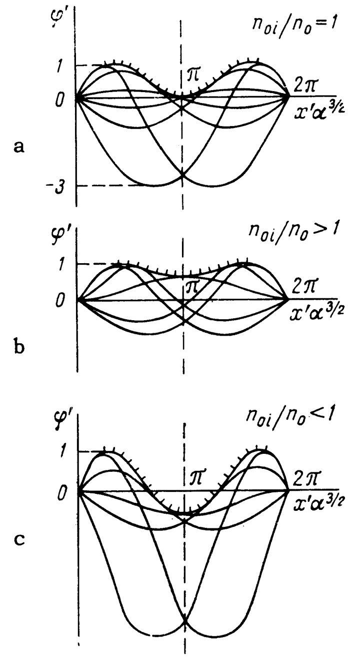


FIG. 3

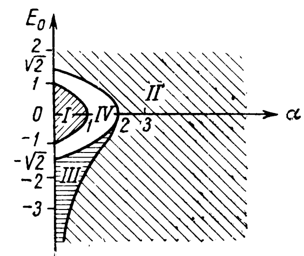


FIG. 4

FIG. 3. The first periods of the periodic distributions of $\varphi'$ with respect to $x'\alpha^{3/2}$ for various values of $|E_0'| < \sqrt{1-\alpha}$ [$\sqrt{1-\alpha}$ is the maximum value of $|E_0'|$ for periodic modes]; $x'\alpha^{3/2} = \omega_0 x v_0^{-1}(n_{0i}/n_0)^{3/2}$.

## 3. ELECTRON-ION BEAMS WITH ARBITRARY INITIAL DISTRIBUTION FUNCTIONS

Without taking collisions into account, the stationary distributions of the potential in electron-ion beams emitted by a plane are determined by the integral of the system of the Vlasov and Poisson equations:

$$E^2 = 8\pi m\int f_e(v)v\left[v^2 - \frac{2e\varphi}{m}\right]^{1/2}dv$$
$$+ 8\pi M\int f_i(v)v\left[v^2 + \frac{2e\varphi}{M}\right]^{1/2}dv + C \tag{18}$$

(M is the ion mass). The relation $E^2(\varphi)$ is generally speaking multiply-valued and can be represented in the form[2]

$$E^2(\varphi|x) = Y_{es}(\varphi|x) + Y_{is}(\varphi|x) + Y_\infty(\varphi). \tag{19}$$

Here $Y_{es}$ and $Y_{is}$ denote the contributions of the slow (reflected to the emitter) electrons and ions, which vanish after reflection of the slow particles by the first two extrema of the potential, while $Y_\infty$ corresponds to the contribution of the fast particles and has a negative second derivative everywhere. A change of $E_0^2$ by a certain amount $\Delta E_0^2$,

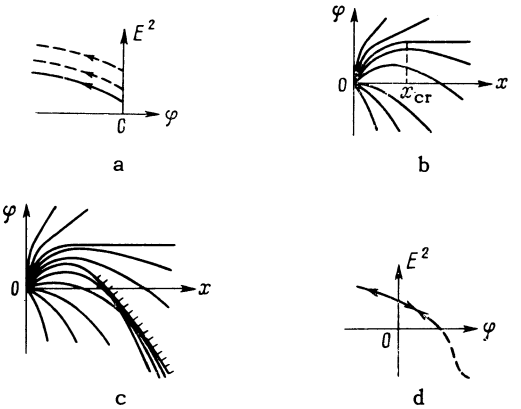


FIG. 5

at constant initial distribution functions $f_e(v)$ and $f_i(v)$ $(v > 0)$, causes a shift of the entire plot of $E^2(\varphi|x)$ as a whole along the ordinate axis by an amount $\Delta E_0^2$, if the number of slow particles of each sign is conserved. On the other hand, if the change in the maximum and minimum values of the potential influences the fraction of the reflected particles, then a redistribution of the contributions $Y_{es}$, $Y_{is}$, and $T_\infty$ takes place, and the plot of $E^2(\varphi|x)$ assumes a different form.

Let us consider first beams consisting of particles of the same sign, for example pure electron beams. When $E_0 < 0$ these beams are stable for arbitrary positions of the collector, the same as the beam of monoenergetic electrons. Indeed, for a specified distribution function $f_e(v)$, modes with large absolute value of $E_0$ correspond to larger absolute values of $E$ for each value of $\varphi$ (Fig. 5a) and accordingly to smaller values of $\varphi$ for a specified value of $x$ (Fig. 5b). Modes in which all the electrons are reflected, that is, modes with $E_0$ larger than a certain $E_{0cr}$, are also stable. The mode with $E_0 = E_{cr}$ corresponds to $E \equiv 0$ after reflection of the electrons ($x > x_{cr}$, Fig. 5b). For $E_0$ in the range $0 < E_0 < E_{cr}$, the modes can be stable everywhere (Fig. 5b), or else can have an envelope for certain values of $E_0$ and be stable only up to the envelope (Fig. 5c). The envelope may appear in the case when the plot of $E^2(\varphi)$ (Fig. 5d) has some part with a negative derivative of sufficiently large absolute value (even if this part of the plot lies below the abscissa axis at the given $E_0$, meaning that the initial distribution function contains a large group of electrons with near equal energies.

For $E_0 > 0$, when the collector is situated sufficiently close to the emitter, the limitation imposed by the collector potential on the growth of the potential may cause part of the slow electrons not to be reflected to the emitter, but to strike the collector. A corresponding change takes place in the variation of the potential between the emitter and the collector for a given $E_0$, compared with the case of total reflection of the slow electrons. We can show, however, that even such modes with monotonic potential are stable.

Modes in pure ion beams are similar to those considered above and differ only in the signs of $E_0$ and $\varphi$.

Let us proceed to consider modes in beams with particles of both signs. The modes with monotonic potential, which correspond at sufficiently large distances from the collector to total reflection of the electrons (ions), are stable for arbitrary collector positions (Fig. 6a). Similar modes, having for a sufficiently large distance from the collector an extremum of the potential (that is, $E_0 < 0$ for reflected electrons and $E_0 > 0$ for ions (Fig. 6b)),

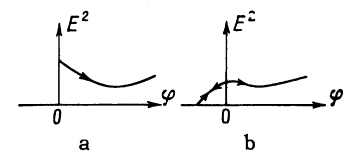


FIG. 6

are certainly stable only when the collector lies ahead of the potential extremum (that is, when there is still no reflection of the corresponding particles). In the presence of a potential extremum, the modes can be either stable or unstable. To determine the stability it is necessary here to calculate $d\varphi_c/dE_0$ for specified $f_e(v)$ and $f_i(v)$ and for a specified distance to the collector $x_c$ in each concrete case.

Let us consider now modes in which a purely periodic structure appears at a sufficiently large distance to the collector, that is, modes with fast particles of both signs. These modes are stable when the collector is situated ahead of the first potential extremum. If there are no particles reflected near the first extremum of the potential, then the modes become unstable between the first and second extrema. Indeed, in this case the minimum value of the potential decreases with increasing $E_0^2$, and the maximum increases (Fig. 7a, dashed curves). This means that modes with nearly equal values of $E_0$ have a point of intersection ahead of the second extremum of potential (Fig. 7b), and since the instability condition $\partial\varphi(x, E_0)/\partial E_0 < 0$ is satisfied near the point of intersection, the mode will become unstable at some neighboring point.

If enough particles are reflected near the first extremum of the potential, then we can choose, in principle, distribution functions $f_i(v)$ and $f_e(v)$

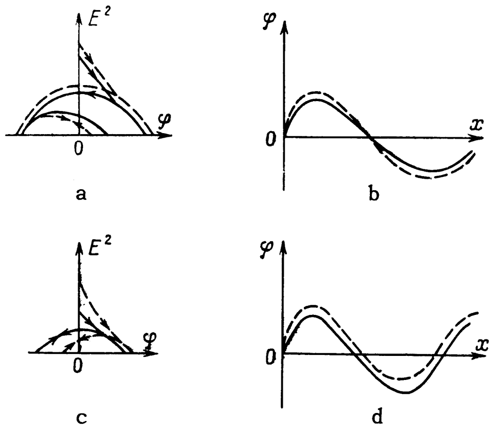


FIG. 7

such that the modes will be stable up to the second extremum and beyond for values $E_0$ in a certain interval $\Delta E_0$ near the given value of $E_0$. Let us consider by way of an example a case with $E_0 > 0$ and consequently with reflected electrons near the first extremum (Fig. 7c). With increasing $E_0$ the maximum of the potential increases, but the minimum also increases, owing to the decrease in the function $Y_\infty(\varphi)$(see [2]) as a result of the decrease in the number of electrons passing through the potential barrier. Thus, the appearance of a point of intersection of close modes is not unavoidable in this case (Fig. 7d), but in order for the mode to be continuously stable at sufficiently large distances from the emitter (several times the Debye radius), it is necessary to choose quite exactly the initial distribution functions $f_e(v)$ and $f_i(v)$, such as to ensure, in particular, equality of the two periods in the close modes. For random functions $f_e(v)$ and $f_i(v)$, the mode will as a rule become unstable even after the first or second extrema of the potential.

In order to determine in greater detail the stability properties of the modes at specified functions $f_e(v)$ and $f_i(v)$, it is necessary to calculate the distributions $\varphi(x, x_c, E_0)$ for different $E_0$ and a given $x_c$ (the distribution becomes dependent on $x_c$ when the collector is sufficiently close to the emitter, when the collector position influences the fraction of the reflected electrons).

## CONCLUSION

Let us formulate the deductions of this paper.

1. Instability of planar electron-ion beams with fixed potential difference between the emitter and a certain point of the beam (collector or grid) is the result of feedback through the voltage source that maintains constant the potential difference, for $d\varphi_c/\partial E_0 > 0$, where $\varphi_c$ is the collector potential (with the usual sign) and $E_0$ is the field at the emitter. If the relaxation time in the source circuit is $\tau > \omega_0^{-1}$, then the stability develops within a time of the order of $\tau$, but if $\tau < \omega_0^{-1}$, then the development of instability is delayed only by the inertia of the plasma and occurs within a time of the order of $\omega_0^{-1}$ (the case considered by Pierce[1]).

2. The stable modes are those with monotonic potential between the emitter and the collector (grid)(that is, for a plasma-gap length shorter than or of the order of the Debye radius). Modes with one extremum of potential can be either stable or unstable. The presence of slow particles, generally speaking, improves the stability. Modes with two and more potential extrema (that is, with a plasma gap longer than several Debye radii) are as a rule unstable.

3. For the stability of beams with fixed difference of potentials between the construction elements it is necessary either to choose the parameters of the apparatus such as to make the distance between neighboring elements not larger than the Debye radius (for example, by introducing intermediate grids), or weaken the feedback by increasing the time $\tau$ (by introducing inductance in the source circuit), or else by using non-planar geometry.

In conclusion I am sincerely grateful to A. I. Morozov for suggesting the topic and M. V. Nezlin for a fruitful discussion.

[1] J. R. Pierce, Journal of Applied Physics **15**, 721 (1944).

[2] V. M. Smirnov, Radiotekhnika i élektronika **8**, 1729 (1963).

Translated by J. G. Adashko
126