QUASILINEAR THEORY OF PLASMA CYCLOTRON INSTABILITY

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The solution of the system of quasilinear equations for cyclotron instability is obtained for the case of oscillations with a one-dimensional spectrum. The energy of the electromagnetic oscillations excited in a plasma with anisotropic ion velocity distribution by an ion beam passing through the plasma is determined with the aid of the obtained formulas. The case of simultaneous excitation of several oscillation modes is also considered.

1. THE dynamics of plasma instabilities can be investigated in many cases by using a quasilinear theory in which the main nonlinear effect is assumed to be the variation of the distribution function of the resonant particles, for which the following relation is satisfied

$$k_z v_z = \omega_k + n\omega_H \tag{1}$$

(v_z is the particle velocity in the direction of the external magnetic field $\mathbf{H}_0$; ω_k and k_z are the frequency of the wave and the projection of its wave vector on the direction of $\mathbf{H}_0$; $\omega_H = eH_0/mc$, $n = 0, \pm 1, \ldots$).

Under certain conditions, the resonant particles can intensify the waves by exchanging energy with them. The change in the distribution function of the resonant particles under the influence of the wave amplified by them leads to stabilization of the instability and to the occurrence of a stationary oscillation spectrum. For the quasilinear theory to be applicable it is necessary that this stabilization occur at sufficiently low amplitudes, for which the nonlinear interaction of the wave is still negligible.

A solution of the system of quasilinear equations for the distribution function of the resonant particles and the spectral density of the oscillation energy was obtained so far for the case of Langmuir oscillations and a one-dimensional spectrum.[1, 2] In this case the elementary effect causing the instability is Vavilov-Cerenkov radiation ($kv_{res} = \omega_k$). In this paper we investigate in a quasilinear approximation the instability of transverse plasma oscillations propagating along the external magnetic field. This instability is connected with cyclotron radiation of resonant particles due to the anomalous and normal Doppler effects ($kv_{res} = \omega_k \pm \omega_H$). An example of this type of instability is the cyclotron instability of a plasma with anisotropic velocity distribution function, investigated in the linear approximation by Sagdeev and Shafranov.[3] 1)

2. Let us consider a circularly-polarized transverse wave propagating along an external electromagnetic field: $E_\pm = E_x \pm iE_y$. The dispersion equation for such a wave is

$$k^2c^2 = \omega_k^2(\varepsilon_{xx} \mp i\varepsilon_{xy}), \tag{2}$$

where ϵ_{xx} and ϵ_{xy} are defined by the formulas given in the paper of Stepanov and Kitsenko[6] with $k_\perp = 0$. We shall henceforth assume that the resonant particles transferring energy to the oscillations are ions.2)

As usual, we neglect the contributions of the resonant particles to the real part of the tensor ϵ_{ik} and assume, in addition, that the following conditions are satisfied for the plasma particles:

$$kv_{T\alpha} \ll |\omega_{H\alpha} - \omega_k|, \qquad kv_{T\alpha} \ll \omega_k, \qquad \alpha = i, e, \tag{3}$$

which are necessary in order to neglect thermal motion in the real part of ϵ_{ik}. In this case we obtain from (2) the following dispersion equation

$$D(\omega_k, k) = \mp \frac{i\pi}{2}\frac{\omega_{0i}^2\omega_{Hi}}{N_0}\int d\mathbf{v}\, v_\perp \delta(kv_z - \omega_k \pm \omega_{Hi}) \times \left[\frac{\partial f_0}{\partial v_\perp} \pm \frac{kv_\perp}{\omega_{Hi}}\frac{\partial f_0}{\partial v_z}\right], \tag{4}$$

1)We note that, besides the indicated instability, there can occur in an anisotropic plasma also an aperiodic instability which has a nonresonant character in the sense that the energy required for excitation of the oscillations is transferred to all the particles of the plasma[4]. A quasilinear theory of such an instability was considered in[5].

2)The analysis in the present article can be applied without special difficulty to the case when the oscillations are excited by resonant electrons.

where

$$D(\omega_k, k) = k^2c^2 - \omega_k^2 + \omega_{0i}^2 \frac{\omega_k}{\omega_k \mp \omega_{Hi}} + \omega_{0e}^2 \frac{\omega_k}{\omega_k \mp \omega_{He}}, \quad (4')$$

f_0 is the distribution function of the resonant particles, $\omega_{0\alpha}^2 = 4\pi e^2 N_0/m_\alpha$, and N_0 is the plasma density. We shall henceforth confine ourselves to an examination of the wave E_-. All formulas for the other polarization direction can be obtained by making the substitutions $\omega \to -\omega$ and $k \to -k$.

Assuming that the growth increment is sufficiently small, $\gamma/\omega \ll 1$, the frequency ω_k of the wave is determined by solving the equation

$$D(\omega_k, k) = 0. \quad (5)$$

For a specified k this equation has four real roots $\omega = \omega_{kj}$, $j = 1, \ldots, 4$ (see Fig. 1). The corresponding growth increments γ_{kj} are

$$\gamma_{kj} = -\frac{\pi}{2}\frac{\omega_{0i}^2}{N_0}\frac{\omega_{kj}}{\partial D/\partial\omega_{kj}}\frac{1}{|k|}\int dv_\perp v_\perp \left[\frac{v_\perp}{v_z}\frac{\partial f_0}{\partial v_z} - \frac{\omega_{Hi}}{\omega_{kj}}\left(\frac{\partial f_0}{\partial v_\perp} - \frac{v_\perp}{v_z}\frac{\partial f_0}{\partial v_z}\right)\right]\Bigg|_{v_z=(\omega_k+\omega_{Hi})/k}. \quad (6)$$

When $j = 1$ excitation of the wave is impossible, since $\omega_1 > kc$ and the resonance condition $kv_{res} = \omega_k + \omega_{Hi}$ cannot be satisfied.

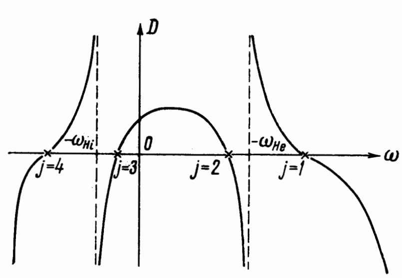

FIG. 1. Roots of the dispersion equation $D(\omega_k, k) = 0$ for $k = \text{const}$.

When $j = 2$ and 3, the dispersion equation can be simplified by putting $|\omega_j| \ll -\omega_{He}$. As a result we obtain

$$\omega^2[1 + \omega_{0i}^2 / \omega_{Hi}(\omega + \omega_{Hi})] = k^2c^2. \quad (7)$$

If $|\omega_j| \ll \omega_{Hi}$, we obtain from (7) the dispersion equation of the Alfven wave

$$\omega_j^2 = k^2c_A^2 \quad (k \ll \omega_{Hi}/c_A), \quad (8)$$

where we put

$$c_A^2 = c^2v_A^2/(v_A^2 + c^2), \qquad v_A^2 = c^2\omega_{Hi}^2/\omega_{0i}^2.$$

For large k ($k \gg \omega_{Hi}/c_A$) the root ω_2 coincides with the frequency of the helical wave

$$\omega_2 = k^2c^2\omega_{Hi}/\omega_{0i}^2 \quad (8')$$

(it is assumed that $|\omega_2\omega_{He}| \ll \omega_{0e}^2$), and the root ω_3 tends to $-\omega_{Hi}$ with increasing k.

Using (7) and determining v_{res} from the resonance condition, we can obtain a simple relation between the velocity of the resonant particles and the phase velocity of the wave amplified by them (see also [7]):

$$v_{res} = v_{ph}{}^3c_A{}^2/v_A{}^2(v_{ph}{}^2 - c_A{}^2) \quad (v_{ph} = \omega/k). \quad (9)$$

A plot of $v_{res}(v_{ph})$ for $v_{res} > 0$ is shown in Fig. 2. The region $c_A \le v_{ph} < c$ corresponds to the root ω_2, while the region $-c_A \le v_{ph} \le 0$ corresponds to ω_3. We note that in the first region there are two oscillation branches with different values of v_{ph} and the same value of v_{res} ($v_{ph}^{I} < v_{ph}^{II}$), and for small v_{ph} the first branch goes over into the Alfven wave and for large v_{ph} the second coincides with the helical wave. It is also important that when $v_{ph} > 0$ there exists a minimum value of the resonant-particle velocity:

$$v_{min} = \sqrt{27/4}\, c_A{}^3/v_A{}^2, \quad (10)$$

whereas in the region $v_{ph} < 0$ the velocity v_{res} decreases to zero. For the same reason, when oscillations with $v_{ph} > 0$ are excited the anisotropy of the ionic velocity distribution function causes the energy of the excited oscillations to be exponentially small if $v_A^2 \gg v_{Ti}^2$, and when a beam with sufficiently small thermal scatter passes through the plasma, there is actually no excitation of the oscillation branches in question if $v_{min} > u_0$ (u_0 is the beam velocity).

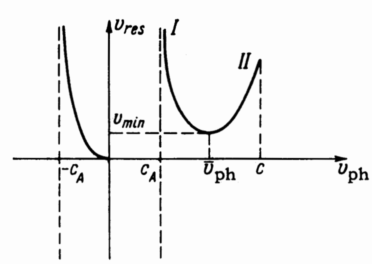

FIG. 2. Velocity v_{res} of the resonant particles vs. the phase velocity of the oscillations excited by them, for $v_{res} > 0$ and $\omega \ll |\omega_{He}|$.

When oscillations are excited in a plasma with anisotropic distribution of the ion velocities, the first term in expression (6) for the increment always exerts a stabilizing influence, but at certain values of ω the second term in (6) may become predominant and lead to instability. Then, if the condition

$$\int f_0\,dv_\perp > -\int \frac{v_\perp^2}{v_z}\frac{\partial f_0}{\partial v_z}dv_\perp \quad \text{when}\quad v_z = v_{res} \quad (11)$$

is satisfied (the longitudinal thermal energy exceeds the transverse energy), instability sets in when $\omega > 0$ ($j = 2$); if the inequality sign in (11) is reversed, the unstable waves are those with $\omega < 0$, that is, with $j = 3$. The instability is connected with the anomalous Doppler effect when $\omega > 0$ and with the normal Doppler effect when $\omega < 0$.

If the ion velocity distribution function is Maxwellian with two different temperatures

$$f_0 = N_0\frac{m_i}{2\pi T_\perp}\left(\frac{m_i}{2\pi T_\parallel}\right)^{1/2}\exp\left\{-\frac{m_i v_\perp^2}{2T_\perp}-\frac{m_i v_z^2}{2T_\parallel}\right\}, \quad (12)$$

then we have from (6) the following relation for the growth increment:

$$\gamma_j = -\left(\frac{\pi}{2}\right)^{1/2}\frac{\omega_{0i}^2}{\partial D/\partial\omega_j}\left(\frac{m_i}{T_\parallel}\right)^{1/2}\left(\frac{T_\parallel - T_\perp}{T_\parallel}v_{res} - v_{ph}\right)$$
$$\times\exp\left\{-\frac{m_i v_{res}^2}{2T_\parallel}\right\}. \quad (13)$$

Using (9) and (13), we can obtain the following condition on the phase velocities of the unstable waves

$$\frac{v_{ph}^2 - c_A^2}{v_{ph}^2}\frac{v_A^2}{c_A^2}\frac{T_\parallel}{T_\parallel - T_\perp} \leqslant 1. \quad (14)$$

When $T_\parallel > T_\perp$ the unstable waves are those with $v_{ph} > c_A$, and if the condition

$$\frac{T_\parallel - T_\perp}{T_\parallel} \leqslant \frac{2}{3}\frac{v_A^2}{c_A^2} \quad (14')$$

is satisfied, then the instability occurs in this region only on the first branch, and with further increase of the temperature anisotropy the second branch also becomes unstable; in the case of strong anisotropy, $T_\perp/T_\parallel \ll 1$, all waves with $v_{ph} < c$ are unstable. If $T_\parallel < T_\perp$, the waves with $v_{ph} < 0$ are stable, and with increasing temperature anisotropy the limit of unstable v_{ph} shifts from $-c_A$ to zero.

We shall consider also the case when the instabilities are connected with passage of an ion beam through the plasma. We shall assume that the ion beam is sufficiently spread out, $kv_T \gg \gamma$. Then γ is given by (6). Assuming that the beam and plasma ion distribution function is Maxwellian:

$$f_0 = N_0\left(\frac{m_i}{2\pi T}\right)^{3/2}\left[\exp\left(-\frac{m_i v^2}{2T}\right)\right.$$
$$\left.+\frac{N_1}{N_0}\exp\left(-\frac{m_i v_\perp^2}{2T}-\frac{m_i(v_z-u_0)^2}{2T}\right)\right] \quad (15)$$

(N_1 and N_0 are the densities of the beam and of the plasma, $N_1 \ll N_0$), we obtain from (6) the following formula for the increment:[3)]

$$\gamma_j = -\sqrt{\frac{\pi}{2}}\,\omega_{0i}^2\frac{1}{\partial D/\partial\omega_j}\left(\frac{m_i}{T}\right)^{1/2}\left\{\frac{N_1}{N_0}(u_0 - v_{ph})\right.$$
$$\left.\times\exp\left[-\frac{m_i}{2T}(u_0 - v_{res})^2\right] - v_{ph}\exp\left[-\frac{m_i}{2T}v_{res}^2\right]\right\}. \quad (16)$$

In the case in question instability sets in only when $j = 2$, i.e., it is connected with the anomalous Doppler effect. In addition, simultaneous excitation of both branches—Alfven and helical—is possible when the beam passes through the plane. However, if $u_0 \gg v_A$, then the growth increment for the Alfven wave is much larger, since

$$\left|\frac{\partial D}{\partial\omega^{I}}\right| \Big/ \left|\frac{\partial D}{\partial\omega^{II}}\right| \ll 1, \qquad v_{ph}^{I} \ll v_{ph}^{II}$$

and the excitation of the second branch can be neglected.[4)]

When $j = 4$ the excitation of the wave is impossible, for in this region $v_{ph} \gg c$, and it follows from (13) and (16) that the excitation takes place either when $v_z \gg v_{ph}T_\parallel/(T_\parallel - T_\perp)$ or when $u_0 > v_{ph}$.

3. We now proceed to derive and investigate the equations of the quasilinear approximation. The equation for the change of the "background" distribution function f_0 under the influence of the oscillations is obtained, as usual, by averaging the kinetic equation without the collision integral:

$$\frac{\partial f_{0\alpha}}{\partial t} = -\frac{e_\alpha}{m_\alpha}\left\{\left\langle \mathbf{E}\frac{\partial f_{1\alpha}}{\partial \mathbf{v}}\right\rangle + \frac{1}{c}\left\langle [\mathbf{vH}]\frac{\partial f_{1\alpha}}{\partial \mathbf{v}}\right\rangle\right\}. \quad (17)^*$$

The angle brackets denote here averaging over distances that are large compared with the wavelength of the oscillations, f_1 is the oscillating addition to the distribution function, due to the oscillations, and **E** and **H** are the electric and magnetic field of the oscillations. Representing f_1, **E** and **H** in the form of a superposition of plane waves, and using for the connection of the ampli-

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3)The plasma electrons are not at resonance with the oscillations excited by the beam, since the condition $kV_{Te} \ll \omega_{He}$ is satisfied for these oscillations.

4)We do not consider in this paper excitations of the electronic branch of the oscillations, $\omega \sim |\omega_{He}|$. In the case of instability due to the anisotropy of the ionic temperatures, the anisotropy necessary for excitation of this branch should be very strong, $T_\perp/T_\parallel \lesssim m_e/m_i$. In the case of excitation by means of a spread-out ion beam, the growth increments of the electronic branch (III), which corresponds to large values of the phase velocity ($v_{ph}^{III} > v_{ph}^{II} > v_{ph}^{I}$), are small even when $u_0 \sim v_A$, and excitation of this branch can be neglected.

*$[\mathbf{vH}] = \mathbf{v}\times\mathbf{H}$.

tudes f_k and E_k of these expansions the usual formula of the linear theory

$$f_{k\alpha}=\frac{e_\alpha}{im_\alpha}\frac{E_k e^{i\theta}}{\omega_k-kv_z+\omega_{H\alpha}}\left[\left(1-\frac{kv_z}{\omega_k}\right)\frac{\partial f_{0\alpha}}{\partial v_\perp}+\frac{kv_\perp}{\omega_k}\frac{\partial f_{0\alpha}}{\partial v_z}\right] \quad (18)$$

($v_\perp$, v_z, and θ are cylindrical coordinates in velocity space), we obtain after averaging the following equation for $f_{0\alpha}$:

$$\frac{\partial f_{0\alpha}}{\partial t}=\frac{e^2}{2m_\alpha^2}\left\{\frac{1}{v_\perp}\frac{\partial}{\partial v_\perp}\left[v_\perp\sum_k\frac{|E_k|^2}{|\omega_k|^2}\times\frac{\gamma_k}{(kv_z-\omega_k-\omega_{H\alpha})^2+\gamma_k^2}\left(|\omega_k-kv_z|^2\frac{\partial f_{0\alpha}}{\partial v_\perp}+[\omega_{H\alpha}+2(\omega_k-kv_z)]kv_\perp\frac{\partial f_{0\alpha}}{\partial v_z}\right)\right]+\frac{\partial}{\partial v_z}\left[\sum_k\frac{|E_k|^2}{|\omega_k|^2}\frac{kv_\perp\gamma_k}{(kv_z-\omega_k-\omega_{H\alpha})^2+\gamma_k^2}\times\left(-\omega_{H\alpha}\frac{\partial f_{0\alpha}}{\partial v_\perp}+kv_\perp\frac{\partial f_{0\alpha}}{\partial v_z}\right)\right]\right\}. \quad (19)$$

As is customary in the quasilinear theory (see [8]), it is necessary to break down the distribution function f_0 into two parts—the distribution functions of the resonant and nonresonant particles. Using conditions (3), we obtain from (19) the following equation for the distribution function of the nonresonant particles:

$$\frac{\partial F_{0\alpha}}{\partial t}=\frac{e^2}{2m_\alpha^2}\sum_k\frac{|H_k|^2}{c^2k^2}\frac{\gamma_k\omega_k^2}{(\omega_k+\omega_{H\alpha})^2}\frac{1}{v_\perp}\frac{\partial}{\partial v_\perp}\left(v_\perp\frac{\partial F_{0\alpha}}{\partial v_\perp}\right). \quad (20)$$

It follows from this equation that the excitation of the oscillations is accompanied by transverse diffusion of the nonresonant particles in velocity space. The oscillations increase the transverse energy of these particles by an amount

$$\Delta\mathscr{E}_\alpha=\Delta\left[\int\frac{m_\alpha v_\perp^2}{2}F_{0\alpha}\,d\mathbf{v}\right]=\frac{1}{8\pi}\sum_k\frac{|H_k|^2}{k^2c^2}\omega_k^2\frac{\omega_{0\alpha}^2}{(\omega_k+\omega_{H\alpha})^2} \quad (21)$$

(we neglect the contribution of the initial noise).

For resonant particles ($kv_z\approx\omega_k+\omega_H$), Eq. (19) can be greatly simplified by taking the limit as $\gamma\to 0$. As a result we obtain the following equation for f_0^{res} [9-11] (we shall henceforth omit the suffix "res" for f_0):

$$\frac{\partial f_0}{\partial t}=\frac{\pi e^2}{2m_i^2}\sum_k\frac{|H_k|^2}{k^2c^2}\omega_{Hi}^2\left[\frac{1}{v_\perp}\frac{\partial}{\partial v_\perp}v_\perp-\frac{kv_\perp}{\omega_{Hi}}\frac{\partial}{\partial v_z}\right]\times\delta(kv_z-\omega_k-\omega_{Hi})\left[\frac{\partial f_0}{\partial v_\perp}-\frac{kv_\perp}{\omega_{Hi}}\frac{\partial f_0}{\partial v_z}\right]. \quad (22)$$

The equation for the time variation of $|H_k|^2$ is written in the quasilinear approximation in the form

$$\frac{\partial|H_k|^2}{\partial t}=\frac{\pi}{N_0}\frac{|H_k|^2}{\partial D/\partial\omega_k}\omega_{0i}^2\omega_{Hi}\int v_\perp\delta(kv_z-\omega_k-\omega_{Hi})\times\left[\frac{\partial f_0}{\partial v_\perp}-\frac{kv_\perp}{\omega_{Hi}}\frac{\partial f_0}{\partial v_z}\right]d\mathbf{v}. \quad (23)$$

Equations (22) and (23) form a closed system of quasilinear-approximation equations for the cyclotron instability. The difficulty involved in investigating this system of equations lies in the fact that, unlike the case previously considered, [1, 2] these equations are not one-dimensional in velocity space and, furthermore, in the general case it is possible for several branches of oscillations to be excited simultaneously. However, we shall confine ourselves throughout, with the exception of Sec. 5, to an analysis of the case when only one branch of the oscillations is excited in the plasma, and v_{ph} is a single-valued function of v_{res}. As already noted earlier, for this purpose it is necessary to satisfy the condition (14′), if the instability is connected with temperature anisotropy, or else the condition $u_0\gg v_A$ in the presence of a beam. In the case under consideration, (22) and (23) can be greatly simplified and reduced to one-dimensional equations by changing over to the independent variables[5)]

$$w=v_\perp^2+v_z^2-2\int_{v_{min}}^{v_z}v_{ph}(v_z')\,dv_z',\quad v=v_z \quad (24)$$

(for concreteness we are considering in this section the instability in the region $v_{ph}>0$).

Integrating in (22) and (23) with respect to k and v with allowance for the δ-function, and going over to the variables (24), we obtain the following system of equations

$$\frac{\partial f_0}{\partial t}=\frac{e^2}{4m_i^2c^2}\frac{\partial}{\partial v}\left[v_\perp^2(w,v)\frac{|H_k|^2}{|v-d\omega/dk|}\frac{\partial f_0}{\partial v}\right], \quad (25)$$

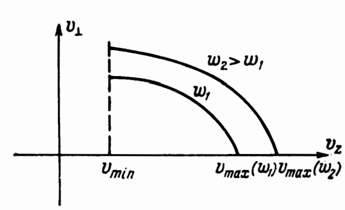

FIG. 3. The lines w = const.

[5)]An analogous transformation of variables, which reduces the equation for f_0 to one-dimensional form, was previously used by Andronov and Trakhtengerts.

$$\frac{\partial |H_k|^2}{\partial t} = -\frac{\pi^2}{N_0}\omega_{0i}^2 \frac{|H_k|^2}{\partial D/\partial\omega_k}\frac{k}{|k|}\int_{w_{min}(v)}^{\infty} v_\perp^2(w, v)\frac{\partial f_0}{\partial v}dw$$

$$(v = (\omega + \omega_{Hi})/k = v_{res}). \quad (26)$$

Equation (25) describes the diffusion of the resonant particles along the lines w = const. The velocity v varies on these lines (see Sec. 3) from v_{min} to $v_{max}(w)$, where $v_{max}(w)$ is defined by

$$w = v_{max}^2 - 2\int_{v_{min}}^{v_{max}} v_{\text{ph}}dv. \quad (27)$$

The minimum value of w for a given v, $w_{min}(v)$ is determined from this same equation by making the substitution $v_{max} \to v$.

From (25) and (26) we can obtain the energy integral, which determines the spectral density of the oscillation energy in terms of the variation of the distribution function of the resonant particles. Integrating (25) with respect to w and expressing

$$|H_k|^2\int v_\perp^2\frac{\partial f_0}{\partial v}dw$$

by means of (26) we obtain

$$\int_{w_{min}(v)}^{\infty}\frac{\partial f_0}{\partial t}dw = -\frac{1}{16\pi^3 m_i c^2}\frac{\partial}{\partial v}\left[\frac{\partial D/\partial\omega_k}{|v - d\omega/dk|}\frac{\partial |H_k|^2}{\partial t}\right]. \quad (28)$$

Integrating (28) with respect to t and v, we obtain the following energy integral:

$$|H_k|^2 = -16\pi^3 m_i c^2\frac{|v - d\omega/dk|}{\partial D/\partial\omega_k}\int_{v_{\text{lim}}}^{v} dv'\int_{w_{min}(v')}^{\infty} dw\,[f_0(t, w, v') - f_0^0(w, v')]. \quad (29)$$

The contribution of the initial noise, $\sim |H_k|^2$ for t = 0, is neglected as usual; v_{lim} is the lower limit of the spectrum: $|H_k|^2_{v_{\text{lim}}} = 0$; $f_0^0(w, v) = f_0(0, w, v)$.

The diffusion of the particles under the influence of the oscillations leads as $t \to \infty$ to occurrence of a stationary state with a "plateau" on the distribution function

$$\partial f_0^\infty / \partial v = 0 \quad \text{for} \quad v > v_{\text{lim}}^\infty; \quad (30)$$

$$|H_k^\infty|^2 = 0, \quad \partial f_0^\infty/\partial v \neq 0 \quad \text{for} \quad v \leqslant v_{\text{lim}}^\infty. \quad (31)$$

The initial distribution function $f_0^0(v)$ for w = const is shown in Fig. 4. For sufficiently large w, when $v_{max}(w) > v^*$, the distribution function has a segment with positive derivative $\partial f_0^0/\partial v$, which, in accordance with (26), leads to instability. As a result of the development of the instability, the resonant particles diffuse into the region of smaller v, and a "plateau" f_0^∞ appears on the distribution function. The diffusion of the particles in the velocity region in which $f_0^\infty < f_0^0$ is accompanied by absorption of energy at the expense of the energy of the oscillations generated by the particles with large values of w, for which $f_0^\infty > f_0^0$ at the same value of v (see Fig. 4). With decreasing v, the interval of values of w in which $f_0^\infty < f_0^0$ increases and energy is absorbed, so that the quantity

$$\int_{w_{min}}^{\infty}[f_0^\infty(w) - f_0^0(w, v)]\,dw$$

decreases.

The limit v_{lim}^∞ of the oscillation spectrum as $t \to \infty$ is determined from the following equation:

$$\int_{w_{min}(v_{\text{lim}}^\infty)}^{\infty} f_0^\infty(w)\,dw = \int_{w_{min}(v_{\text{lim}}^\infty)}^{\infty} f_0^0(w, v_{\text{lim}}^\infty)\,dw. \quad (32)$$

If the limit of the spectrum were below v_{lim}^∞ as given by (32) then, in accordance with (29), an interval of v would exist with negative oscillation. Actually, however, the oscillation energy attenuates to zero in the region $v < v_{\text{lim}}^\infty$ (the energy of the oscillations generated when particles with large w, for which $f_0^\infty > f_0^0$, diffuse into this region of values of v is completely absorbed by the particles with smaller w, but this energy is insufficient for establishment of a "plateau" for all w). Thus a stationary state, defined by relations (31), arises when $v < v_{\text{lim}}^\infty$.

Since $\partial |H_k^\infty|^2/\partial v = 0$ when $v < v_{\text{lim}}^\infty$, we find from (29) that the condition (32) for the distribution function $f_0^\infty(w, v)$ should be satisfied for all $v < v_{\text{lim}}^\infty$. This condition, however, is insufficient for the determination of $f_0^\infty(w, v)$. For this purpose it is necessary to consider with the aid of (25) and (26) the time evolution of the instability, and this is a very complicated problem. We note only that the distribution function for $v < v_{\text{lim}}^\infty$ can-

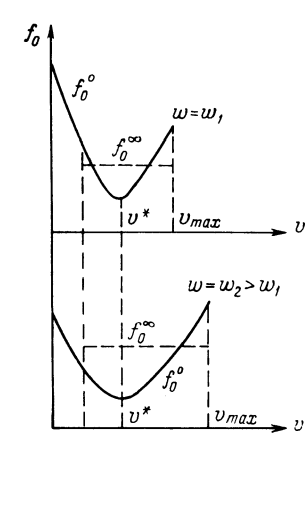

FIG. 4. Change in the distribution function of the resonant particles during the instability.

not coincide with the initial function f_0^0, for otherwise an unstable discontinuity would exist on the distribution function at $v = v_{lim}^\infty$. The growth increment at the discontinuity is equal to

$$\gamma = A \int_{w_{min}(v_{lim}^\infty)}^{\infty} (w - w_{min})[f_0^\infty(w) - f_0^0(w, v_{lim}^\infty)]\,dw$$

($A > 0$, and the region in which $f_0^\infty > f_0^0$ corresponds to large w).

In many cases, however, we can neglect the diffusion of the resonant particles in the region $v < v_{lim}^\infty$ (see Sec. 4 below). Then we get from (25), inasmuch as the diffusion coefficient also vanishes at $v = v_{max}(w)$ ($v_\perp = 0$), a relation that determines $f_0^\infty(w)$ when $v > v_{lim}^\infty$ (the law of conservation of the number of particles on the line w = const):

$$f_0^\infty(w)[v_{max}(w) - v_{lim}^\infty] = \int_{v_{lim}^\infty}^{v_{max}(w)} dv\, f_0^0(w, v). \quad (33)$$

If v_{lim}^∞ as given by (32) is smaller than the minimum value of the resonant velocity (see (10)), it is necessary to take v_{min} as the lower limit of the spectrum. When $v < v_{min}$, the distribution function remains unchanged.

Using (21) and (29), we can readily obtain the following energy conservation law: the energy given up by the resonant particles during the instability is equal to the energy acquired by the nonresonant particles and by the field. Summing with respect to k in (29) we get

$$-\sum_k |H_k|^2 \frac{\partial D}{\partial \omega_k} \frac{\omega_k}{k^2} = 8\pi^2 c^2 \int_{v_{lim}}^{\infty} dv\, v_{ph}(v) \int_{v_{lim}}^{v} dv'$$
$$\times \int_{w_{min}(v')}^{\infty} dw\, [f_0(t, w, v') - f_0^0(w, v')]. \quad (34)$$

Integrating in (34) by parts and using the law of conservation of the total number of particles for $v \geq v_{lim}$, which follows from (28), we obtain an equation for the change in the energy of the resonant particles:

$$\Delta\mathscr{E}_{res} = \int \frac{m_i}{2}(v_\perp^2 + v_z^2)(f_0^0 - f_0)\,d\mathbf{v}$$
$$= -\sum_k \frac{|H_k|^2}{8\pi} \frac{\partial D}{\partial \omega_k} \frac{\omega_k}{k^2 c^2} \quad (35)$$

In the derivation of (35) we used also the fact that the diffusion occurs only along the lines w = const.

The energy of the electromagnetic field is

$$\mathscr{E}_f = \frac{1}{8\pi} \sum_k |H_k|^2 \left(1 + \frac{\omega_k^2}{k^2 c^2}\right),$$

therefore we obtain with the aid of (21), (35), and (4′) the following energy conservation law:

$$\mathscr{E}_f + \sum_\alpha \Delta\mathscr{E}_\alpha = \mathscr{E}_{res}. \quad (36)$$

4. In this section we consider in greater detail cases when the instability of the cyclotron waves is due to the anisotropy of ion temperatures or to the interaction between and ion beam and a plasma, and we determine with the aid of the general formulas of the preceding section the spectrum of the oscillations excited by such an instability.

The initial distribution function of the beam-plasma system is written, in terms of the variables w and v, in the form

$$f_0^0 = N_0 \left(\frac{m_i}{2\pi T}\right)^{3/2} \exp\left[-\frac{m_i w}{2T} - \frac{m_i}{T} c_A v\right]$$
$$\times \left\{1 + \frac{N_1}{N_0} \exp\left[\frac{m_i}{T} u_0 v - \frac{m_i}{2T} u_0^2\right]\right\}. \quad (37)$$

We assume that $u_0 \gg c_A, \sqrt{T/m_i}$, so that Alfven waves with $v_{ph} \approx c_A$ are excited.

The distribution function (37) has a minimum at

$$v = v^* = \frac{u_0}{2} + \frac{T}{m_i u_0} \ln\left[\frac{N_0}{N_1} \frac{c_A}{u_0}\right];$$

for larger v we get $\partial f_0^0/\partial v > 0$ and instability sets in. Using (33), we obtain

$$f_0^\infty \approx N_1 \left(\frac{m_i}{2\pi T}\right)^{3/2} \frac{T}{m_i u_0 v_{max}(w)}$$
$$\times \exp\left(-\frac{m_i w}{2T} + \frac{m_i u_0 v_{max}(w)}{T}\right). \quad (38)$$

In this formula $w_{max}(w) = c_A + (c_A^2 + w)^{1/2}$. The distribution function f_0^∞ has a maximum at $w + w^*$, determined by the relation

$$v_{max}(w^*) = u_0.$$

From now on the region $w \approx w^*$, in which $v_{max} \gg c_A$ and $v_{max} \gg v_{lim}^\infty$, will be important for the calculation of the oscillation energy, a fact we have already used when determining f_0^∞.

To calculate the integral with respect to w in (29) we use the saddle point method. As a result we obtain

$$\int_{w_{min}(v)}^{\infty} dw\, [f_0^\infty(w) - f_0^0(w, v)] = \frac{N_1}{(2\pi)^{3/2}} \left(\frac{m_i}{T}\right)^{1/2} \left\{\left(\frac{8T}{m_i u_0^2}\right)^{1/2}\right.$$
$$\times \Phi\left(\left(\frac{m_i}{8T}\right)^{1/2} \frac{w_{min}(v) - w^*}{u_0}\right) - \exp\left(-\frac{m_i (v - u_0)^2}{2T}\right)$$
$$\left.- \frac{N_0}{N_1} \exp\left(-\frac{m_i v^2}{2T}\right)\right\}. \quad (39)$$

We have introduced here the symbol

$$\Phi(z)=\int_z^\infty e^{-u^2}du.$$

The lower limit of the oscillation spectrum at $t\to\infty$ is determined by relation (32) and is equal to

$$v_{\lim}{}^\infty=\left(\frac{2T}{m_i}\right)^{1/2}\ln\left[\frac{N_0}{N_1}\left(\frac{m_i u_0^2}{T}\right)^{1/2}\right]. \qquad (40)$$

We note that the diffusion of the particles in the region $v < v_{\lim}^\infty$ can in this case be neglected, since the width of this region is of the order of $v_{\lim} \ll u_0$. An allowance for the particle diffusion when $v < v_{\lim}^\infty$ would lead to small corrections $\sim v_{\lim}^\infty/u_0$ in Eq. (33) for f_0^∞.

Using (39), we obtain from the general formula (39) the following relation for the spectral energy density of the Alfven waves excited by the beam:

$$|H_k{}^\infty|^2+|E_k{}^\infty|^2=8\pi^{3/2}N_1 m_i v^3\frac{c_A}{\omega_{Hi}u_0}\left(1+\frac{c_A{}^2}{c^2}\right)\Phi\left(\left(\frac{m_i}{2T}\right)^{1/2}\times(v-u_0)\right)\qquad (k=\omega_{Hi}/v). \qquad (41)$$

For small v, the quantity $|H_k^\infty|^2$ increases in a power-law fashion ($\sim v^3$), reaching a maximum at $v\simeq u_0$; with further increase of v it decreases exponentially with a characteristic decrement $\Delta v \sim (T/m_i)^{1/2}$. Summing in (41) over all k, we find that the total energy of the electromagnetic oscillations as $t\to\infty$ remains smaller than the beam energy in a ratio c_A/u_0:

$$\sum_k\frac{|E_k{}^\infty|^2+|H_k{}^\infty|^2}{8\pi}\approx\frac{1}{4}N_1 m_i c_A u_0\left(1+\frac{c_A{}^2}{c^2}\right). \qquad (42)$$

The smallness of the oscillation energy is attributed to the fact that in the highest order in the parameter $c_A/u_0 \ll 1$ the particle energy $m(v_\perp^2+v_z^2)$ is conserved on the lines w = const, along which the diffusion takes place, and the diffusion merely transfers the energy in the beam from the longitudinal to the transverse component. A change in the total energy in the beam appears in the next higher order in the parameter c_A/u_0.

We now consider a case when the instability is due to the temperature anisotropy. We assume first that $T_\parallel \gg T_\perp$ and condition (14′) is satisfied, so that only one branch of the oscillations with $v_{ph} > 0$ is unstable. In terms of the variables w and v the initial distribution function takes the form

$$f_0{}^0=N_0\frac{m_i}{2\pi T_\perp}\left(\frac{m_i}{2\pi T_\parallel}\right)^{1/2}\exp\left[-\frac{m_i w}{2T_\perp}+\frac{m_i v^2}{2T_\parallel T_\perp}\times(T_\parallel-T_\perp)-\frac{m_i}{T_\perp}\int_{v_{min}}^{v}v_{\rm ph}(v')\,dv'\right]. \qquad (43)$$

A plot of this function is shown in Fig. 4. In the case which we consider, that of v^* for which f_0^0 is minimal, is determined from the equation

$$v^*=\frac{T_\parallel}{T_\parallel-T_\perp}v_{\rm ph}(v^*). \qquad (44)$$

Using (9) we get

$$v^*=c_A\frac{T_\parallel}{T_\parallel-T_\perp}\left(1-\frac{c_A{}^2}{v_A{}^2}\frac{T_\parallel-T_\perp}{T_\parallel}\right)^{-1/2}. \qquad (44')$$

It can be shown that an appreciable contribution to the oscillation energy is made only by those values of w for which $v_{max}(w)-v \lesssim T_\parallel/m_i v^*$. We shall henceforth assume that the following condition is satisfied

$$\frac{T_\parallel}{m_i v^*}\ll\left(\frac{T_\perp}{m_i}\frac{T_\parallel}{T_\parallel-T_\perp}\right)^{1/2}. \qquad (45)$$

Introducing the dimensionless variables

$$u=\alpha(v-v^*),\qquad u_{1,2}=\alpha(v_{max,\lim}-v^*),$$

where

$$\alpha=\left(\frac{m_i}{2}\frac{T_\parallel-T_\perp}{T_\parallel T_\perp}\right)^{1/2},$$

we write f_0^0 for $u \ll 1$ in the form

$$f_0{}^0(w,v)=f_0{}^0(w,v^*)\left[1+u^2\left(1-\frac{T_\parallel}{T_\parallel-T_\perp}\frac{dv_{\rm ph}}{dv^*}\right)\right]. \qquad (46)$$

We note that this expansion is not applicable if $v^*\to v_{min}$, when $dv_{ph}/dv^*\to-\infty$. Inasmuch as $u\sim \alpha T_\parallel/m_i v^*$ and

$$\frac{dv_{\rm ph}}{dv^*}\approx\frac{c_A{}^2}{v_A{}^2}\frac{T_\parallel-T_\perp}{T_\parallel}\left(3\frac{c_A{}^2}{v_A{}^2}-2\frac{T_\parallel}{T_\parallel-T_\perp}\right)^{-1}\approx-\frac{1}{3^{3/4}}\left[\frac{c_A}{v^*-v_{min}}\right]^{1/2},$$

the condition for applicability of the expansion (46) is

$$v^*-v_{min}\gg c_A\left(\frac{T_\parallel}{m_i v^{*2}}\frac{T_\parallel}{T_\perp}\right)^2. \qquad (47)$$

Using (46) we obtain for f_0^∞ the relation

$$f_0{}^\infty(w)=f_0{}^0(w,v^*)\left[1+\frac{1}{3}(u_1{}^2+u_1u_2+u_2{}^2)\times\left(1-\frac{T_\parallel}{T_\parallel-T_\perp}\frac{dv_{\rm ph}}{dv^*}\right)\right]. \qquad (48)$$

With the aid (46) and (48) we can readily calculate from the general formulas the energy of the oscillations and the limit of the spectrum as

$t \to \infty$. As a result we get

$$v_{\lim}^{\infty} = v^* - \frac{2}{3}\frac{T_\parallel}{m_i v^*},$$

$$|H_k^\infty|^2 = \frac{32}{3}\left(\frac{\pi}{2}\right)^{3/2} N_0 |T_\parallel - T_\perp| \left(1 - \frac{T_\parallel}{T_\parallel - T_\perp}\frac{dv_{\mathrm{ph}}}{dv^*}\right)$$
$$\times \frac{|v - d\omega/dk|}{v^*}\frac{c^2}{|\partial D/\partial\omega_k|}\left(\frac{m_i^3}{T_\perp^2 T_\parallel}\right)^{1/2}(v - v_{\lim}^\infty)^2$$
$$\times \exp\left\{-\frac{m_i}{2T_\parallel}v^*(2v - v^*)\right\} \tag{49}$$

$$(k = \omega_{Hi}/[v - v_{\mathrm{ph}}(v)]). \tag{50}$$

The maximum of $|H_k^\infty|^2$ is reached when $v \approx v_{\lim}^\infty + 2T_\parallel/m_i v^*$, and $|H_k^\infty|^2$ decreases exponentially with further increase of v.

At $t \to \infty$, undamped oscillations exist only when $v \geq v_{\lim}^\infty$. If t is finite, however, oscillations that lead to particle diffusion exist also when $v < v_{\lim}^\infty$. Let us determine the width of the region below $v_{\lim}^\infty$ in which the particle diffusion takes place. To this end we estimate with the aid of (29) the contribution made to the oscillation energy by the particles for which $f_0^\infty > f_0^0$. As already noted, only the diffusion of these particles is accompanied by energy transfer to the oscillations. We also assume here that a "plateau" on the distribution function is established for these particles. We thus obtain an upper bound for $|H_k^{\max}|^2$—the maximum value of $|H_k(t)|^2$ in the region $v < v_{\lim}^\infty$:

$$|H_k^{max}|^2 \lesssim \exp\left[-\frac{m_i u^* v^*}{\alpha T_\parallel}\right]|H^\infty|^2; \tag{51}$$

Here $|H^\infty|^2$ is the maximum in the spectrum of $|H_k^\infty|^2$ in the region of the plateau ($v > v_{\lim}^\infty$),

$$u^* = {}^1\!/_2|u_2| + (3u^2 - {}^3\!/_4 u_2^2)^{1/2}.$$

It follows from (51) that the width of the region in which

$$|H_k^{max}|^2 \sim |H^\infty|^2,$$

is

$$\Delta v \sim T_\parallel / m_i v^*.$$

Outside this region the quantity $|H_k^{\max}|^2$ is exponentially small, and particle diffusion can be neglected. Inasmuch as $\Delta v \sim |v_{\lim}^\infty - v_{\max}|$, (48) and (50) yield in the general case only estimates.

With the aid of (50) we can readily obtain the following estimate for the total oscillation energy:

$$\sum_k \frac{|E_k^\infty|^2 + |H_k^\infty|^2}{8\pi}$$
$$\sim N_0|T_\parallel - T_\perp|\left[\frac{kc^2}{|\partial D/\partial\omega_k|}\left(1 + \frac{\omega_k^2}{k^2c^2}\right)\right]\Bigg|_{v_{res}=v^*}$$
$$\times\left(1 - \frac{T_\parallel}{T_\parallel - T_\perp}\frac{dv_{\mathrm{ph}}}{dv^*}\right)\frac{1}{(v^* - v_{\mathrm{ph}}^*)}\left(\frac{T_\parallel}{m_i v^{*2}}\right)^{3/2}$$
$$\times \exp\left(-\frac{m_i}{2T_\parallel}v^{*2}\right). \tag{52}$$

We note that since $m_i v_A^2 \gg T_\parallel$, the oscillation energy is exponentially small even for the strongest anisotropy. This is connected with the small number of the resonant particles interacting in this case with the waves.

If $v_{\lim}^\infty$ is sufficiently close to $v_{\min}$ ($v_{\lim}^\infty - v_{\min} \ll T_\parallel/m_i v^*$), then the diffusion in the region of velocities $v < v_{\lim}^\infty$ is negligible, since the particle distribution function does not change when $v < v_{\min}$. In this case the obtained formula for $|H_k^\infty|^2$ is exact. When $v_{\lim}^\infty < v_{\min}$, the lower limit of the spectrum coincides with $v_{\min}$. In this case we have for the spectral density of the oscillations the formula

$$|H_k^\infty|^2 = \frac{16}{3}\left(\frac{\pi}{2}\right)^{3/2}$$
$$\times N_0|T_\parallel - T_\perp|\frac{c^2|v - d\omega/dk|}{|\partial D/\partial\omega_i|v^*}\left(\frac{m_i^3}{T_\perp^2 T_\parallel}\right)^{1/2}$$
$$\times(v - v_{min})\left[2(v - v_{min}) + \frac{2T_\parallel}{m_i v^*} - 3(v^* - v_{min})\right]\cdot$$
$$\times\exp\left[-\frac{m_i}{2T_\parallel}v^*(2v - v^*)\right]. \tag{53}$$

If the transverse ion temperature exceeds the longitudinal temperature, the instability is connected with the normal Doppler effect and occurs in the region $v_{\mathrm{ph}} < 0$. The initial distribution function $f_0^0(w, v)$ has a maximum at a value v^* defined by (44). At large v the derivative $\partial f_0^0/\partial v < 0$, which leads in this case to instability. Calculation of f_0^∞ and $|H_k^\infty|^2$ is carried out in analogy with the corresponding calculations for case $T_\parallel > T_\perp$ and leads to the same formula (50) for $|H_k^\infty|^2$. In this case, however, we have for strong temperature anisotropy, $T_\perp \gg T_\parallel$,

$$v^* \sim v_A(T_\parallel / T_\perp)^{3/2} \ll v_A,$$

and the oscillation energy increases somewhat compared with the case $T_\parallel > T_\perp$, when $v^* \gtrsim v_A$ (as before, we are considering the case when $m_i v_A^2 \gg T_\parallel$). It follows from (45), however, that in order for the formula obtained for $|H_k^\infty|^2$ to be valid the temperature anisotropy must not be too large, $T_\perp \ll (T_\parallel^2 m_i v_A^2)^{1/3}$, and consequently we have $v^* \gg \sqrt{T_\parallel/m_i}$ under the conditions when formula (50) is applicable.

5. In the preceding sections we have considered the case when only one of the two branches of the

oscillations of $v_{ph}(v)$ existing when $v_{ph} > 0$ is excited, corresponding to the smaller values of the phase velocity $v_{ph}^{I} < v_{ph}^{II}$. The "plateau" on the distribution function, resulting from the excitation of this branch, is stable with respect to excitation of a second branch of oscillations. Indeed, substituting in the expression for the growth increment of this branch of oscillations

$$\gamma = C \int_{w_{min}^{II}}^{\infty} v_\perp^2 \frac{\partial f_0}{\partial v} dw^{II},$$

where

$$C = -\frac{\pi^2}{2N_0}\omega_{0i}^2 \frac{1}{\partial D/\partial\omega_k} > 0,$$

and the derivative with respect to v is taken at

$$w^{II} = v_\perp^2 + v_z^2 - 2\int_{v_{min}}^{v_z} v_{ph}^{II}(v_z')\,dv_z' \equiv \text{const},$$

the distribution function

$$f_0^\infty(w^I) = f_0^\infty\left[w^{II} + 2\int_{v_{min}}^{v}(v_{ph}^{II} - v_{ph}^{I})\,dv'\right],$$

we obtain

$$\gamma = 2C \int_{w_{min}^{II}}^{\infty} v_\perp^2 (v_{ph}^{II} - v_{ph}^{I}) \frac{\partial f_0^\infty}{\partial w^{II}} dw^{II}$$

$$= -2C(v_{ph}^{II} - v_{ph}^{I}) \int_{w_{min}^{II}}^{\infty} f_0^\infty dw^{II} < 0. \tag{54}$$

In many cases, for example when $u_0 \sim v_A$, simultaneous excitation of both branches of oscillations is possible. In these cases the quasilinear system of equations (22)–(23) cannot be reduced to the one-dimensional system (25)–(26). However, it is possible to write the quasilinear equations in a form similar to that used in (25) and (26). Breaking up the sum over k in (22) into two terms corresponding to the two branches of the oscillations, and going over in each of the terms to the variables v and w^β, where w^β is defined by formula (24) in which v_{ph} is replaced by v_{ph}^{β}, we obtain

$$\frac{\partial f_0}{\partial t} = \frac{e^2}{4m_i^2c^2}\sum_{k,\beta}\frac{\partial^\beta}{\partial v}\left\{v_\perp^2 \frac{|H_k^\beta|^2}{|v - d\omega^\beta/dk|}\frac{\partial^\beta f_0}{\partial v}\right\} \tag{55}$$

(the summation over β is over both oscillation branches, and $\partial^\beta/\partial v$ denotes the derivative with respect to the v with w^β constant).

From (23) we obtain similarly

$$\frac{\partial |H_k^\beta|^2}{\partial t} = -\frac{\pi^2}{N_0}\omega_{0i}^2 \frac{|H_k^\beta|^2}{\partial D/\partial\omega_k^\beta}\int_{w_{min}^{\beta}}^{\infty} v_\perp^2(w^\beta, v)\frac{\partial^\beta f_0}{\partial v} dw^\beta. \tag{56}$$

The system (55)–(56) has two stationary solutions,

$$|H_k^{II}|^2 = 0, \quad \partial^I f_0/\partial v = 0$$

or

$$|H_k^{I}|^2 = 0, \quad \partial^{II} f_0/\partial v = 0.$$

Only the first of these, as already noted, is stable (see (54)). Thus, the entire energy of the oscillations excited in the plasma is pumped over, in final analysis, into the first branch.

In the case under consideration, when both branches are simultaneously excited, the oscillation energy can be determined with the aid of the energy integral, in analogy with (29). To derive this integral we use the system (22)–(23). It is necessary to integrate over all v in (22). The derivation of the energy integral is the same as before (see the derivation of (29)). As a result we obtain

$$32\pi^3 m_i c^2 \int_{v_{lim}}^{v_z} dv_z' \int_0^\infty v_\perp (f_0 - f_0^0)\,dv_\perp$$

$$= -\sum_{k,\beta}\frac{\partial D}{\partial\omega_k^\beta}\frac{1}{|v - d\omega^\beta/dk|}|H_k^\beta|^2. \tag{57}$$

As $t \to \infty$, when the entire energy is pumped over into the first branch, we obtain for the determination of $|H_k^I|^2$ an equation that coincides with (29). A "plateau" $\partial^I f_0^\infty/\partial v = 0$ is established on the distribution function at $t \to \infty$. However, since simultaneous excitation of both branches of the oscillations causes the particle diffusion to be no longer one-dimensional, it is impossible to obtain in the general case a conservation law similar to (33). On the other hand, if the phase velocities of both branches are sufficiently close,

$$\varepsilon = \frac{v_{ph}^{II}(v) - v_{ph}^{I}(v)}{v_{ph}^{I}(v)} \ll 1,$$

then the particle diffusion is close to one-dimensional and we obtain for the determination of f_0 an equation that differs from (33) in small terms of the order of ϵ.

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