PROPAGATION OF LIGHT THROUGH A SCHWARZSCHILD SINGULAR SPHERE

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An analytic solution is obtained for the problem: How does an observer see the emitting surface of a gravitating sphere expanding with parabolic velocity inside its singular sphere?

1. AS is well known, empty spherically symmetric space-time is not confined to the region $1 < r < \infty$, $-\infty < t < \infty$ of variation of the Schwarzschild coordinates r, t with the metric

$$ds^2 = -\frac{r-1}{r}dt^2 + \frac{r}{r-1}dr^2 + r^2(d\theta^2 + \sin^2\theta\, d\varphi^2). \quad (1)$$

This region (shaded in the figure, see below) borders on two regions,[1] called the interior of the singular sphere, in which also a coordinate system with the metric (1) can be introduced,[2] the coordinate r now taking values $0 < r < 1$. At the boundary between the regions there is no singularity of the internal geometry, but the metric tensor (1) has a singularity here. Therefore, for the solution of the problem of light propagation through this boundary use has been made[3] of the Lemaitre coordinate system (cf. e.g.,[4]) which has no singularity on the boundary but is more complicated, so that only a numerical integration was possible.

In the present paper the problem is the same as in[3], namely: how an observer sees the emitting surface of a gravitating sphere expanding with parabolic velocity, and is solved analytically in a coordinate system with the metric (1). To overcome the difficulty encountered in identifying the parts of a geodesic which are separated by the singular sphere, the equation of the geodesic on one side of the singular sphere is regarded as the analytic continuation of its equation on the other side of the singular sphere. In other words, we use a coordinate system with the metric (1) obtained, for example, from the Lemaitre coordinate system by means of an analytic transformation of coordinates. The resulting assignment of complex values of the coordinate t to events inside the singular sphere is not criminal, since it has no physical meaning and arises from considerations of mathematical orderliness in the calculations. The canonical parameter, which can readily be verified to be r on radially isotropic geodesics, of course remains real.

It must be pointed out that the method of identification used here is of course not the only possible one.

2. The equation of the world line of an emitter located on the surface of a sphere expanding with parabolic velocity can be obtained[4] by fixing the Lemaitre radial coordinate R in the equation which gives the transition from the Schwarzschild system to the Lemaitre system. In this way we get for an emitter in the equatorial plane:

$$t_1 = \frac{2}{3}\sqrt{r_1}(r_1+3) + \ln\frac{\sqrt{r_1}-1}{\sqrt{r_1}+1} - R,$$

$$\varphi_1 = \text{const}, \qquad \theta = \frac{\pi}{2}. \quad (2)$$

Here R plays the role of an additive constant; t_1, r_1, φ_1 are running coordinates of the emitter, and we use, for example, the sheet of the complex plane of r_1 such that $\operatorname{Im}(t_1) = +i\pi$ when $0 < r < 1$.

The equations of the world line of the observer can obviously be

$$r_2 = \text{const} > 1, \qquad \varphi_2 = 0, \qquad \theta = \pi/2$$

(r_2, φ_2 are running coordinates of the observer).

The equation of the isotropic world line of a light pulse connecting the event (t_1, r_1, φ_1) (emission) with the event (t_2, r_2, φ_2) (registration) is given in[5]:

$$t_2 - t_1 = \int_{r_1}^{r_2} \frac{r^2\sqrt{r}\,dr}{(r-1)[r^3-(r-1)\rho^2]^{1/2}}, \quad (3)$$

$$\varphi_1 = -\int_{r_1}^{r_2} \frac{\rho\,dr}{[r(r^3-(r-1)\rho^2)]^{1/2}}, \quad (4)$$

where the integration is taken along the real axis of r with the appropriate detour around the singular point of the integrand at $r = 1$. The constant ρ

fixes the "impact parameter" of the ray; through it the angle α between the direction of travel of the light pulse and the radial direction is determined. In fact, an elementary displacement dr, $d\varphi$ has as its projections on the coordinate axes [(on the basis of (1)] segments of lengths $[r/(r-1)]^{1/2}\,dr$, $r\,d\varphi$. Using the fact that we are interested in a displacement along an isotropic geodesic, we get from (4)[1)]

$$\tan\alpha = [r(r-1)]^{1/2}\frac{d\varphi}{dr} = \rho\left[\frac{r-1}{r^3-(r-1)\rho^2}\right]^{1/2}. \quad (5)$$

To calculate the Doppler shift of the frequency of the light we apply an argument analogous to that given in[6]. It is easy to see that the frequency ν (or the quantum energy $h\nu$) of a registered (or emitted) light pulse is proportional to the scalar product of the unit vector e^i ($e^ie^jg_{ij}=1$) tangent to the world line of the observer (or the emitter) and the wave vector K^i ($K^iK^jg_{ij}=0$) tangent to the world line of the light pulse. In order that the proportionality constant not change along the world line of the light, and that it be permissible to compare the frequency values at different points of this line, the wave vector K^i must undergo parallel displacement along this world line (isotropic vectors cannot be compared as to length).

According to (1) and (2), the components of the unit vector $e_1{}^i$ of the emitter are

$$e_1{}^i = \{r_1/(r_1-1),\ 1/\sqrt{r_1},\ 0\}.$$

The components of the unit vector of the observer are obviously

$$e_2{}^i = \{[r_2/(r_2-1)]^{1/2},\ 0,\ 0\}.$$

The isotropic vector obtained by differentiation of (3) and (4) with respect to r is not one corresponding to parallel displacement, but is obviously proportional to one that is. Denoting the scalar proportionality constant by ψ, we get

$$K^i = \left\{\frac{r^2\sqrt{r}\,\psi}{(r-1)[r^3-(r-1)\rho^2]^{1/2}},\ \psi,\ \frac{\rho\psi}{[r(r^3-(r-1)\rho^2)]^{1/2}}\right\}.$$

To determine $\psi(r)$ we require that

$$dK^i = -K^j\Gamma_{jk}{}^i dx^k, \quad (6)$$

where for convenience we have used dx^k to mean an elementary vector displacement along the isotropic geodesic $x^0=t(r)$, $x^1=r$, $x^2=\psi(r)$ given by (3) and (4). Since

$$K^i = \psi(r)\frac{dx^i}{dr}, \qquad \frac{d^2x^i}{dr^2}+\Gamma_{jk}{}^i\frac{dx^j}{dr}\frac{dx^k}{dr} = \lambda(r)\frac{dx^i}{dr},$$

(6) reduces to the form

$$d\psi/dr + \lambda\psi = 0,$$

which can be integrated by elementary means and gives, up to a constant factor,

$$\psi = \left[1-\frac{r-1}{r^3}\rho^2\right]^{1/2}.$$

The result is that we get for the ratio of the frequencies of the registered and the emitted light the expression

$$\frac{\nu_2}{\nu_1} = \frac{K^i(r_2)e_2{}^j g_{ij}(r_2)}{K^i(r_1)e_1{}^j g_{ij}(r_1)}$$
$$= \frac{r_1(r_1-1)}{r_1^2-[r_1^3-(r_1-1)\rho^2]^{1/2}}\left[\frac{r_2}{r_2-1}\right]^{1/2}, \quad (7)$$

which is, indeed, given not as a function of the time of observation t_2 but as a function of the parameter r_1; the connection between r_1 and t_2 is found by eliminating t_1 from (2) and (3):

$$t_2 = \frac{2}{3}\sqrt{r_1}(r_1+3)+\ln\frac{\sqrt{r_1}-1}{\sqrt{r_2}+1}$$
$$-R+\int_{r_1}^{r_2}\frac{r^2\sqrt{r}\,dr}{(r-1)[r^3-(r-1)\rho^2]^{1/2}} \quad (8)$$

(the arbitrary constant R serves to fix the initial time of the phenomenon).

Equations (5), (7), and (8) give the complete evolution of the picture observed at r_2 (with the exception of the intensity), because when r_1 and ρ are eliminated from these equations there remains the dependence of the observable spectral shift on the angle of observation α and the time of observation t_2.

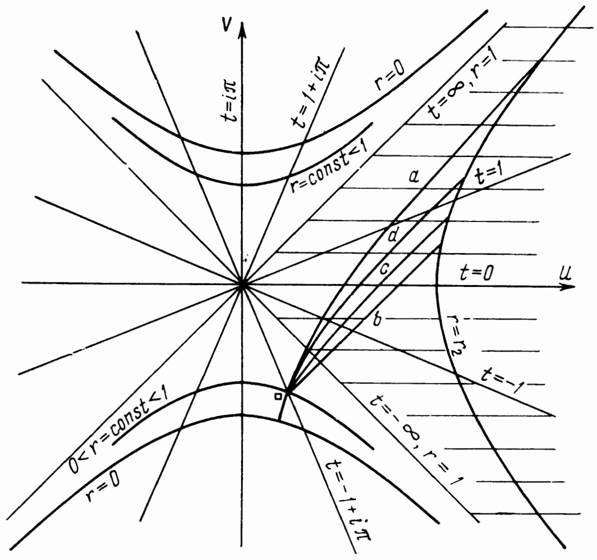
The figure shows the u, v, plane,[1] on which are traced some coordinate lines of the Schwarzschild coordinate system (t, r), and also the world lines of the emitter (a), the observer ($r=r_2$), and of light pulses emerging from an arbitrary event □ (see figure) with $\rho=0$ (b) and $\rho>0$ (c, d). The world line of the light pulse with the largest possible value $\rho=\rho_{max}$(d) comes out from the event □ tangent to the world line of the observer (a). In other words, the radial component of the velocity of the pulse with ρ_{max} is equal to the speed of expansion of the emitting sphere—i.e., such pulses are emitted tangentially to the surface of the sphere. This condition gives

$$e_1{}^0/e_1{}^1 = K^0(r_1,\rho_{max})/K^1(r_1,\rho_{max})$$

or

$$\rho_{max} = r_1.$$

1)This formula gives the correct result if on the world line of the observer $r=\text{const}$, $\varphi=\text{const}$. Since such an observer cannot be realized for $r<1$, this formula does not apply there.



The angular diameter $2\alpha_{max}$ of the visible luminous disk is now determined by means of (5) as a function of the parameter r_1 (for $\rho = r_1$).

Equation (7) shows that in the course of time the frequency ratio ν_2/ν_1 for the center of the disk ($\rho = 0$) decreases monotonically from infinity (at the time of the first flash) according to the law (r_1 increases from zero to r_2)

$$\left.\frac{\nu_2}{\nu_1}\right|_{\rho=0} = \frac{1+\sqrt{r_1}}{\sqrt{r_1}}\left(\frac{r_2}{r_2-1}\right)^{1/2}$$

and for the edge of the disk ($\rho = r_1$) it is constant and equal to $[r_2/(r_2-1)]^{1/2}$. The frequency ratio of the light emitted by the luminous sphere as it passes through the singular sphere ($r_1 = 1$) and received by a very distant observer is equal to two; the light received by such an observer from the edge of the disk has its natural color at all times.

These last conclusions agree with the results given in [3].

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